Introduction to interfacial waves
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Lecture - 60
Derivation of the Stokes travelling wave (contd..)

We were looking at the Derivation of the Stokes wave. We had seen that first we obtained the equations at linear order these turned out to be homogeneous equations the right hand side was 0. Then we went to order epsilon square the next order and then we found that at this order the equations became inhomogeneous.

This is similar to what we did earlier in the course for non-linear oscillators except that now we are dealing with partial differential equations. Recall that we have the Laplace equation at every order we also have two versions of the Bernoulli equation. The second version of the Bernoulli equation is obtained by taking the total derivative on the Bernoulli equation and then using the kinematic boundary condition to obtain another equation.

So, we had two versions of the Bernoulli equation the Bernoulli equation itself and a modified Bernoulli equation. Using this we went up to second order where we found that the equation governing phi 2 the second order velocity potential phi 2 was governed by again a Laplace equation. And then there were 2 other boundary conditions, but now these were all applied at z is equal to 0 still but they were inhomogeneous.

And what appeared on the right hand side where all quantities which came from the previous order. So, quantities like phi 1 eta one and their various derivatives. So, this is what we have found so far so let us proceed further. So, let me summarize the equations that we have found until now.

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$O(\epsilon)$:

$$\nabla^2 \phi_1 = 0 \rightarrow (A1)$$

$$\left(\frac{\partial \phi_1}{\partial z}\right)_0 + \eta_1 = 0 \rightarrow (B1)$$

$$\left[\frac{\partial^2 \phi_1}{\partial z^2} + \left(\frac{\partial \phi_1}{\partial \tau}\right)\right]_0 = 0 \rightarrow (C1)$$

$O(\epsilon^2)$:

$$\nabla^2 \phi_2 = 0$$

$$\left(\frac{\partial \phi_2}{\partial z}\right)_0 + \eta_2 = -\left(\frac{\partial^2 \phi_1}{\partial \tau \partial z}\right)_0 \eta_1 - \frac{1}{2}\left\{\left(\frac{\partial \phi_1}{\partial x}\right)^2 + \left(\frac{\partial \phi_1}{\partial \tau}\right)^2\right\}$$

$\partial^2 \phi$

Note that the last term is applied at $z = 0$ viz. $-\frac{1}{2}\left\{(\frac{\partial \phi_1}{\partial x})^2 + (\frac{\partial \phi_1}{\partial z})^2\right\}_0$

So, at order epsilon we have found the equations are our familiar equations. At z is equal to 0 I am going to skip writing the z is equal to 0 because at every time we are going to write apply it at z is equal to 0 only. So, I am just going to indicate that by a 0 and then a modified Bernoulli equation this whole thing also at 0 is equal to 0 and we had called this A1 B1 and C1.

A1 is my governing equation B1 C1 are really boundary conditions. At order epsilon square we had grad square phi 2 equal to 0. Then, we had del phi 2 by del tau also at 0 plus eta 2 is equal to minus del square phi 1 by del z del tau and 0 into eta 1. And then we have a modified Bernoulli equation.

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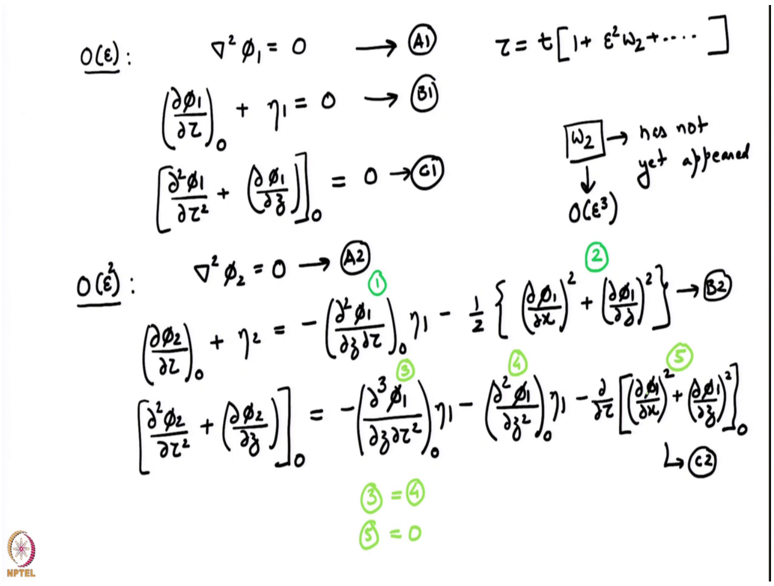

The whole thing applied at z is equal to 0 all of these are applied at z is equal to 0. Recall that the z is equal to 0 condition is applied only to phi and its derivatives eta whether its eta 1 eta 2 eta 3 and so on these are not functions of z. So, it is only on phi 1 phi 2 phi thr3ee and their various derivatives that the z is equal to 0 condition applies. Also at 0 and we had we can call this A2 B2 and C2.

We had understood in the last video how did we get the right hand sides it is by repeated Taylor series expansion and making sure that we have everything up to order epsilon square. As I mentioned earlier A1 B1 C1 are homogeneous whereas, in order epsilon square B2 and C2 are inhomogeneous note there on the right hand side of B2 and C2 what appears are all terms coming from the previous order.

So, just like whatever we have done until now using perturbative techniques we have to solve at any given order before we can proceed to the next order. So, here also we will have to solve the equations at order epsilon and then use them in order to simplify the equations and order epsilon square. The solution that will appear at order epsilon will be used to determine the right hand sides of the equations and order epsilon square and the same pattern will repeat.

Again note that the left hand side the operators which appear on the left hand side of the equations both at order epsilon and order epsilon square are very identical it is just the right hand side which keeps changing and in this particular problem we will have to also write down the equations at order epsilon cube. We will do that shortly, but note that the right hand side we will just keep getting more and more lengthy we will keep adding more and more terms on the right hand side.

Whereas, the structure of the left hand side the left hand side operator will remain identical at every order. Also note that the quantity omega 2 has not yet appeared. In our equations recall that omega 2 was I expect omega 2 to give me my frequency correction how does the frequency depend on amplitude and so omega 2 should appear in a non-linear at a non-linear order.

However, order epsilon square on the right hand side of the equations you can see that there is no omega 2 as yet. So, in order to determine omega 2 we will actually have to go to order epsilon cube, we will do that shortly. Let us now solve the equations at order epsilon first let us write down the solutions and use it to simplify the equations at order epsilon square.

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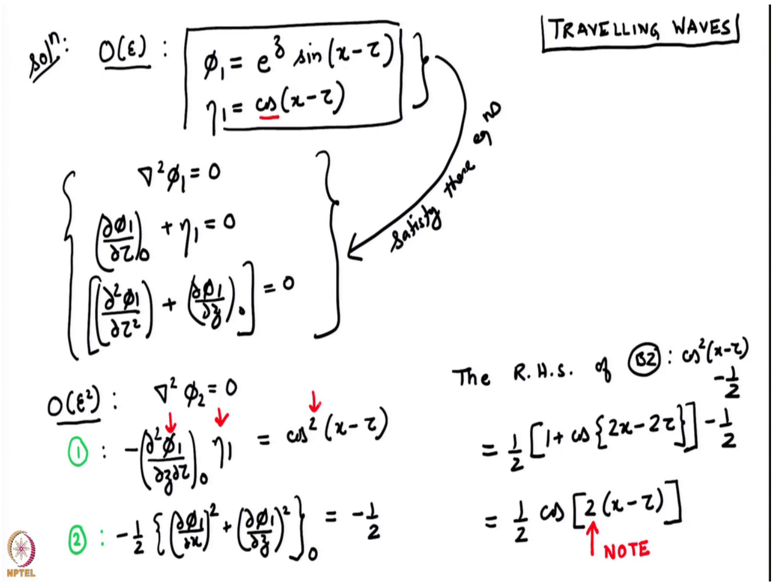

So, order epsilon so this is solution. The solution at order epsilon is easy we are basically looking at travelling wave solutions. This entire procedure can also be done for standing waves, but let us work on travelling waves. So, at order epsilon our solution is familiar so, in non dimensional way it is e to the power z sin of x minus tau and eta 1 is cos of x minus tau.

This is a solution to the equation we are not writing down the most general solution to the equation, but you can go and check that these satisfy the equations that we have written earlier. Recall that we have written the equations at previous order are grad square phi 1 equal to at this order or grad square phi 1 equal to 0 and del phi 1 by del tau at 0 plus eta 1 is equal to 0 and del square phi 1 by del tau square plus del phi 1 equal to 0.

You can go back and check that these two satisfy these equations. This is this should be clear where we have got these solutions from if you go back into one of the early videos in the course where we discuss travelling wave solutions, you will find that we have written solutions of this form.

Again whether eta one is cosine or whether phi 1 is sin that depends on which term we have we had written it we had written the most general solution and you can go back and see that one specific instance of the most general solution is what we have written here. And you can see by substituting and checking that the solution that we have written is a solution to the equations that A1 B1 and C1. So, these are the equations A1 B1 and C1 ok.

So, we have found the solutions to equations A1 B1 and C1 that we already know we are just writing it in a non dimensional sense. So, that is why instead of writing e to the power k z we are writing e to the power z; z is a non dimensional distance ok. Similarly, k x is written as x and then there is a omega t ok which is also non dimensionalized, ok.

So, now this is tells us the solution at this order so now, let us go to the next order. So, order epsilon square; set order epsilon square we of course, have the Laplace equation which is grad square phi 2 is equal to 0 that is homogeneous. So, this is nothing to be done there, but the other two boundary conditions have to be simplified because what appears on the right hand side of those boundary conditions equation B2 and C2.

So, equations B2 and C2 you can see that the right hand side depends on phi 1 and its various derivatives as well as eta 1. We now know what is phi 1 in eta 1 and so, we need to substitute and work out the precise functional form of the right hand side of equations B2 and C2 only then can we solve these equations. So, our next task is to work out the functional forms of the right hand side let me give some numbers to these terms. So, I will call this term 1 and this is term 2.

So, now I am looking at term 1 and term 1 is minus del square phi 1 by del z del tau at 0 into eta 1, this is just what I have written as term 1 in green. So, I need to work out the form for that term and this you can show easily from knowing these two solutions, this you can show easily is just cos square x minus tau. Similarly, term 2 is minus half of del phi 1 by del x whole square plus del phi 1 by del z whole square evaluated at 0 z is equal to 0.

And term 2 will just turn out to be minus half there will be a cos square x minus tau and a sin square x minus tau they will add up to give you 1 and so you will just be left with the minus half prefactor. So, the right hand side of B2 equation B2 is cos square x minus tau minus half which I can write it as. So, cos square if I multiply it and divide by 2 then I can write it as 1 plus cos 2 x minus 2 tau 1 plus cos 2 theta is 2 cos square theta is this and so this just becomes half cos 2 of x minus tau.

Notice the appearance of this quantity cos 2 x minus tau, notice the appearance of this quantity. We had started with a cos x minus tau here and at non-linear order on the right hand side what has appeared is a cos 2 times x minus tau. Why 2? Because there were a there was a product between two things here ok this is coming from the quadratic non-linearity and that is causing a cos square to appear here and which in turn we have written it as. So, cos square theta we have written it as cos 2 theta ok.

So, we have cos 2 x minus tau appearing in our equations and you can see that is basically a non-linear effect ok. So, if you put a traveling wave of the form cos x minus tau then what appears due to non-linearity is cos 2; 2 times x minus 2 times tau. We will discuss this more, let us now proceed and similarly you can write down the equation C2, similarly we need to write down the right hand side of equation C2.

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$$\left(\frac{\partial^2 \phi_2}{\partial \tau^2}\right)_0 + \left(\frac{\partial \phi_2}{\partial z}\right)_0 = 0$$

$O(\epsilon^2)$: $\quad \nabla^2 \phi_2 = 0$

$$\left(\frac{\partial \phi_2}{\partial \tau}\right)_0 + \eta_2 = \frac{1}{2}\cos[2(x-\tau)]$$

$$\left(\frac{\partial^2 \phi_2}{\partial \tau}\right)_0 + \left(\frac{\partial \phi_2}{\partial z}\right)$$

Note the error : $\left(\frac{\partial^2 \phi_2}{\partial \tau^2}\right)_0 + \left(\frac{\partial \phi_2}{\partial z}\right)_0 = 0$

And so the equation C2 is basically the modified Bernoulli equation 0 and this if you work it out. So, once again you can label them so, I will label them here. So, I will call them term 3 term 4 and term 5. So, you will find the term 3 and term 4 once you substitute and work out the forms of term 3 and term 4 you will find that 3 and 4 will cancel each other whereas, 5 will just be 0. So, overall the right hand side of equation C2 is just 0 ok. So, please work this out try this yourself.

So, 3 will be equal to 4 ok and 5 will be equal to 0, 5 is 0 because there is a del phi 1 by del x whole square plus del phi 1 by del z whole square evaluated at z is equal to 0, that will eventually add up to a sin square plus a cos square ok and that will give you 1 and del by delta of a constant is just 0 ok. So, you will get 5 equal to 0 and consequently the right hand side of equation C2 will be 0 it will just become a homogeneous equation. So, equation C2 is just 0.

So, now we have this thing where at order epsilon square we have grad square phi 2 is equal to 0 that is our Laplace equation then, we have del phi 2 by del tau at 0 plus eta 2 is equal to half cos twice x minus tau, this we have already found before. And then we have del square phi 2 by del tau at 0 plus del phi 2 by del z at 0 is equal to 0, the 0 is a consequence of cancellation.

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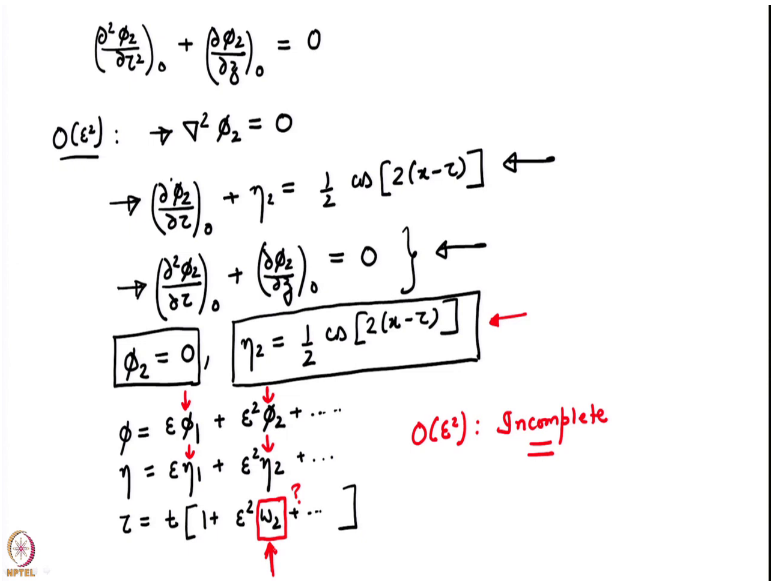

So, these are our my updated equations A2 B2 and C2 where we have worked out the right hand sides for B2 and C2 ok. And you can see that it is only for equation B2 that there is a right hand side equation C2 does not have a right hand side it is just 0. So, you can see that the equation governing phi 2 is this and that.

So, you can think of it as this is a boundary condition this is an equation governing phi 2 and if you know phi 2 you can use equation B2 to determine eta 2. We are going to choose because the right hand side of equation C2 the this equation is 0 we are going to choose phi 2 to be equal to 0. You can think a little bit more as to why this choice is justified.

If we make that choice then equation this; obviously, satisfies the Laplace equation. So, we do not have a problem with that and now we just need to determine eta 2 eta 2 is just determined from this equation phi 2 is identically 0 so all derivatives are 0 so eta 2 just becomes half of cos 2 x minus tau. So, I will put that also in a bracket.

So, you see what has happened is this is telling us that at non-linear order, we had started at linear order with a linear travelling wave solution something that we have already seen previously in the course. And we know what is the velocity potential corresponding to a linear travelling wave we also know what is the dispersion relation.

What we are finding at quadratic order is or at non-linear order is basically that there is a second harmonic which appears at the interface eta 2 whereas, there is no correction to the velocity potential at this order let us see what happens.

So, now you see this you can think of it as phi is equal to epsilon phi 1 plus epsilon square phi 2 this is what we had written, then we have eta is equal to epsilon eta 1 plus epsilon square eta 2 and we also have a tau which is t into 1 plus epsilon square omega 2 plus dot dot dot and they are done here also.

So, you see we have we know this; we know this we also have determined this we also have determined this but we do not yet know the value of this. So, our problem at order epsilon square our problem at order epsilon square is still incomplete.

We need to determine this as I had mentioned earlier omega 2 has not yet appeared in the equations at order epsilon square. Recall the equations A2 B2 C2 on the right hand side we did not have a small omega 2 appearing anywhere in those equations in the equations B2 and C2.

So, we now need to proceed to the next order and do the same exercise at the next order and see whether omega 2 appears in the next order. We will find that omega 2 does appear at the next order we are not going to solve the equations of that order we will just find that there are some resonant forcing terms which appear at that order. We will have to get rid or eliminate those terms the resonant forcing terms just had we had done that in for the for a non-linear pendulum or a non-linear oscillator.

Pay attention that now we are doing this in the context of partial differential equations. Earlier when we are looking at non-linear oscillators with single degree of freedom whether it was duffing oscillator whether it was a non-linear pendulum we were looking at ordinary differential equations of course, in the method of multiple scales we had converted it into equivalently into partial differential equations.

But these are really partial differential equations because now we are looking at a fluid system ok we are looking at what is the non-linear modification to a linear travelling wave ok we are trying to calculate that. And we have calculated, what is the non-linear modification a harmonic of the primary wave that we have imposed is appearing the second harmonic?.

But we have not yet determined the correction to the frequency of the wave or in other words the correction to the phase speed of the wave which is determined by this small omega 2 for this we need to proceed to the next order and see how to determine omega 2 from that order so let us continue.

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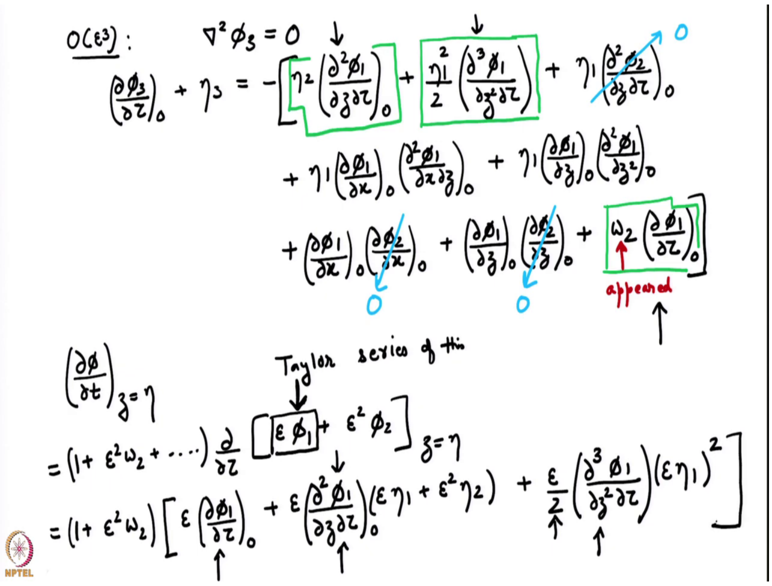

So, at order epsilon cube once again we will have the Laplace equation. Now, for phi 3 our Bernoulli equation will pick up a number of terms on the right hand side. I am just going to write those terms and then I will explain a few of them, as to how it works. The idea is similar to what we had done earlier, but now we have to go to a few more terms in the Taylor series expansion ok.

So, I will pick up one term and that will lead to 3 terms which appear on the right hand side I will explain how those 3 terms have come and very similar manner you can understand how the remaining terms have come there are number of terms on the right hand side of this equation so let me write those terms first.

So, minus so I will put a bracket because all the terms on the right hand side are minus. So, minus eta 2 del square phi 1 everything again gets applied at 0, this is just a consequence of the Taylor series approximation. The right hand side now becomes quite lengthy so it will take some time to write it ok. So, those are the terms on the right hand side you can see that this is quite lengthy there are total of 8 terms which have appeared.

Now, before we start working on them we have to do the same thing as before. We have to work out the forms of each of those terms from what we know about the previous order. In particular there is a simplifying feature at the previous order phi 2 is 0 so some of these terms will go to 0. So, I am just going to set those terms to 0 where phi 2 appears.

So, we have this term which is 0 we have this term which is 0 and we have this term which is 0. So, that eliminates 3 terms from my right hand side and I am left with total of 5 terms. Now, I will just explain I am just going to put in boxes some of these terms ok so because there are so many terms so it is a good idea to explain where they have come from and how do we write this Taylor series expansion and this.

Note that a omega 2 something that we wanted this is the reason why we came to this order and this has appeared now at this order. This was this had not yet appeared at order epsilon square now we have a small omega 2 on the right hand side, so, let see. So, recall that in the Bernoulli equation we have a term in the Bernoulli equation before we did all the expansions we had a term del phi by del t at z is equal to eta.

We express del by del t in terms of del by del tau del by del t was this into del by del tau and phi itself was expanded as epsilon phi 1 plus epsilon square phi 2 and this whole thing was applied at z is equal to eta. So, I am going to show you how this expansion can be used to generate those terms this and that so let see how.

So, we will have 1 plus epsilon square omega 2 plus let me not write the dot dot dot because that is not important here. And then we have epsilon del phi 1 by del tau now this will be applied at z is equal to eta, but now I am going to do a Taylor series expansion on this.

So, del phi by del tau at z is equal to eta is del phi by del tau at z is equal to 0, plus because there was a multiplying factor. So, I again put an epsilon here the next term would be the second derivative of the derivative with respect to z of this quantity again applied at z is equal to 0 so it will become a second derivative del z into del tau once again at 0.

And now I have to multiply it with eta; eta itself is epsilon eta 1 plus epsilon square eta 2. This is just coming from Taylor series expansion of this term Taylor series of this and I have gone up to the second term in the Taylor series expansion. Note that all the derivatives are evaluated at z is equal to 0 even when we go up to non-linear order ok. So, this is the first term similarly we can do a similar exercise with the second term ok.

So, let us do that and so we will have so, now let us work on the. So, I have we have missed out one more term so we will need to write the Taylor series approximation up to one more term ok. So, this is the next term would be half and then there would be a del cube phi 1 by del z square into del tau.

So, this is just the del by del z of this into eta square and eta is basically epsilon eta 1 plus epsilon square eta 2. I do not want to go up to order epsilon to the power 4 so I am just writing eta as epsilon eta 1 and there is a factorial 2 here which I have put here ok and this is whole squared ok so this is whole square. So, all of this, this, this and this is just coming from this term alone ok and I will put a bracket here.

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The derivation will be continued in the next video