Introduction to interfacial waves

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Lecture - 07
Vibrations of clamped membranes

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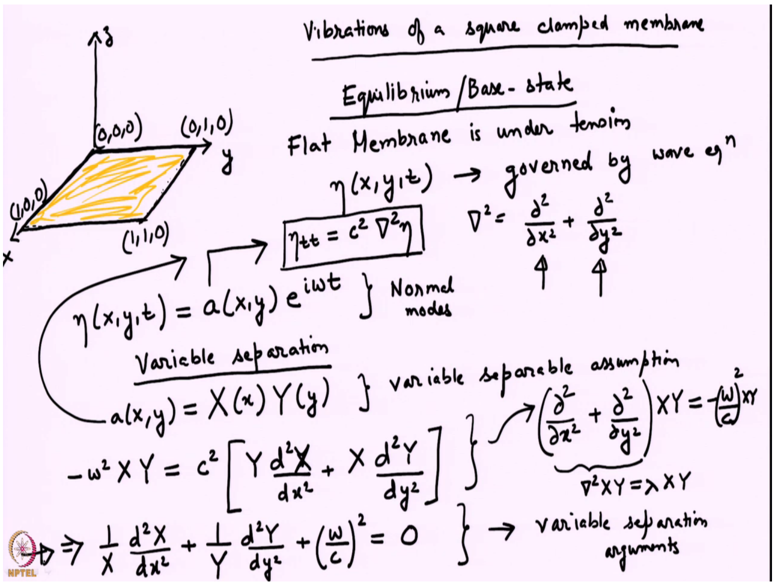

We were looking at Vibrations of a clamped square membrane in two dimensions. It was governed by a linear wave equation a 2D wave equation. We had started with doing normal modes on it and using the kind of coordinate system. It is this was a Cartesian coordinate system and since the membrane was clamped at all ends it allowed us to do variable separation.

So, we said that the eigen mode can be written in variable separation form as some capital X which is a function of small x into capital Y which is a function of small y. Substituting that we had obtained an equation of this form. Now, once again I would like to highlight that whenever we do normal mode analysis it always leads us to an eigen value problem.

Even this has the structure of an eigen value problem. You can readily see this that this can be written in the form because Y is just a function of small y. So, I can write it in the form del square by del x square del square plus del y square of XY is equal to of XY. So, this operator is our familiar Laplacian operator.

So, the Laplacian of XY is equal to lambda times XY; XY is our eigen function a and so, grad square of a is equal to lambda of a. So, lambda here is basically our minus omega square by c square. So, you can see that boundary conditions like before we will discretize the values of lambda; only certain values of lambda will allow a non trivial solution to this equation which satisfies those boundary conditions.

So, that will determine our eigen frequencies and in turn will determine our eigen modes of the system. Once again we will write our final answer as a linear superposition of all the eigen functions. Here there will be a here again there will be a summation from 1 to infinity. But here because there are 2 eigen functions in 2 directions there will be a double summation. So, let us work on that.

So, you can see that this is an eigen value problem and so, now, we are working on this equation where I have divided the equation in the this equation and where I have divided throughout by XY. So, now, let us work on this equation.

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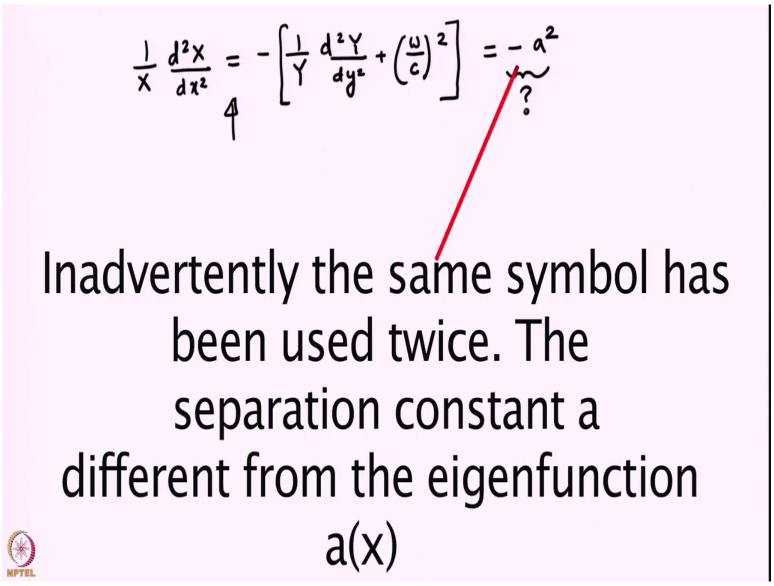

So, this equation can be further written using variable separation arguments. I would like to separate out everything which depends on small x on one side and everything which depends on small y on the other. Now, we can see that we have a pure function of small x on the left hand side and a pure function of small y on the right hand side, omega by c is a constant.

So, that is not a function of y and so, because small x and small y can be varied independently. So, the only way this equality can hold good is if each of these expressions is equal to a constant and it is the same constant. We will be choosing our constant to be a negative constant. I encourage you to think what happens if I choose a positive constant.

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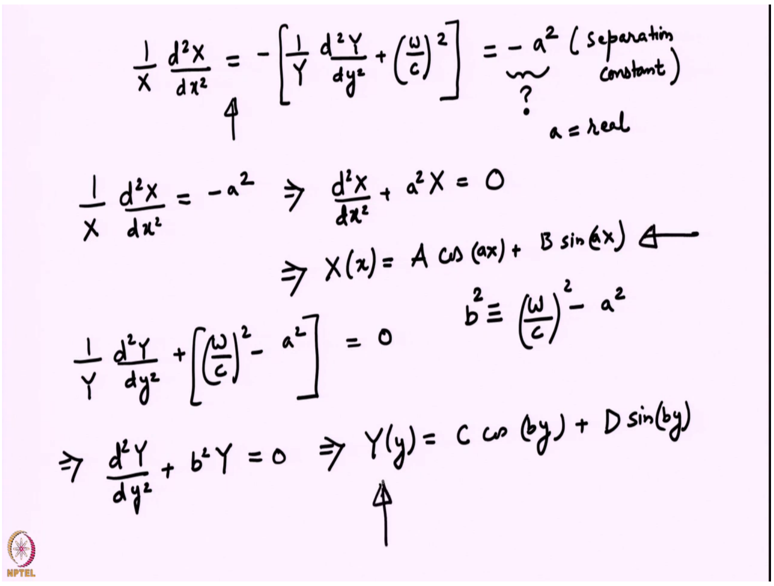

So, I am choosing a negative. So, this is also what is known as a separation constant; a is a real number and so, a square is real and minus of a square is a negative constant ok. So, we are choosing a negative value of the separation constant. So, now, if I equate the first if I write it like this; this is equal to minus a square and so, I get an equation for X. And this equation this is just a linear constant coefficient equation easy to solve and so, A cos ax plus B sin ax.

In general these constants a and b will once again be complex constants because our eigen function are in general complex. So, we will take that into account at the end when we write down the answer in terms of real quantities. So, this is my expression for capital X as a function of small x, now let us work on the second part.

So, I have 1 by Y d square Y by d y square plus omega by c whole square minus a square is equal to 0. And this can just be written as d square Y by d y square plus b square into Y is equal to 0, where I have defined; I have defined b square to be equal to omega by c; it is a new variable. I have just called omega by c whole square minus a square as b square. The square is for convenience.

This, similarly, the square in the separation constant is also for convenience; otherwise we will have to keep carrying a square root every time. So, again this is also very easy to solve; C cos by plus D sin by. So, we have got expressions for capital X and small x. Now, let us go over to the boundary conditions, which says that the membrane is clamped at all ends.

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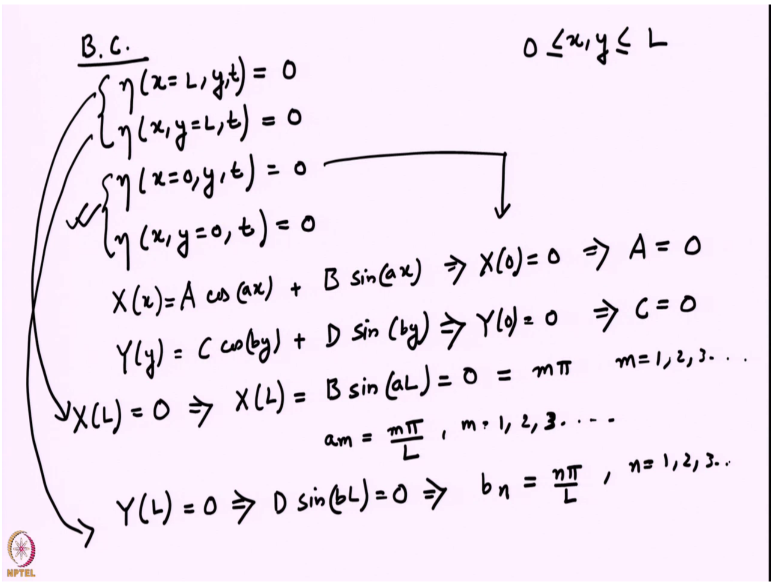

So, my boundary conditions says eta at x is equal to L and all y is equal to 0, for all time ok. So, maybe I should write because eta is a function of time. So, I do not have to write this thing. Then I have eta x, y is equal to L at all time is equal to 0, eta x is equal to 0, y at all time is 0; eta x, y is equal to 0 at all time is 0. Each of this ensures that one side of the membrane is clamped. Remember that x and y both go from 0 to L.

Now, let us work on the lower two boundary conditions because those are at x equal to 0 and y equal to 0. So, if you recall our expression for eta. So, the only x part of eta is proportional to this capital X and the y part of eta is proportional to capital Y, this is the only small y dependency and this is the only small x dependency. So, it is equivalent to imposing these boundary conditions only on capital X and capital Y. If we do that then we obtain.

So, X of small x was obtained earlier to be A cos small ax plus B sin small ax and Y of y was C cos by; I am just writing it again for convenience. So, this would imply, so, this condition would imply that X of 0 is 0. So, this would just straight away imply that A is 0. This would imply similarly that Y of 0 is 0; this would imply C is 0. This is just telling us that we only are going to get a sine series and that is because of the boundary conditions, but now just like earlier we had a sine series.

We are going to have a sine series now, but it is going to be a two dimensional sine series a sine along x and an another sine along y. So, these two boundary conditions are done, now we need to focus on these two the other two. So, let us work on those. Now, X of L; small x equal to L is also 0, this comes from here the first one of those two and this implies X was already equal to. So, is equal to B sin a L equal to 0.

So, this is just going to discretize the value of small a. We are going to set this equal to some m pi. Once again m is equal to 1, 2, 3 up to infinity and then this is just going to tell us that a m is equal to m pi by L. I have attached a subscript to m because a cannot be any value, but only integral multiples of pi over L and so on.

The second of those conditions tells us that Y of L when small y is equal to L is 0, this implies D sin b L equal to 0. And this also implies that once again this will discretize the value of small b and we should not use the same index m. So, I will use another index. So, I will use n and so, b n is equal to n pi over L; like m, n over also goes from 1, 2, 3 up to infinity.

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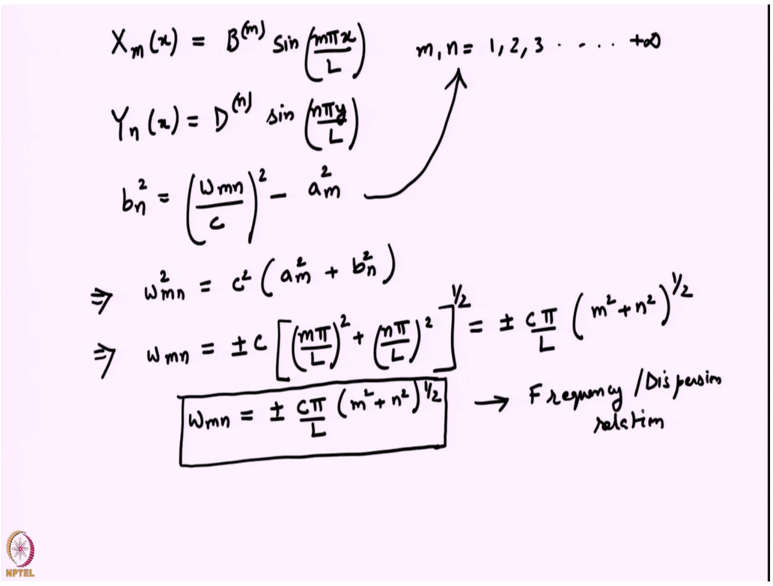

So, let us now write down what are the eigen functions that we have found. The our eigen functions look like X m of x is along the x part of the eigen function is B m sin m pi x over L and Y n over x is equal to D n sin n pi y over L and m and n go from 1, 2, 3 up to infinity. What are the allowable? So, this together when they are taken a product you get the eigen modes.

Now, the eigen mode will require a double index m, n in order to indicate it because there are two discrete indexes. What about the eigen frequencies? We have seen earlier that b square was defined as omega by c whole square minus a square. We have seen now, from the boundary conditions that b and a are discretized.

So, b has a discrete index n, a has a discrete index m and consequently this omega is going to have two discrete indices m and n. So, let me rewrite this again. So, we are going to have b n square is equal to omega mn by c whole square minus a m square, once again m and n are subject to that restriction.

So, you can see that this tells you that omega mn square is equal to c square into a m square plus b n square. And if I write the square root of this then this is plus minus c a m was basically m pi by; so, this is m pi by L whole square plus n pi by L whole square and this can be written as plus minus. So, there should be a square root here and this can be written as c pi over L m square plus n square to the power half.

So, omega mn is equal to plus minus c pi over L m square plus n square to the power half, it is my frequency relation. You can see that this generalizes what we already knew. If you substitute n is equal to 0 then you will get omega m is equal to or omega m 0 is equal to plus minus c pi over L into m plus minus c pi over L into n and that is exactly what we had found earlier for our analysis of the 1D wave equation.

There we had written c as a function of c has been had been written as square root t by rho. Here rho would be an aerial density of the membrane. So, one way once we do this, so, here our dispersion relation is written in terms of the wave speed c instead of tension and the density of the membrane. So, now, this is our frequency relation or dispersion relation.

Like earlier we can once again write the final answer as a linear combination over all the eigen modes there will be a double summation now 1 over m and another over n.

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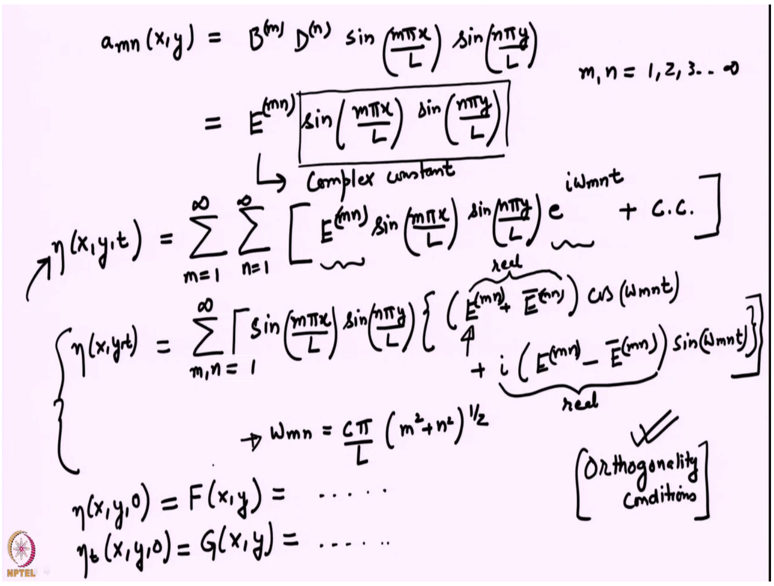

So, let us write the final answer. So, the final answer. So, we have found our eigen function now have two indices; a mn of x comma y is some B of m into D of n into sin m pi x over L sin n pi x n pi y over L and these two constants can be combined into a single constant we will call it E and it will have two indices mn and then the same thing.

So, this is a mn and in general this is a complex constant. So, how do we write the final solution of our 2D wave equation of a membrane which is clamped at all edges? The general solution is once again it is basically summation m is equal to 1 to infinity. So, here m and n both go from 1 to infinity and then there will be a summation over n 1 to infinity.

And then you would have m E mn sin m pi x by L sin n pi y by L e to the power i omega mn t. Omega also has two indices now, mn and then we need to add the complex conjugate of this part; of this part. So, you will add a E mn bar and you will add a e to the power minus i omega mn t and the eigen functions will remain the same. So, you can write it like this.

And so, if you express it in real notation it would just reduce to again a double summation over I am just going to write it as a single summation just to as a shorthand and then sin m pi x over L into sin n pi y over L into you would have E mn plus E mn bar into cos omega mn t plus i times E mn plus E bar mn into sin omega mn into t. So, that is my final answer with omega mn being given by the expression.

So, now we have added the minus part also. So, I can ignore the minus sign in omega. So, it is just C pi by L m square plus n square to the power half. So, together they give me the most general solution to the clamp membrane equation and you can see once again that this part is a sum of E n its complex conjugate and i times this E minus E bar is also. So, both these are real.

So, you can replace E plus E bar and E minus E bar with some constants, but you will have to put some indices on top of that or double index C mn and D mn let us say ok. And so, once again you we have written it down as a linear combination of the eigen modes. Now in order to find out these constants we will have to specify the initial shape of the rectangular membrane.

So, if you give it some perturbation eta of x, y comma 0 this would be some function let us say let us call it some capital F of x comma y and then you will have to substitute t equal to 0 in this expression and you will get a double Fourier series. Once again the coefficients of the double Fourier series will be obtained using orthogonality conditions. That is also not enough. You will also have to specify the velocity of the membrane at time t equal to 0 at every point.

So, once again eta t at x. So, eta t is del eta by del t. So, I am just taking a partial derivative of this expression with respect to time and this will give you at time t equal to 0. This will also give you another function of x comma y. This is exactly the same as what we had done earlier. Earlier we were doing it for only one space dimension, now we are generalizing to more than one space dimension.

And this will give you again another expression in which you have to have set time equal to 0. The resultant expression, so, you are going to get some expression here another expression here. You can write those expressions yourselves and then use the orthogonality conditions. You will get two double Fourier series, use the orthogonality conditions to determine the coefficients of both of them.

Once you determine the coefficients you will be able to find out what is the value of E mn in the series and that gives you your answer. Here also if you want your modes if you want your membrane to vibrate in a pure normal mode you will have to choose one of these. You will have to choose the initial displacement.

So, for example, if you want the membrane to vibrate in the 1-1 mode then you will have to choose an initial displacement which is proportional to sin pi x by L and sin pi y by L. If you set it up that way with 0 velocity everywhere it will vibrate purely in the 1 comma 1 mode and that mode will have the frequency given by C pi by L into square root of 2 which is given by this expression when I substitute m is equal to n is equal to 1.

There is a countable infinity of modes again and the general answer is expressible as a linear combination of them. Once again it is not necessary that your membrane should vibrate only in a pure normal mode. You can give it any arbitrary perturbation like F of x comma y and if you do that many normal modes will get excited and then the system will vibrate in a linear superposition of all the normal modes.

The resultant motion can look extremely complicated and may not even be periodic it will still be oscillatory ok. So, now having completed and gained some experience with solving these kind of equations using the method of normal modes let us go over to one more coordinate system which is a circular membrane and we will analyze it in cylindrical coordinates.

And we will keep things a little bit simple initially so that we will assume that it is axisymmetric and we will learn about Bessel's equations. This is going to be very useful when we later learn about waves on cylindrical interfaces when we learn about fluid waves on cylindrical interfaces.

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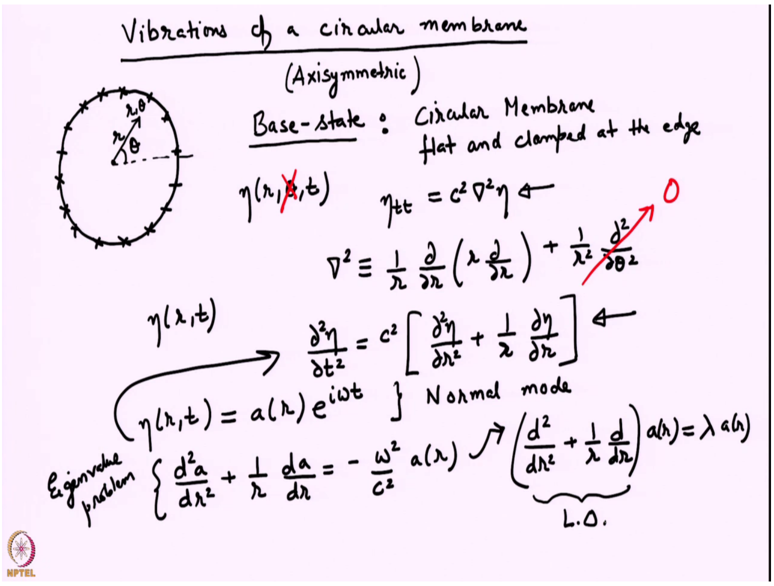

So, our next topic is vibrations of a circular membrane. We had looked at until now we have looked at a rectangular membrane, now we are going to look at a circular membrane. So, this is the center and I am going to use say any point has coordinates r and theta; r is the length of the radius vector up to that point and theta is the angle it makes with the dotted line.

Now to simplify things a little bit I will be making the axisymmetric approximation. This is only an approximation, one does not need to make it, our analysis will become slightly more complicated without this approximation. So, let us. So, you can see that in the base state this is a circular membrane. You can think of it as a tabla membrane or the membrane of a mridangam. It is clamped at all ends. It is entirely clamped at its edges.

So, once again we are using the equivalent of the boundary condition we had used earlier. So, it is clamped continuously around. So, the displacement all along the periphery is always 0. So, circular membrane is flat. So, in the base state or equilibrium state the circular membrane is flat and clamped at the edge. This clamping is true at all times even when there is a perturbation on the membrane. So, the equations governing the membrane can one once again be easily obtained.

So, here I am going to again once again. So, the membrane you can imagine that the membrane displacements is coming out of the plane of the board. So, if I indicate the displacement eta then eta would ideally be a function of r theta and t and so, we are going to have a wave equation for eta once again. Those the small amplitude displacements are once again going to be governed by eta and the equation would be a wave equation.

If I write the generic wave equation in basis free notation then it would be this and this is the scalar Laplacian. So, all I have to do it to adopt it to this problem is to find out the expression for the Laplacian in cylindrical coordinates. Here there is no z. So, I will only have to worry about derivatives with respect to r and theta. So, that would be 1 by r del by del r of r del by del r plus 1 by r square del square by del theta square.

Now because I make the axisymmetric approximation, this term is going to go to 0 that ensures that eta is not a function of theta. So, it simplifies my analysis substantially and so, I have eta which is just a function of r and t and so, my equation governing eta. So, I just have this term with derivatives with respect to r and if I just open it up it would be this.

One can derive this equation, this is the wave equation governing small amplitude displacements of a membrane where there are no non axisymmetric effects. One can derive this equation from more basic principles by applying Newton’s second law of motion to a small part of the membrane. We have avoided that and we have straight away written this equation by writing it using vector notation, alright.

So, now, let us do let us look at we expect oscillatory solutions here as well and so, let us do a normal mode analysis of this problem. So, like usual we set eta of r comma t is equal to some eigen mode a which is a function of r only because eta is not a function of theta into e to the power i omega t. So, this is our normal mode ok.

So, this should lead us to an eigen value problem. So, if you substitute this into this equation it would lead us to this equation. Once again you can see that this is an eigen value problem. If I write this as the operator d square by dr square plus 1 by r d by dr operating on a of r is equal to some lambda times a of r.

So, you can immediately see that this is an eigen value problem where lambda is minus omega square by c square and this is the linear operator. We will analyze this equation. We will find its solutions this is related to the Bessel's equation we will encounter it again when we do based on fluid interfaces later and we will learn about this equation and what are its solutions and we will solve this and determine the frequency relation.

We will also determine the eigen modes of this system and then write the final answer as a linear superposition over the eigen modes.