Introduction to interfacial waves

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Lecture - 09
Introduction to Jacobian elliptic functions

Until now we have looked at linear linearized oscillation problems. So, all the equations which we have looked at now, whether they were single masses or coupled masses, whether they were ordinary differential equations or partial differential equations, always represented linear equations.

So, in the case of coupled masses, we where the masses were oscillating vertically, we had to linearize our equations, but then we really solve the linear ordinary differential equations. In the case of our rectangular membrane as well as circular membrane or partial differential equation was a linear wave equation and so, we could use the method of normal modes to solve this.

We are now going to look at large amplitude oscillations. Now, large amplitude oscillations is more of the rule rather than an exception and so, we need to learn how to deal with large amplitude oscillations. And, what are the new features here, once the oscillation amplitude crosses a certain threshold.

Now, we are going to introduce something called elliptic functions. These are a useful set of mathematical tools, which will help us solve some of the equations exactly. Later on we will start with perturbation methods for systematically go to higher order approximations for solving non-linear equations. So, let us learn what are elliptic functions.

Elliptic functions are something, which actually a generalization of circular trigonometric functions. Just as circular functions are defined on a unit circle, elliptic functions are defined on a ellipse, which is suitably normalized such that one of its axis, in this case the semi minor axis is unity.

Elliptic functions are of use in solving some of the some of the typical non-linear problems, in particular we will be looking at one such problem, of the non-linear pendulum. If you recall that usually the pendulum equation is a non-linear equation which has a sin theta term in it.

Now with the sin theta term it is a non-linear equation. The usual way in which that equation is solved is we Taylor or expand sin theta about theta equal to 0 and this leads to so, sin theta getting approximated as theta. This renders the equation in linear it is a constant coefficient ordinary differential equation and that leads us to the solution of the pendulum equation.

In this once we discuss elliptic functions; we will not make that approximation. For a certain set of initial conditions, we will learn how to solve the pendulum equation exactly. And, then we will try to understand, what are the essential features of a non-linear pendulum, which were I am absent in a linear pendulum.

So, let us first understand, what are elliptic functions. And, then we will use these elliptic functions to solve the pendulum equation.

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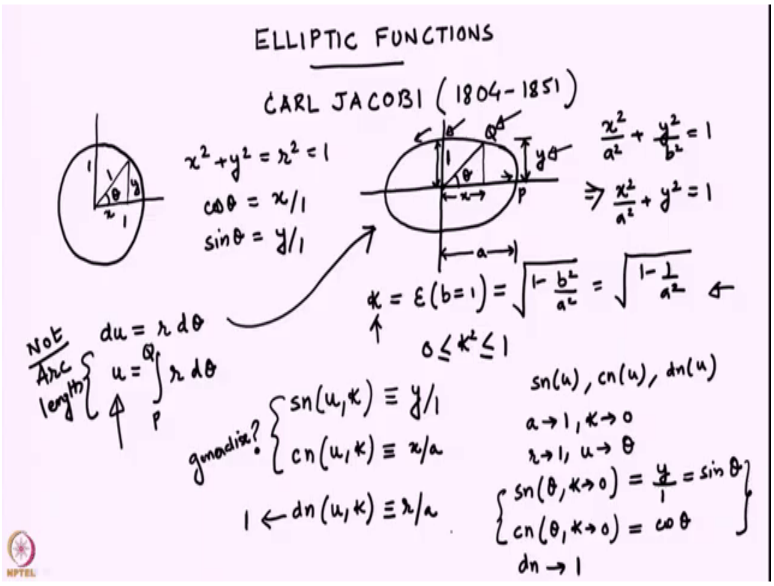

So, there are many kinds of elliptic functions the particular ones that, we will be looking at are called the Jacobian Elliptic functions this will these were, first found by Carl Jacobi a German Mathematician, who lived in the period 1804 to 1851. We will be looking at Jacobian Elliptic functions. Recall that circular functions or circular trigonometric functions are defined on the unit circle. So, the radius is unity. And, so, the coordinate of any point which has x component x and y this is the radius.

So, x square plus y square is equal to in general r square and r is 1 here. So, x square plus y square equal to 1. And, if I define cos theta then cos theta is just x, and sin theta is just y, it is actually x by 1 and y by 1. Now, these are the familiar trigonometric functions. What the elliptic function does is it generalizes this to an ellipse. So, let us draw the ellipse first.

So, this is the center of the ellipse and as I said before it is normalized such that the semi minor axis is of unit length. So, this is of unit length. The semi major axis is a and we draw a radius vector this is x, that is y, this is theta. So, let us now see how are the elliptic functions defined on this ellipse whose semi minor axis is 1.

We all know that the equation of an ellipse is x square by a square plus, y square by b square is equal to 1, where a and b are the semi major and the semi minor axis. In this case b is 1 so, the equation of the ellipse becomes this. Now, one can define an eccentricity of the ellipse, we will call this in the notation of elliptic functions, this is called the modulus of the function.

So, I will call it with a small k, small k is basically defined as the eccentricity of an ellipse, whose minor semi minor axis is 1. And, recall that the definition of eccentricity of an ellipse is square root 1 minus b square by a square, b in this case is 1 and so, this becomes 1 minus a square.

Because a is the major semi major axis, a by definition is greater than 1 and so, you can immediately see that kappa is between 0 and 1, more importantly kappa square is between 0 and 1. Later on when we evaluate these functions, you will see that most of the packages except kappa square as input and so, kappa square needs to be between 0 and 1.

This is one of the arguments of the elliptic functions. The elliptic functions actually have two arguments k or it is let us call it k instead of kappa. So, k actually is a measure of the shape of the ellipse you can see that it is related to a, k is equal to square root 1 minus 1 by a square if you; you can make the by controlling a you can control the shape of the ellipse. If you take a to 1, then k goes to 0, in other words you be the ellipse becomes a circle.

So, by playing around with a 1 can vary k and consequently 1 can vary the modulus of the function. Similar to the trigonometric functions, there is also another argument which is like an angle, but it is not quite an angle it is a generalized angle. So, let us see how the generalized angle is defined. So, the generalized angle u is defined as d u is equal to r d theta, this is all with reference to this ellipse.

So, this is a definition of u so, I can integrate this. So, if I call this point P and this point Q, then u is defined as u at the point Q is defined as the integral from P to Q r d theta. Note that, this is not an arc length, it has this is a generalized angle; it is dimensional because r is dimensional. And, that is why we call it a generalized angle, because it does not have the dimensions of an angle and it is like an angle, but it is not exactly arc length either, it is not arc length.

Now, the elliptic functions, there are many kinds of elliptic functions, we are going to explore only two of them. So, let us look at the elliptic functions. So, we will define following sin we will define something which is called s n. So, it is like a sin, but it uses the alphabets s and n it has two arguments; one is this generalized angle which we have defined here and one is this modulus which is like the eccentricity of the ellipse and controls the shape of the ellipse.

This is defined as y. So, if I am at any point Q on the ellipse, the projection, the vertical projection of it whatever is the length of that, that is sin s n of u comma k. Notice that at this point there is a particular value of u and if you give this function that value of u and tell it the eccentricity of the ellipse, it will return you the value of y at that point.

So, it is actually a normalise so, it is actually give you y by 1, it is y divided by the semi minor axis, then, we have c n, following cosine we have c n. And, this generalizes the cosine and this is defined as x by a. So, y is normalized by the semi minor axis, x is normalized by the length of the semi major axis.

There is one more which is useful when we write down relations for relations between s n and c n. And, that is called d n again it accepts two arguments and that is equal to a non dimensionalized r, r scaled by a.

Now, typically in our applications we will see that k is determined by initial conditions. And, once we have chosen a particular set of initial conditions k will not vary. So, it is common not to write k and write s n only as a function of u, c n only as a function of u and d n only as a function of u. But, we have to remember that they are actually two arguments and one has been suppressed in the notation.

Now, let us try to understand that are these really generalizations of sin and cos. So, even before we do that, you can see that s n and c n the way they have been defined, if you start from this point on the ellipse point P, you can see that y is 0 there. So, s n when u is 0, would be 0, for given value of k. As you go around when you reach the topmost point of the ellipse, y would go to 1 as you go around it will again reduce.

So, you can see that it starts from 0 goes to 1 reduces and then when it goes comes to the other half, it will become negative. So, it is an oscillatory function, it is also a periodic function, but its period is different from a sin. Similarly, c n you can see is also an oscillatory function and a periodic function, you can try to understand how it goes. These are both normalized to 1 so, their maximum and minimum values will go from 1.

So, now let us try to understand in what sense do they generalize the sine and the cosine function that we know in trigonometry, you can see that s n. So, let us take the limit a going to 1. So, you can see that when a goes to 1 look at this figure, when a the semi major axis goes to 1, we are shrinking it and making it equal to the semi minor axis. In that case the ellipse reduces to a circle. The radius vector r in this case which is an ellipse is a function of theta, in the case of a circle just becomes a unit vector.

So, u which is defined as r d theta, in the limit of a going to 1 or k going to 0, you can see if this from here that when a goes to 1 k goes to 0. The ellipse becomes a circle u you can see is integral r d theta in the for a circle r is just unity. So, it comes out of the integration and this just becomes u just becomes integral d theta, which is just theta.

So, u just becomes the angle. So, r goes to 1 and u goes to theta. So, in this limit you can see that s n of theta, kappa goes to 0 or k goes to 0, is just y by 1, but what is y by 1? y by 1 is just in this limit the ellipse is deforming to a cycle and you are taking the ratio of y to the radius of the circle, which is unity in this case. So, y by 1 is just sin theta. So, as k goes to 0 the ellipse deforms to a circle and this s n of theta for k going to 0, goes over to sin theta.

Similarly, you can convince yourself that c n of theta as k goes to 0 is just cos of theta you can see that easily, because it is x by a and a goes to 1. So, it is x by 1 and 1 is nothing, but the hypotenuse of the circle of the right angle triangle and it is also the radius of the circle. So, x by 1 is just cos theta.

So, you can see that these two functions generalize the sin theta and cos theta on an ellipse. And in the limit so s n of theta comma 0 would be a so, in the in that limit it would reduce to sin theta and c n would go over to cos theta. What happens to d n? You can see that d n, so, a goes to 1, r which is the radius also goes to 1, so, d n just goes to a constant which is unity in this limit.

So, d n just goes to a constant. You can see that for an ellipse even d n is an oscillatory function alright. So, now, we have introduced the definitions of these functions, we have an intuition about how these functions are they are oscillatory in nature, for very small values of k, they effectively behave like sine and cosine and d n behaves just like a constant and is independent of theta. Now, let us work out some of the properties of these elliptic functions.

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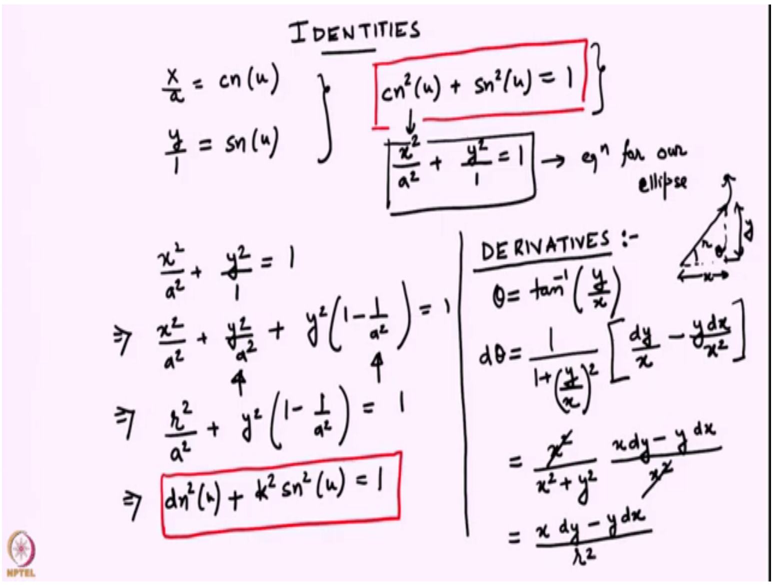

So, one of the first things that is useful to have is identities, just as we have many identities or circular functions sin square plus cos square is equal to 1. Similarly, it will it is useful to work out identities of these elliptic functions. It is immediately obvious from the definition of the elliptic function, because x by a is defined as c n of u, I am suppressing the k as I said before. y by 1 is defined as s n of u, this implies c n square of u plus s n square of u is equal to 1. How do we know this?

It is just because c n square of u is x square by a square and s n square is y square by 1 is equal to 1. And, this must be true, because we are on the ellipse and this is the equation for our ellipse. This is how it should be because c n and s n reduced to sin square theta and cos square theta and we know that on a circle sin square theta plus cos square theta is equal to 1.

So, this generalizes and this is true even on the ellipse, this is the first identity, let us see if we can find more. So, let us start with the equation of our ellipse and let us add and subtract 1 minus y square by a square from the expression. So, I have added and subtracted, I have added here and I subtracted there and I have pulled out y square common. And, then x square plus y square is in the denominator in the numerator I can write this as r square by a square plus y square into 1 minus 1 by a square is equal to 1.

This implies this is just d n square of u, 1 minus 1 by a square we have seen this expression before, it was related to the modulus or the eccentricity of our ellipse and it is the square of the modulus. So, it is k square and so, we have another identity y square is just s n square. So, let me put all of these in boxes. So, this is one identity and that is another. We can find more and we will explore some of these in the assignments.

Now, let us find out if we know how to take derivatives of them. Ok. This will be useful in particular, because we want something called elliptic integrals ok. As we will see shortly when we study the non-linear pendulum, we will be able to integrate the equation, but then the answer will be expressible in terms of an integral. So, these elliptic functions help us express those integrals in terms of known quantities.

So, now let us work on derivatives of elliptic functions. For this we need to do a little bit of algebra, we know that in polar coordinates theta is tan inverse y by x. So, if I have r and we measure theta with respect to the horizontal. And, this point has coordinates x and that length is y, then we know from the right angle triangle, that this is y by x. We are not yet on the ellipse this is just an statement for polar coordinates.

Now, if I take the differential of theta. So, I am asking if I make a small increment in y and a small increment in x, how much does theta change? So, I go from this point to another point, which has a slightly different x and a slightly different y, and consequently a slightly different theta. So, that expression is given by the differential of this the first order differential of this, which is and then the differential of y and x.

Notice that we are not yet on the ellipse. Hence, I am allowed to treat y and x both as independent variables. If, I am on the ellipse then only I have only have one of them as my independent variable. Since, I am not yet on the ellipse I am doing this only for polar coordinates, then y and x are my independent variables, I can go from one point to another and I can vary y and x independent of each other ok.

So, that is why in the expression for d theta on the right hand side, you will see that there is a d y and there is a d x and they are both they both can be chosen independently. So, if I write that then this is d of y by x, which is d y by x minus y d x by x square. And, if I cancel out the x square then this just becomes so, x square if I push it up then this just becomes square plus y square.

And, thus this becomes x d y minus y d x divided by x square. And, I can cancel this and this and write this as x d y minus y d x. And, now I am using mix notation I have x y on the right hand side and theta on the left hand side, I would like to have Cartesian on the right hand side and polar coordinates on the left.

So, x square plus y square is r square. So, I am going to have an r square in the denominator ok. So, let us continue further.

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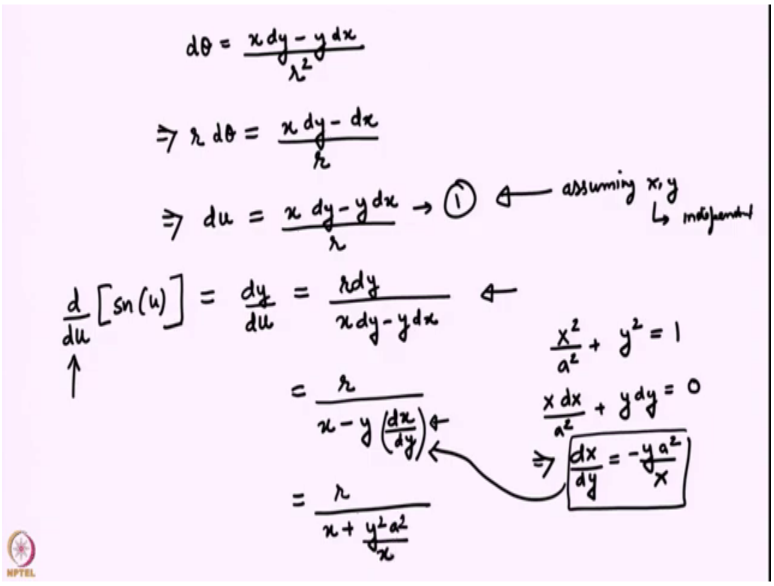

So, we have d theta just rewriting what I wrote on the last page x d y minus y d x divided by r square. Now for our elliptic functions, we need a u, we would like to eventually get on to the ellipse and say that x and y are not independent of each other. But, I can choose only one of them independently and the other is determined from the equation of the ellipse. For that I would like to have my independent variable one of my independent variables as u, d u is r d theta.

So, I can shift 1 r from the denominator here to the other side and then the left hand side is just d u, and this just becomes x d y minus y d x divided by r. I will call this equation 1. Now, let us ask the question what is d by d u of s n u? This is what we are interested in. Notice we should ideally write a partial derivative here, because s n u depends on two arguments u and k.

But, in this we are assuming that all this is for constant k, because we are doing this on a given ellipse. For a given ellipse small a is constant small a is related to the modulus k and so, k is a constant. So, this derivative is being taken just by varying going along the ellipse, but not changing the ellipse itself.

So, I am using ordinary derivatives, but actually it is a partial derivative with del y del u keeping k constant ok. So, what is s n u? s n u by definition is just y. So, this is just d y by d u. We have seen that this can be written as d y and if I use the expression for d u from equation 1 in the denominator, then this just becomes x d y minus y d x divided by r.

But, now I have to be careful because this expression was derived assuming x and y to be independent, that is why I did not I could choose d y and d x and I did not say what is the relation between d y and d x. But, when I put them here I know that this is on the ellipse and so, y and x are not independent of each other or in other words d y by d x is not 0, it is given by the equation of the ellipse.

So, I will keep the r here and I will write this as. So, I am dividing up and down by d y and I get a d x by d y here, d x by d y is not 0 now, because we are on the ellipse. On an ellipse as you go you can choose either x as your independent variable or y is your independent variable. If, you choose y as your independent variable then d x by d y tells me that, if I change y by a certain amount and stay on the ellipse how much does x change.

So, d x by d y can be obtained from the equation of the ellipse and taking derivative of it. So, I am going to do the same now. So, the equation of the ellipse is x square by a square plus y square is equal to 1. We are interested in if I change y by a certain amount how much does x change or d x by d y. So, if I take the derivative there is a 2 in both the terms it will cancel out and so, I will get x d x by a square plus y d y is equal to 0.

And, this can be rearranged and it will give me d x by d y is equal to minus y a square by x. If, I substitute this into that expression then I obtain r divided by x plus y square a square by x.

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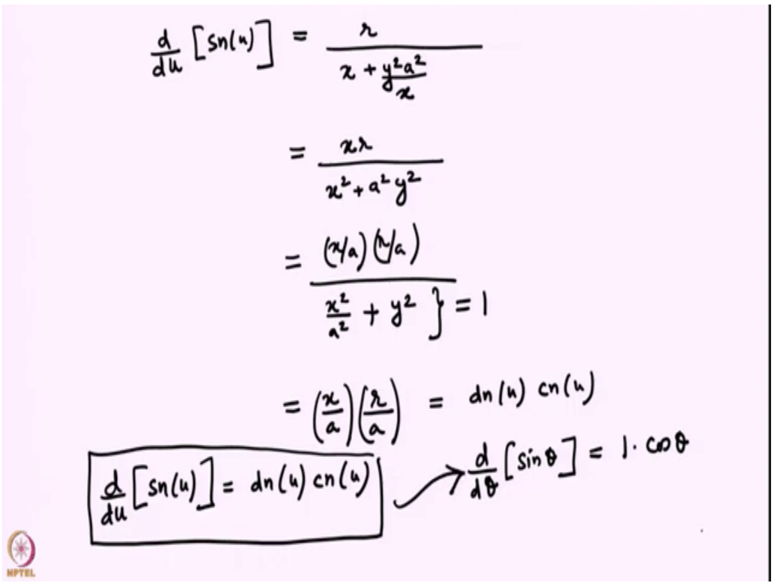

So, we find d by d u of s n u is equal to r divided by x square plus y square a square divided by x, I can multiply and the x will go the in the numerator and I will obtain x square plus a square y square. I would like to express d by d u of s n u in terms of elliptic other elliptic functions.

So, the other elliptic functions are all normalized by certain quantities, if I divide by a square on both sides, then I have by on both numerator and denominator, then I have x by a and r by a, and I have x square by a square plus y square and this quantity, because I am on the ellipse is just equal to 1.

So, this is just x by a, r by a, you can immediately see what is this r by a is d n u, x by a is c n u. So, we find this interesting identity that d by d u of s n u is equal to d n u c n u. Does this make sense when we go to the limit of k going to 0? Yes it does because as I told you earlier in the limit of k going to 0 the ellipse becomes a circle this becomes, d by d theta u goes to theta s n u goes to sin theta, d n u goes to 1 c n u goes to cos theta.

So, we have just recovered our identity d by d theta of sin theta is equal to cos theta. So, this is a generalization of that. Similarly we can also find other identities.