Cardiovascular Fluid Mechanics
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Lecture – 10
Pulsatile Flow

In this lecture we will be looking at the pulsatile flow, as we know that the flow in the cardiovascular system is pulsatile we have in on an average in a healthy human being about that the heart beats about 72 times per minute. So, the flow process or the circulatory process repeats itself about ones per second, the frequency is about ones per second ah. So, the flow is not steady as is assumed in the poiseuille flow for example, rather it the the pressure gradient or the pressure that drives the flow in the circulatory system it changes almost every second and it is periodic in nature. So, the flow is pulsatile and periodic. So, in order to understand or in order to study the flow in the cardiovascular system it is important that we understand and we know what is the flow in a, the pulsatile flow in a rigid channel for example, so that is what we are going to study in this lecture ok.

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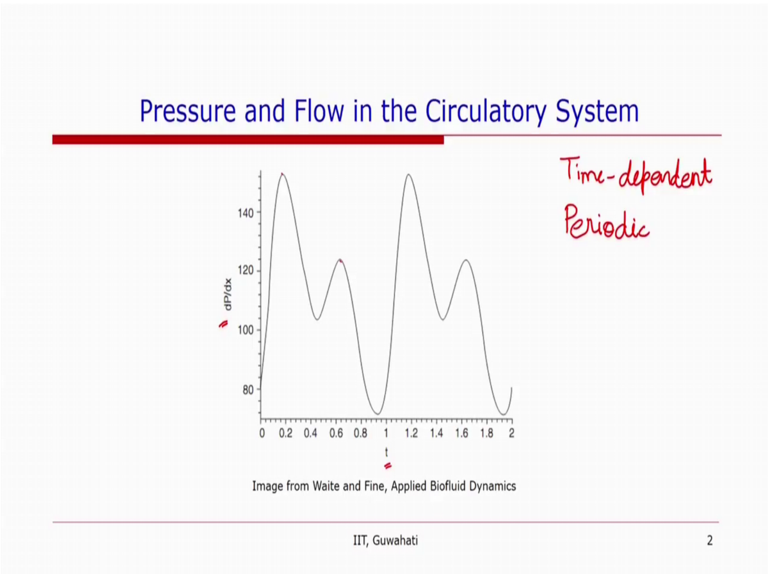

Now, what we see in this graph here is the pressure gradient d p by d x verses time; this is a typical pressures verses time diagram in the circulatory system. So, the pressure goes through certain changes the it peaks first the pressure gradient is highest first and then there is a dip and then again a bit increase and then again and then the process repeat itself. So, the pressure gradient is pulsatile the pressure is pulsatile in nature, it is periodic in nature, it is time dependent, it is not steady and same goes for the pressure gradient.

So, we need to study that because the flow is driven by the pressure gradient in the circulatory system and the pressure and the pressure gradient, they are time dependent, they are pulsatile, they are periodic. So, the associated flow is also going to be pulsatile it is going to be periodic in nature and it will be time dependent. So, while this graph is periodic in nature it is of course, time dependent, but by just looking at the graph we cannot guess the function or time dependent function that the pressure follows.

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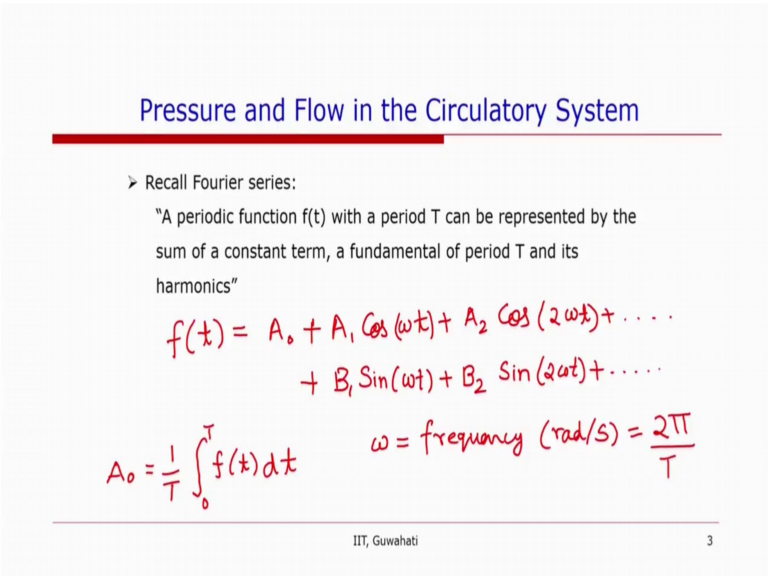

So, here we might recall from our under graduate classes that any periodic function say f t which has a time period of T it can be represented as a sum of one constant term and a fundamental of period T and its harmonics. So, a periodic function can be represented as a sum of f t can be represented as a constant. So, let us say this constant is A 0 plus some cosine terms, let us say A 1 cos omega t plus A 2 cos 2 omega t plus. So, on plus B 1 sin omega t plus B 2 sin 2 omega t plus. So, on where omega is the frequency then the unit is radian per second, it can be 2 pi over t where t is the time period. So, now, this any signal which is periodic and time dependent that can be represented as the sum of a constant term plus sin and cosine terms and these constants A 0, A 1 ah, A 2, B 1, B 2 they can be evaluated by different methods. So, one of the methods is where for example, one calculates that A 0 is equal to 1 over t integral 0 to t f t d t and similar terms for A 1 and B 1 there you will have 2 by t 0 to t f t sin omega t d t and so on so forth ok.

So, the point I am trying to make here is that a periodic signal which is arbitrary can be decomposed into sinusoidal or and sin and cosine terms. Now, this sin and cosine terms the coefficients of these sin and cosine terms can be obtained ah by using fourier transform.

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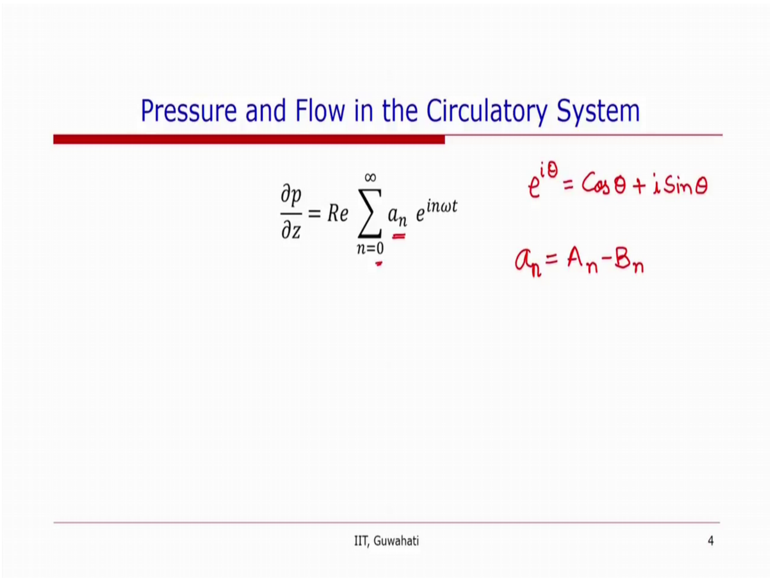

Now, as you might remember that sin and cosine terms can be also represented as a complex exponential terms. So, you can write e to the power I theta is equal to cos theta plus I sin theta. So, for differentiation and integration purposes it is easier to represent the fourier series in terms of exponential term, you can write e to the power I theta is equal to cos theta and I sin theta and. So, let us say that the pressure gradient del p over del z can be represented as sum of a n which is a coefficient in to exponential term e to the power I n omega t where n is an integer which value will change from 0 to infinity at n is equal to 0 the first term will be constant and e to the power I n omega t. So, a n will be ah, a n minus b n if you want to relate it from the previous terms. So, the pressure gradient can be represented as the sum of different harmonics are n is equal to 1, 2, 3, 4 for a can be represented as a sum of the harmonics. So, our target or our goal here now is to understand or to obtain a relationship for pulsatile flow where the pressure gradient is say exponential and the del p by del z is, del p by del z constant and into a function of time which is exponential function ok.

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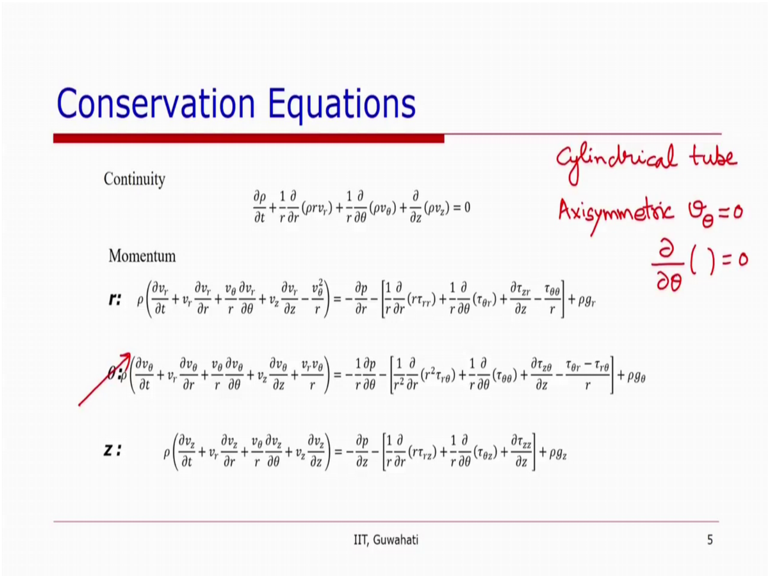

So, we will start with the conservation equations that we have been doing as of until now, the mass conservation which is also called continuity, momentum r theta and z. So, you see they are all in cylindrical coordinates and the stresses the equation has been written in terms of stresses, we will assume the flow in a cylindrical tube and flow is in the axial direction ah. So, the flow is going to be axisymmetric and when flow is axisymmetric; that means, v theta is equal to 0 and the terms the gradient of the variables in the theta direction will be 0. So, this will end up that we will not have any term in the theta coordinate and the respective or the v terms containing v theta or del by del theta or the gradient in the theta direction will be eliminated in the equations r and z directions also.

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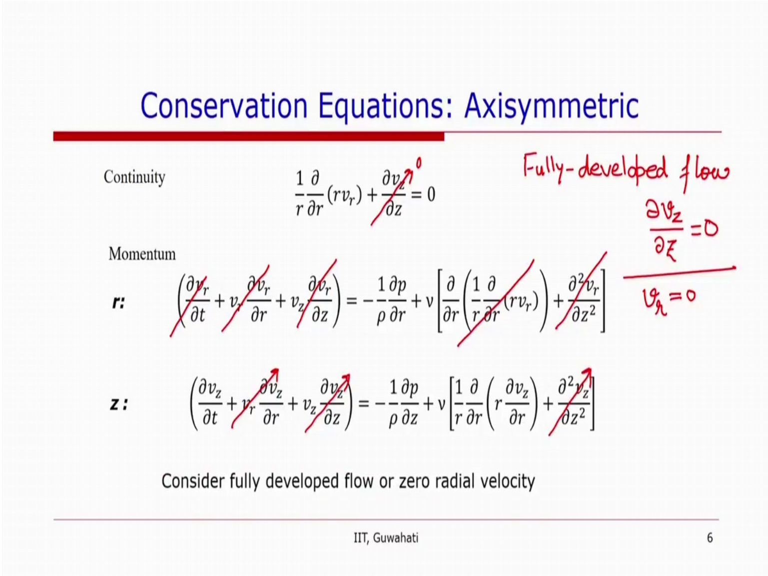

So, we will end up this set of equations for the axisymmetric, continuity r and z and ah. So, this continuity equation we have only 2 terms del v by del z and the radial term and similarly and r and z directions, now we assume at this point that flow is fully developed. So, if the flow is fully developed then we might recall that del v z by del z that is the gradient of the axial velocity in the axial direction is 0 or the velocity profile does not change along the axial direction.

So, if del v by del z is equal to 0 then from here we can see that v r is also going to be 0 or if we assume v r is equal to 0. So, if v r is equal to 0 then this term is going to be 0 and del v by del z is equal to 0. So, we have 2 conclusions from here that the flow is fully developed and v r is equal to 0. So, we are going to neglect the terms remember that v r is equal to 0. So, this is 0 this term is also 0, v z is there ah, but v r is equal to 0. So, this term is also 0 now because this term has v r. So, this term will go to 0 and this term will also go to 0. So, effectively what we are left with by is minus 1 over rho del p by del r is equal to 0 where rho is non 0. So, we will have del p by del r is equal to 0, in this the z momentum equation we have del v by del z del t. So, it will be non 0, but this term will be gone because v r is equal to 0 again the flow is fully developed. So, this term is 0 this term will be there, but again del v by del z del v z by del z is equal to 0. So, this term will also be 0.

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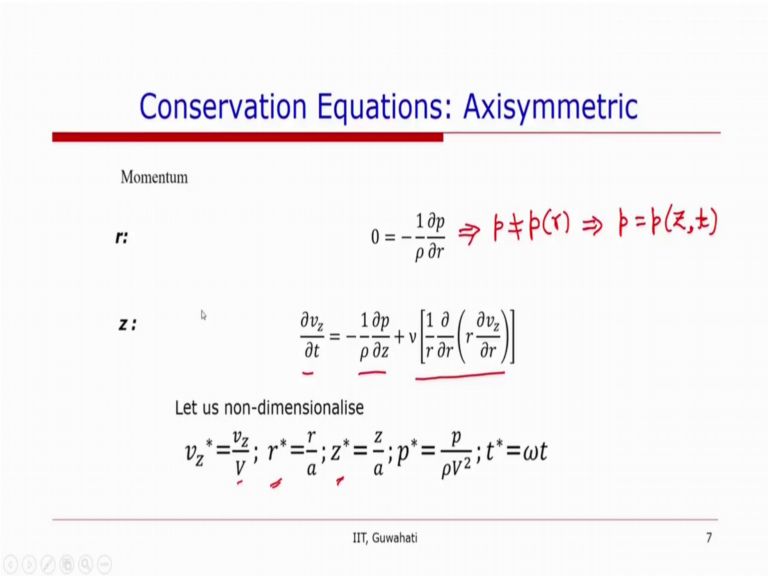

So, now, we will have our momentum equation in the r direction which is del p by del r is equal to 0. So, that simply tell us that p is independent of, r p is not a function of r and p is anyway not a function of theta, p is a function of time and p is a function of the axial direction. So, p is a function of time and p is a function of z and for the z direction momentum equation we have del v z over del t is equal to minus 1 over rho del p by del z nu, 1 over r del by del r r v del r del v z by del r. So, there are 3 terms this is transient term, this is pressure gradient term and the third term is the viscous term.

So, effectively now we are left with the reject term let us non dimensionalise the terms in this v z momentum equation and the scale for non dimensionalisation let us take a velocities scale v to non dimensionalise velocity, the both the length or the distance terms r star and z star r non dimensionalise by the channel radius. So, r by a z by a p star, pressure will be non dimensionalise by the dynamic pressure rho v square and t is non dimensionalise by the time step or say frequency.

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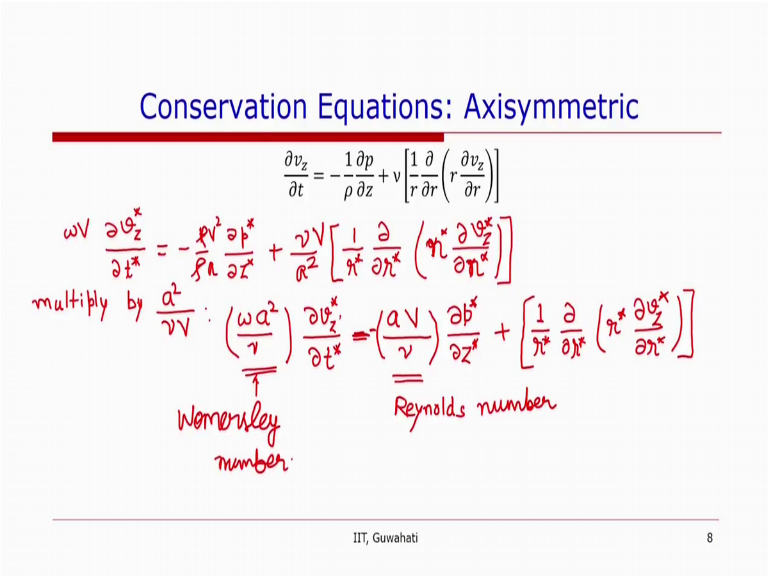

So, it is omega t now we substitute this in the equation, then what we will get is del v z star and it will be multiplied by v divided by del t star and it will be omega will come from here is equal to minus 1 over rho del p star. So, because of p star you will have rho v square term there divided by del z star. So, for z star it will be multiplied by a plus nu lets collect all the terms of a here ah. So, we will have 1 over r star and there will be 1 a for r star, del by del r star and this will be a square into there will be 1 a for r in the numerator and in the another a for r in the denominator. So, we will have both of them cancelling out. So, we will simply have r star del by del r star now for v z we will have v z star and 1 v here.

Let us now eliminate some of the terms. So, we can make this 1. So, we can multiply by a square over nu v, if we do that then we will have first term as omega v will cancel out omega a square by nu into del v z star over del t star is equal to. So, when we multiply this by nu rho and rho will cancel out a square. So, there will be only one a because 1 a will cancel out and only 1 v divided by nu. So, you might have recognised this non dimensional number by now and this is multiplied by there is a minus sign here, del p star by del z star plus nu square a square by nu v and this will be 1. So, we will have simply 1 over r star del over del t star, r star del v z star over del r star ok. So, we can see that there are 2 non dimensional group whereas, this is r e or Reynolds number and this group is a new dimensionalise group which is known as womersley number ok.

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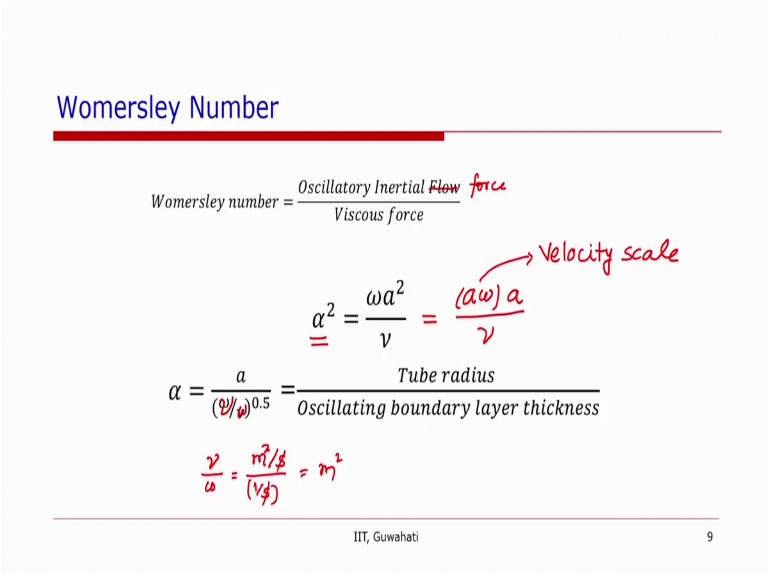

So, womersley number if we look at it is represented by alpha square, this alpha square is equal to omega a square by nu as we have seen in the non dimensionalisation of the differential equation and if we rearrange it then it terms out that it is a omega into a by nu. So, a omega is a velocity scale and a is the length scale and nu is the kinematic viscosity. So, it resembles a Reynolds number and this velocity scale is the oscillatory velocity. So, it is the ratio of inertial forces and viscous forces, but this inertial force is the oscillatory inertial force and the viscous force. So, this is not flow this is force you can also rearrange it alpha is equal to a over ome[ga]- nu over omega, sorry this should be nu over omega and this is you might see that nu over omega the units there, nu is meter square per second and omega is also one over second.

So, second over will cancel out and nu over omega is equal to meter square. So, this nu over omega power 0.5 is going to be meter and this is oscillatory boundary layer thickness. So, alpha is also a ratio of the channel radius and the boundary layer thickness. So, that says that, at low womersley number the viscous effects are going to dominate and the boundary layer is going to be thick whereas, at large womersley number, the at large womersley number the boundary layer will be thin and alpha will be large. So, when alpha is large boundary layer is thin viscous effects are negligible we will come back to this.

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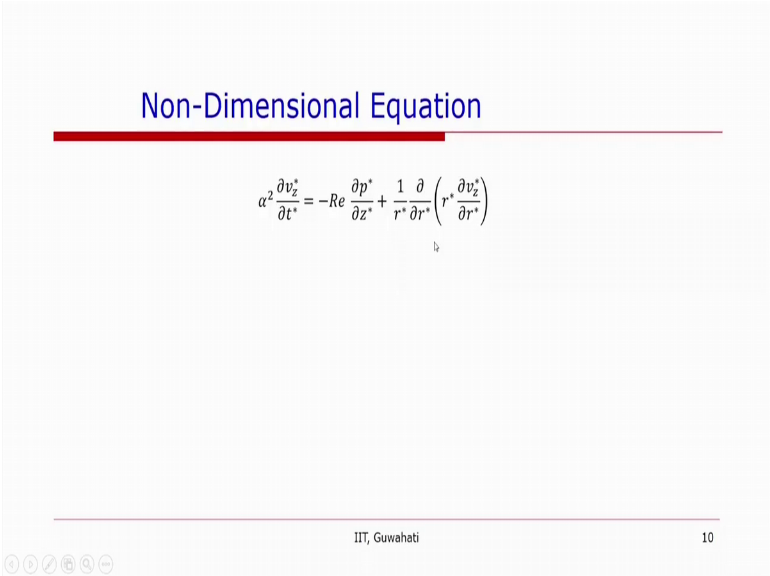

So, anyway after non dimensionalisation if we go back to our equation this equation has been written again in this form alpha square del v z square over del t star del v z star over del t star is equal to minus r e Reynolds number del p star by del z star, plus 1 over r star del over del r star r star del v z star over del r star one might be worrying about the that the Reynolds number is generally dependent on or defined in terms of the radius. So, one can do that and then there will be a 2 term coming in picture there.

So, as I was talking about that if alpha square is small then in that case the viscous term will be significant, but when the alpha is small and it is less than one then alpha square will be further small and this term will go away; that means, at low Reynolds number it is not this term is 0. So, you will have that the pressure gradient term it the pressure gradient term and the viscous term they will balance each other whereas, if the inertial force is high or the womersley number is large in that case the viscous force will be negligible or the viscous term.

Sorry, the viscous term will go away, you can take this on this side and then one can see that the viscous term will go away and the viscous term goes away then the transient inertial term or the oscillatory inertial term that will be balanced by the pressure or pressure gradient will be balance by the inertial force or the transient inertial force. So, ok

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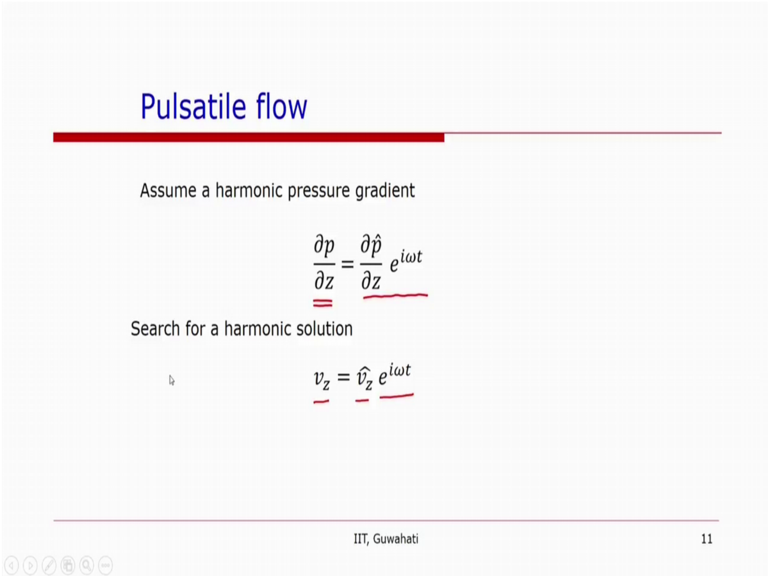

Now, our objective is to understand or to find out a relationship between the pressure gradient and flow or say axial velocity because v theta is equal to 0 and v r is equal to 0. So, the only direction in which velocity is non 0 is the axial direction. So, then we can integrate it to find out the flow rate ah. So, we our objective is to find out that what is the flow rate; The relationship between flow rate and the pressure gradient? So, let us assume a harmonic pressure gradient that del p by del z is equal to del p cap by del z e to the power I omega t and we because the pressure gradient is harmonic. So, the force coming from this let us assume the solution is v z is equal to v z cap I omega t. So, our objective is to find out this v z in terms of the pressure gradient ok. So, first we will look at the 2 asymptotic cases, the first one where the womersley number is small.

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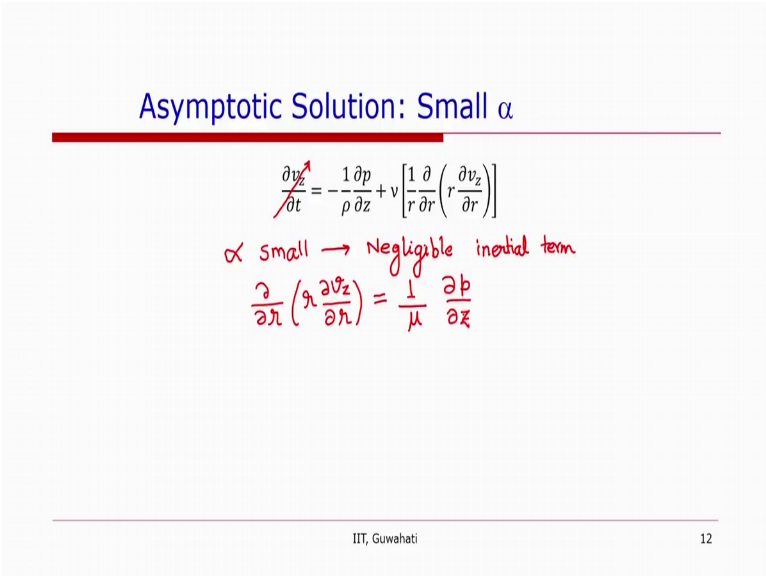

So, as we discussed just few minutes back that at low womersley number when alpha is small; that means, the inertial term is negligible. So, when we neglect the inertial term what we will end up with that the pressure gradient is balanced by the viscous term. So, this term will go away and we will have del over del r of r del v z over del r, is equal to 1 over rho into nu which will be mu the dynamic viscosity into del p over del z we have I think missed one r here. So, 1 over r when it goes here it becomes r here.

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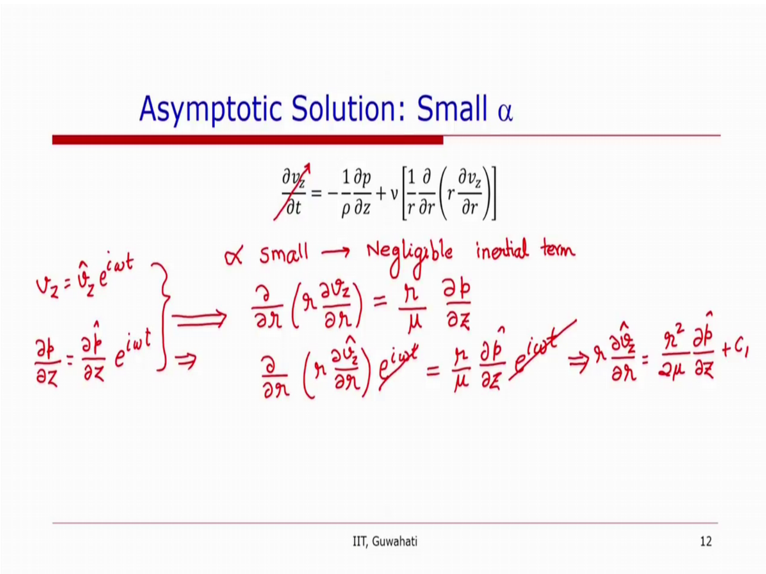

Now, we substitute that del p by del z is equal to del p cap over del z e to the power I omega t and similarly for v z. So, we can also substitute v z is equal to v z cap e to the power I omega t we substitute both of this in this equation then we will get del over del r, r v z cap over del r because e to the power the exponential term does not depend on the radius. So, we can take this out of the differentiation equal to r over mu, del p cap over del z e to the power I omega t now because the exponential terms they can cancel out. So, we will just write this and integrate it and what we will get is r, del v z over del r is equal to we have miss a this is correct r del v z over del r is equal to r square over 2 mu del p cap over del z plus a integration constant let us say c 1, now we take this r on the other side.

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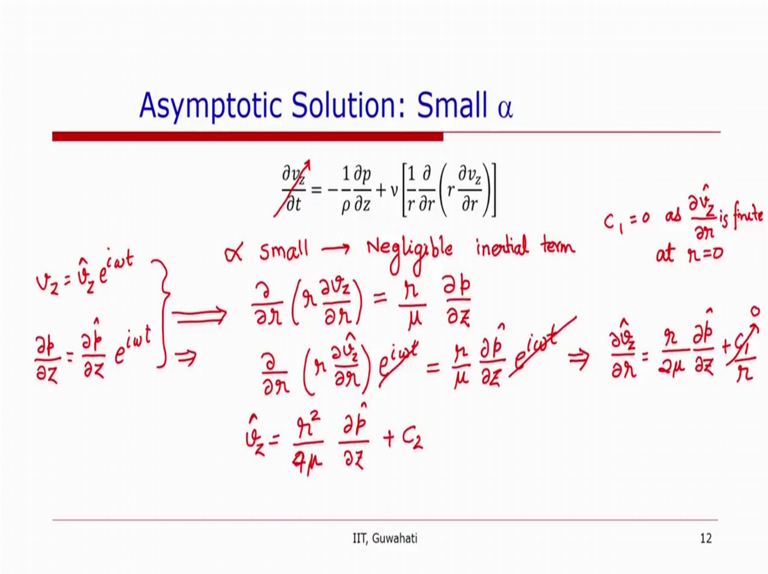

So, let us delete this and this will go away and we will have c 1 by r now this, c 1 has to be 0, as del v z over del r is finite at r is equal to 0. So, if we substitute that then c 1 is going to be 0. So, this term will go away and then we further integrate it. So, we will get v z is equal to r square over 4 mu del p cap over del z plus c 2. So, v z is equal to r square by 4 mu del p by del z plus c 2 and now we use the boundary conditions.

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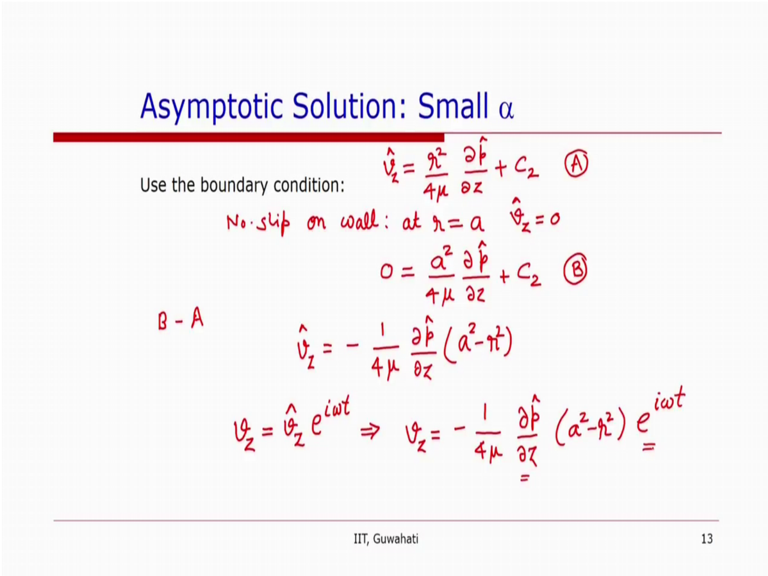

So, we had v z cap is equal to r square over 4 mu del p cap over del z plus c 2, we need to find c 2 and the boundary condition, no slip boundary condition on wall which says that at r is equal to a which is wall v z cap is going to be 0.

So; that means, this is 0 and a square by 4 mu del p cap over del z plus c 2, if we subtract these the this equation let us say equation b and equation a and if we do b minus a then we will end up with v z is equal to minus because it will be minus. So, we will do b minus a, b v z is equal to minus 1 over 4 mu, del p cap over del z and a square minus r square. So, if we substitute this in v z if remember we had v z is equal to v z cap e to the power I omega t.

So, we will have v z is equal to minus 1 over 4 mu del p over del z a square minus r square into e to the power I omega t. Now, you might see because of this negative sign ah you can say that v this term and this term they make the pressure gradient. So, because of this minus sign v z and del p by del z or the flow velocity and the pressure gradient they are out of phase 180 degree out of phase.

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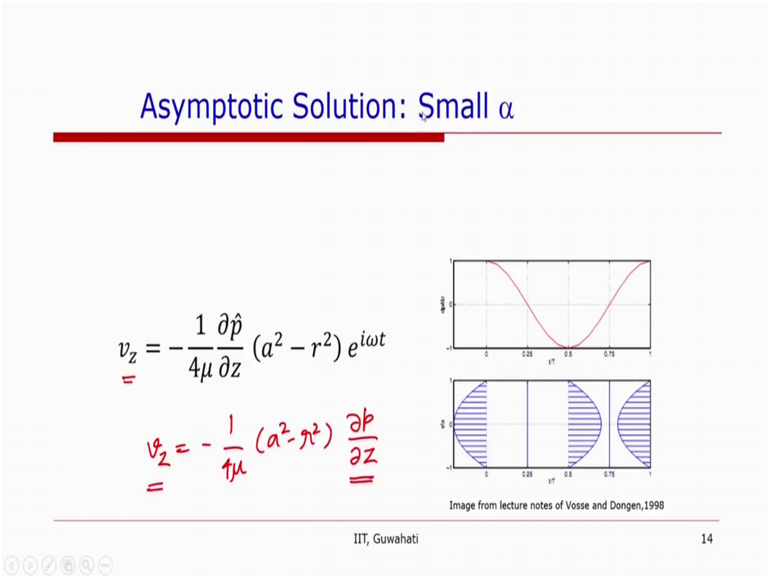

So, they are completely opposite this is the relationship that we just derived and if we plot del p by del z which will be a say sinusoidal function that is 0.5 t by t it is minimum and then it is say 0 at this now the. So, it is at d p by d z is equal to 0 it is 0 and then it is maximum at t by t. So, it is going to be sinusoidal function anyway. So, at the minimum when d p by d z is minimum it is negative minus one then at that point you will see that the velocity is in the forward direction and parabolic velocity profile when obtains.

When d p by d z is maximum; then, the velocity is in the negative direction and the entire velocity is negative at every point whereas, when d p by d z is equal to 0 then the velocity is also 0 at 0.75 we can see and same 0.25. So, they are the velocity and d p by d z they are 180 degree out of phase one can also write this as v z is equal to minus one over 4 mu a square minus r square del p by del z. So, del p by del z and v z they are out of phase let me not confuse you from this sign ok. So, we have looked at asymptotic solution at small womersley number which suggest that its small womersley number the expression is very much similar to what we have for a poiseuille flow, but the velocity and the pressure gradient they are out of phase.

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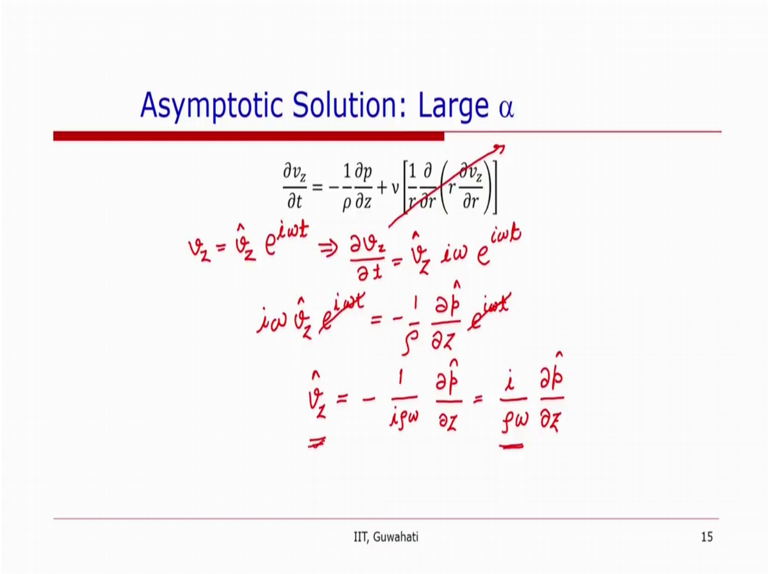

Now, let us look at large alpha or when womersley number is large then as we suggested that as at large womersley number, the inertial forces will be dominant and the viscous forces can be neglected. So, this viscous force can be neglected as we followed earlier let us substitute v z and del p by del z here. So, we know that v z is equal to v z cap, I omega t that gives me that del v z over del t is equal to sorry this is v z cap e to the power I omega t and del v z over del t will be v z cap I omega exponential I omega t.

So, let us substitute this here and what we will get is I omega v z cap exponential I omega t is equal to minus 1 over rho del p cap over del z exponential I omega t. Now, I omega t I omega t will cancel out and what we will have is v z cap is equal to minus 1 over I rho omega del p cap over del z, we can change minus 1 to I square. So, we will get I over rho omega del p cap over del z.

So, what we see from here that in this case velocity is plug flow that is there is no gradient of velocity in the radial direction, the velocity is uniform and which is understandable because the velocity profile comes because of the viscous term and we have neglected the viscous term. So, the velocity profile is plug flow and as you can see from this that this term has I. So, the velocity v z will be 90 degree out of phase or 90 degree phase difference between the pressure gradient and the flow velocity.

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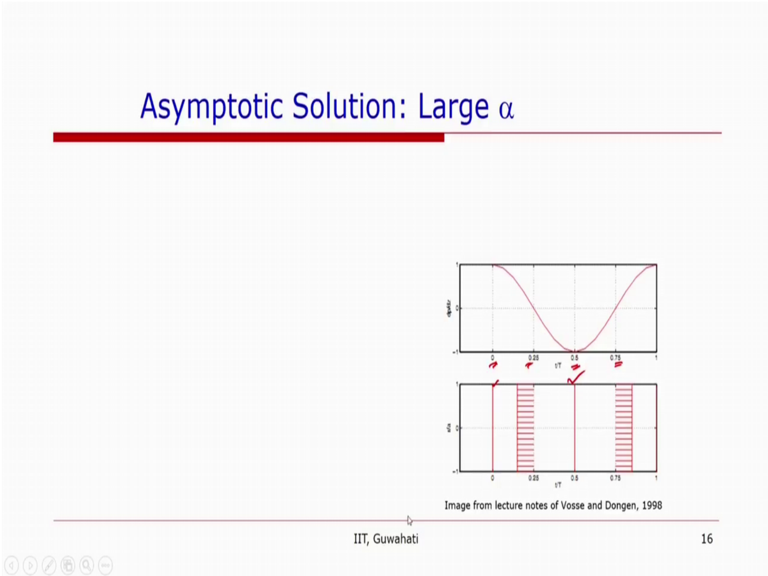

So, as you can see from this graph here, the similar graph for the pressure gradient as we saw for small alpha and for ah velocity at large womersley number we can see the velocity profile is parabolic and wherever there is velocity is maximum the velocity. Ah sorry, wherever the pressure gradient is maximum on those pressure maximum or minimum the velocities are 0 there whereas, when the velocity when the pressure gradients are 0 when the pressure gradient is 0 at those time instants velocity have magnitude as maximum and the velocity profile is plug flow; that means, the velocity is uniform everywhere in the channel cross section ok.

So, now, having looked at the solution at 2 limits, 2 asymptotic limits when the womersley number is small and when the womersley number is a large. Let us look at the solution and try to find out if we can find the solution for the entire range of womersley number.

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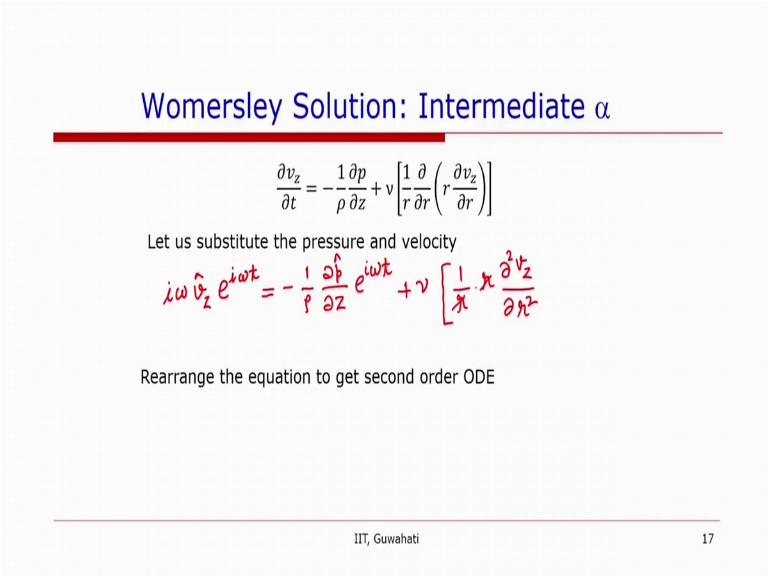

So, we again come back to the v z momentum equation that we have obtained after reducing the navier stokes equations to the axisymmetric ones and then neglecting the terms based on that v r is equal to 0 and del v z over del z is equal to 0 that is flow is fully developed. So, then we have got this equation.

Now, in this equation as we have been doing earlier for the 2 cases 2 asymptotic cases let us now substitute v z and del p by del z. So, if we do that again we will have del v z over del t is equal to I omega v z cap exponential I omega t that is equal to minus 1 over rho del p cap over del z e to the power I omega t plus nu 1 over r and let us break this into 2 terms. So, what we will get is into r del 2 v z over del r 2 and this r and r will cancel out.

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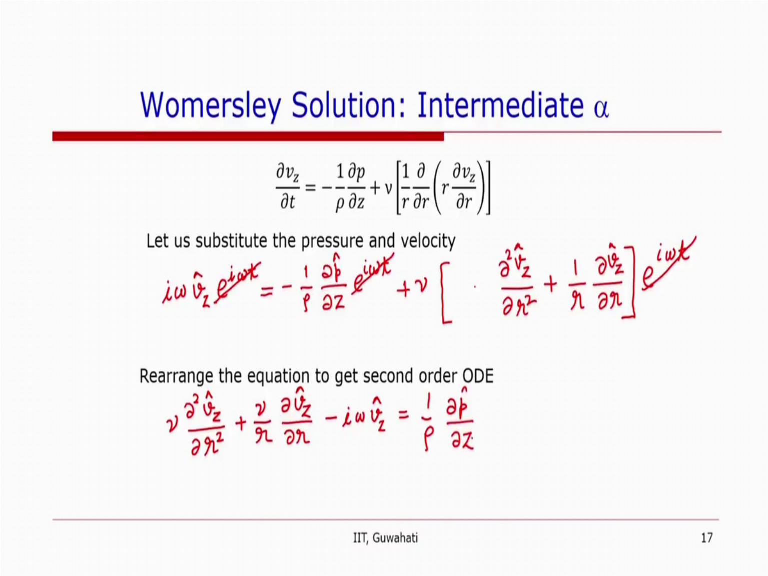

So, we will just get rid of both of them plus we will have one over r and the differentiation of r is 1. So, we will have del v z over del r into e to the power I omega t now all the exponential terms in the 3 equations are same. So, they can be cancelled out and let us rearrange this to as a second order differential equations. So, we can write this as this will be v z cap. So, we can write this as nu del 2 v z cap over del r 2 plus nu over r del v z cap over del r minus I omega v z cap is equal to. So, because we have brought this on that side and then it will be equal to 1 over rho del p cap over del z ok.

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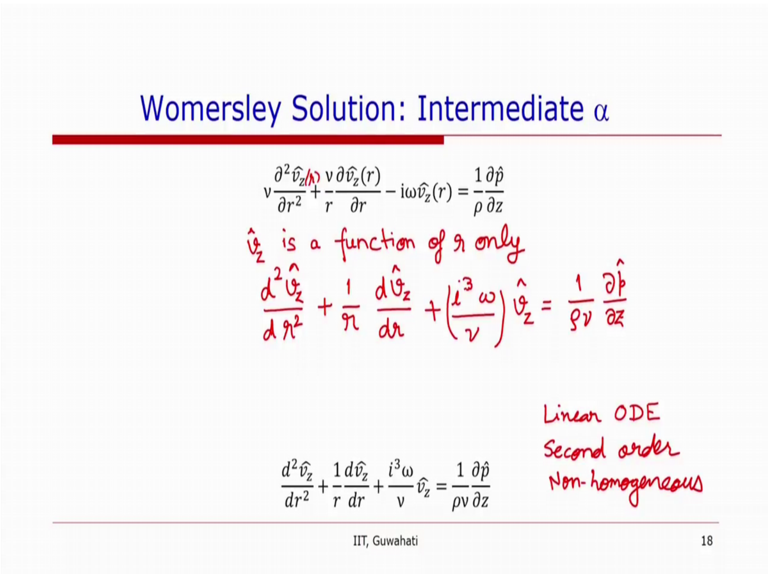

So, this is the equation that we got in the previous slide.

So, now let us rearrange it to this equation. So, first we need to divide by nu and v z cap is a function of r only. So, we can have this as total derivative. So, we can write this as d 2 v z cap over d r 2 plus 1 over r d v z cap over d r this because we have been writing this as a function of r. So, this is also a function of r. So, d v z over d r plus if we want to change this I to I cube. So, minus 1 is equal to I square. So, this will be I cube omega by nu v z cap is equal to 1 over rho into nu or we can also write this as mu del p cap over del z.

So, we have obtained this relationship, now this is if you look at this is a linear o d e or terms have power one only and this is second order and non homogeneous. So, we have to go back if we want to find out the solution of this equation, we have to go back to our under graduate mathematics where we learnt how to solve non homogeneous second order linear ordinary differential equation.

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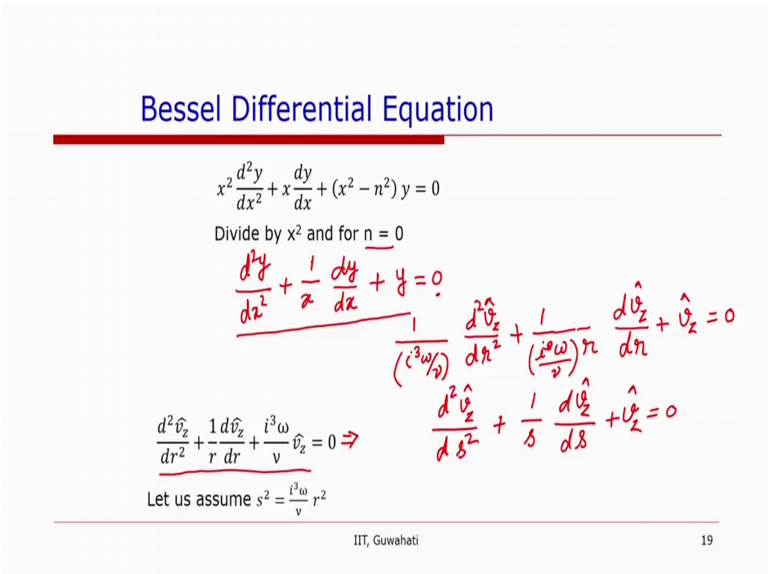

So, if we go back to this you might have heard or you might have read about bessel equations, if you have done a heat transfer course where you have looked at some of the complex equations there also you might encounter bessels of functions otherwise you would have definitely done these in your under graduate mathematics. So, if the ordinary differential equation a homogeneous differential equation, if it looks like this x square d 2 y by d x 2 plus x d y by d x plus x square minus n square y is equal to 0 and if we divide by this by x square and take n is equal to 0. So, we will end up with d 2 y by d x 2 plus 1 over x d y by d x plus y is equal to 0, this is a homogeneous equation and when the equation is non homogeneous this term in place of 0 it will become non 0.

So, now let us compare the form of this equation with the equation that we have derived. So, d 2 y, by d x 2 d 2 v z by d r 2 1 over x d y by d x 1 over r d v z over del r or d v z over d r plus I cube omega by v v z is equal to 0 if we take the homogeneous part of our equation. So, if we want to have the similarity in the 2 equations then we will need to change our variables a bit. So, let us do that and if we assume or another variable let us call this s and if we assume that s square is equal to I cube omega by nu into r square. So, then we can write this equation as let us divide the entire equation by I cube omega by nu. So, we will have 1 over I cube omega by nu d 2, v z over d r 2 plus, 1 over I cube omega by nu into r d v z cap over d r plus v z cap is equal to 0.

Now, we have seen that s square is equal to we have assumed that s square is equal to I cube omega nu into r square. So, from that let us just make this look better. So, this is d r 2 now we can have this d 2 v z cap by d s 2 plus 1 over s there are 2 s here also one here and one in the differentiation. So, 1 over s d v z cap over d s plus, v z cap is equal to 0. So, now, we have both the equations the bessel differential equation for n is equal to 0 and the equation that we have obtained for intermediate womersley number they look very similar. So, the next thing is to look at what is the solution of this ordinary differential equation and if we look at the books this is the form of the equation.

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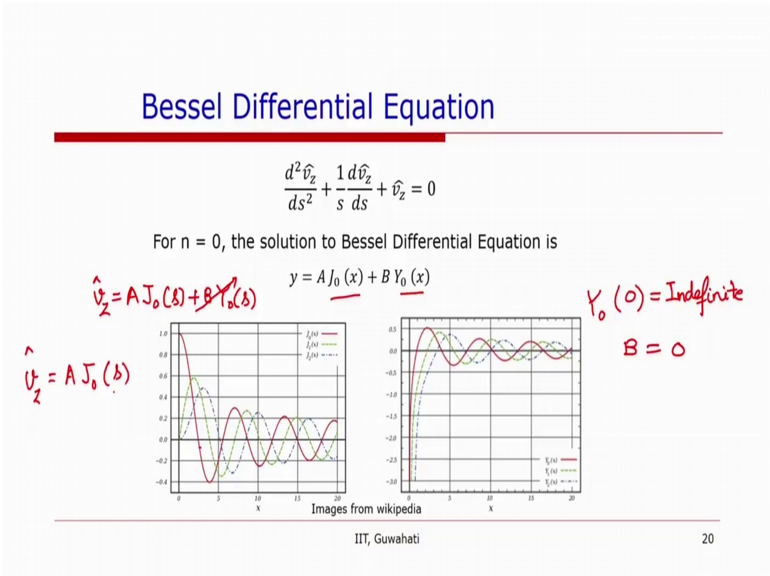

For n is equal to 0 the x y differential equation will have the solution y is equal to a j naught x plus b y naught x we have we have a and b they are constants whereas, j naught x and y naught x are called bessel functions and j naught x is the bessels function of first kind and y naught x is the bessel function of the second kind. So, if we look at the nature of the graphs or the values of this j naught x and y naught x then we with respect to x, then we see that at x is equal to 0 j naught x or the bessel function of first kind is the value is 1. Whereas, it is indefinite at x is equal to 0; that means, y naught at 0 is indefinite whereas, our velocity at s is equal to 0; that means, at r is equal to 0 will be finite right.

So, let us write down the solution based on this the solution for v z cap and what we will have by copying from this a j naught s plus b y naught s now at as we have just seen that y naught 0 is indefinite. So, B is equal to 0. So, that this term goes away. So, we have that v z is finite at r is equal to 0. So, our solution is v z is equal to or v z cap is equal to A j naught s ok, now we have this because our solution our equation is non homogeneous equation. So, what we have obtained is the complementary solution and this complementary solution is a v z complementary is equal to A j naught s.

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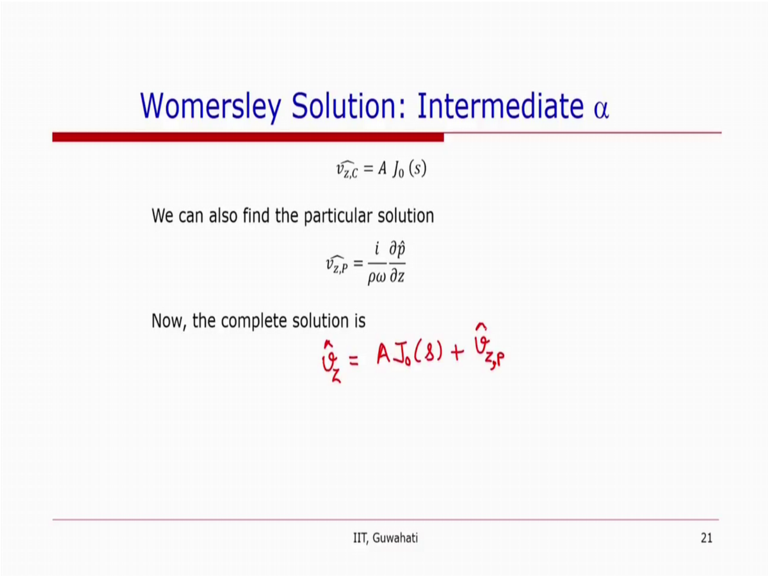

We can also find the particular solution and I will live it to you to find how to find out the particular solution and this will turn out it will turn out that v z cap ah, now the particular solution will be I by rho omega del p by del z which is the last term we had.

So, the total solution will be or the complete solution for v z will be because it is a linear equations. So, both the solution can be super imposed. So, we will have v z is equal to complementary solution which is a j naught s plus v z particular solution.

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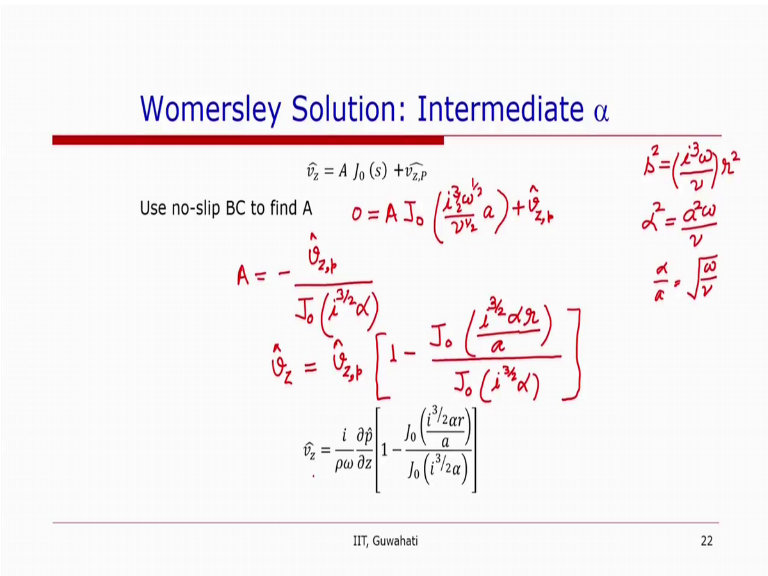

So, that is what the complete solution is, now what we need to do that a is unknown in this remember what was the particular solution, the particular solution we had is v z p is equal to I by rho omega del p by del z. So, we will use this later on and if we use the no slip boundary condition to find a; that means, v z is equal to 0 at A j naught s we had s square is equal to I cube omega by nu r square that is what our definition of s was. So, at r is equal to a the value of s will be I cube omega by nu a square plus v z p cap now from this we can also remember that alpha square which is womersley number is equal to a square omega by nu.

So, this is I to the power 3 by 2 omega power 1 by 2 and nu power 1 by 2 or we can say that alpha by a is equal to root of omega by nu. So, we substitute this, then we will find a is equal to minus v z p divided by j naught I to the power 3 by 2 this will be sorry this will be a only not a square. So, we will have alpha over A and A A will cancel out. So, I to the power 3 by 2 a. So, we will have v z is equal to v z particular we can take this out 1 minus, now we substitute the value of a here. So, that will be j naught and the s if you replace s then it will be I to the power 3 by 2 in place of root omega by nu we can write alpha over a and r here for s and this is divided by j naught I to the power 3 by 2 alpha.

So, this is the value of v z cap and the relationship that we can find out now what we have found is v z cap and the v z will be v z cap into I omega t. So, this is the relationship between the velocity and the pressure gradient for pulsatile flow fully developed flow in a rigid tube, we can also find out the asymptotic solution by replacing alpha is equal to for very small alpha and for very large alpha.

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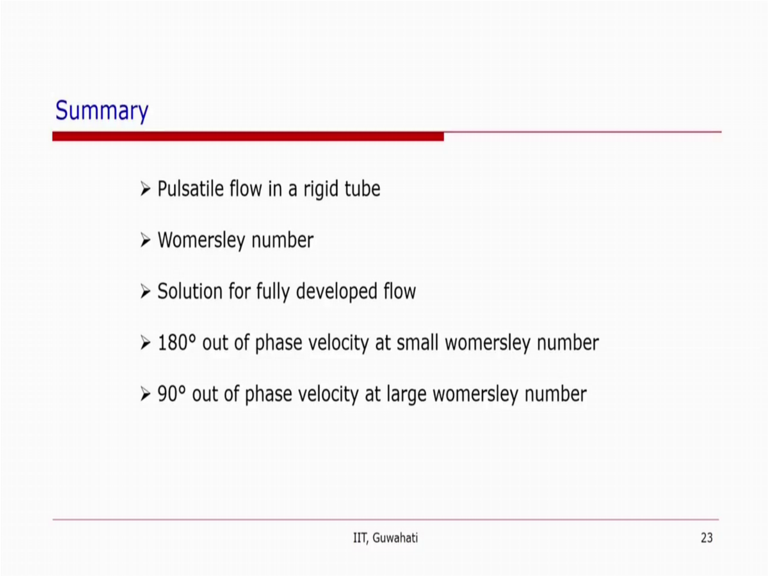

So, in summary today what we have looked at is that using fourier series any periodic signal it can be decomposed into sum of sin or cosine terms or what we call harmonics and we will try to take or we will try to do one example for this case. then we have encountered our new non dimensional number which is very important with respect to or in cardiovascular fluid mechanics which is known as womersley number, it is the ratio of oscillatory inertial flow or the oscillatory inertial force and the viscous force and. So, it is kind of oscillatory Reynolds number and then we have obtained the solution for pulsatile flow for a harmonic or for a sinusoidal function or for an exponential complex function for fully developed flow.

Assuming that we obtained or we saw that at low small womersley number at small womersley number or low womersley number the expression of the velocity profile is very similar to what we obtain for a poiseuille flow, but the ah, but it is time dependent and it is 180 degree the velocity is 180 degree out of phase with the pressure gradient. Whereas, at large wave womersley number the flow profile is same everywhere in the cross section that is no effect of viscosity ah, flow profile look like a plug flow it is uniform everywhere in the cross section and it is at 90 degree out of phase with the womersley number.

So, wherever the pressure gradient we have it is 90 degree out of phase with the pressure gradient. So, wherever the pressure gradient is maxima or minima then because it is sinusoidal function. So, when the pressure gradient is at is at maxima or minima the velocity is 0 and if it is 0 then if the pressure gradient is 0 then the velocity is maxima or minima. We will also try to look at some of the applications and try to have a field of numbers and some examples of this womersley solution in the context of cardiovascular fluid mechanics.

Thank you.