Introduction to interfacial waves

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Lecture - 06
Normal modes of a string fixed at both ends

Recall that we were doing normal mode analysis on the one-dimensional linear wave equation. And we had found that the eigen mode in the discrete case goes over to an eigen function, we had called eigen function a of X.

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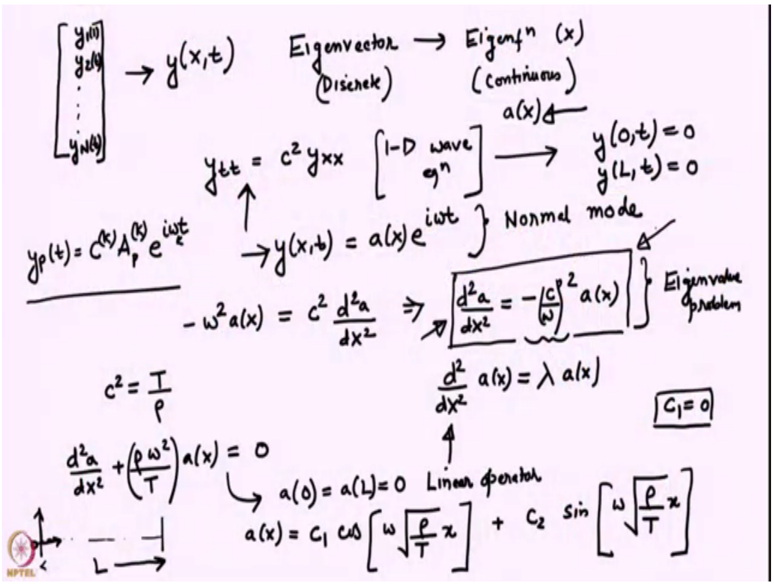

And we had found that in this case as well the eigen function satisfies a an eigen value problem it comes from an eigen value problem. And in this case only for certain values of the frequency, we will there be non-trivial eigen functions. And this because there are particular boundary conditions, which have to be satisfied by the eigen modes ok.

So that will cause only particular values of the frequency to be allowed. Now, we had satisfied the zero displacement boundary condition at the left boundary that had caused the cos term to drop out.

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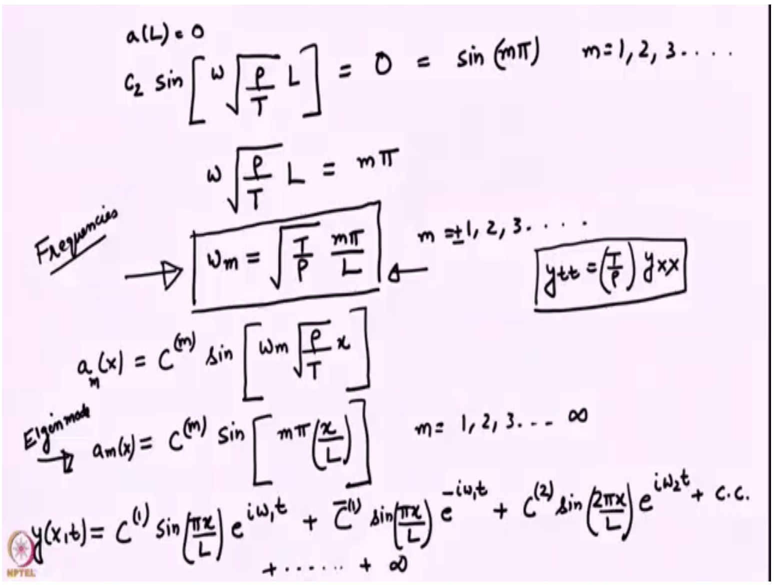

We had also satisfied the zero displacement condition a of L equal to 0 on the right side that had caused the frequencies to be only certain discrete multiples of square root T by rho into pi by L.

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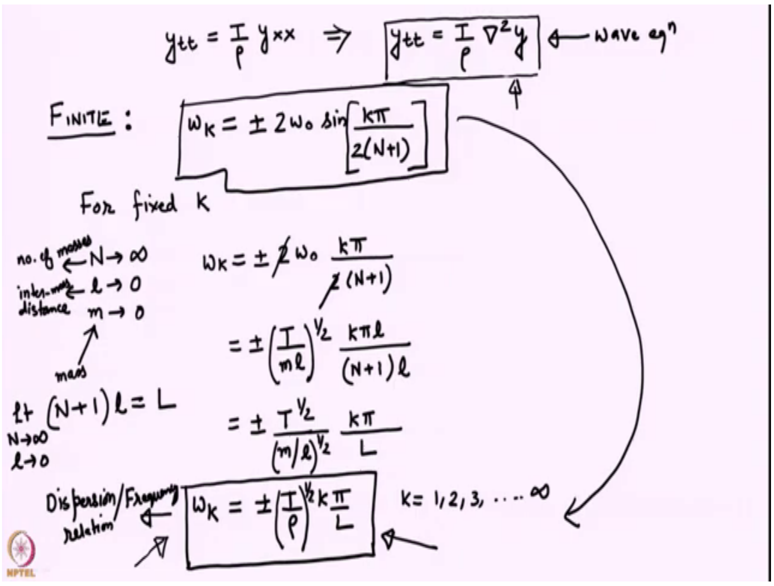

Now, as I had told you before if you compare this formula with the one that we had derived earlier by taking the limit of the finite frequency relation and applying capital N going to 0 and small l going to 0, then we had recovered this. This relation as you can see is I have missed a T here half this relation as you can see is the same as what we have got there.

Now, you can also notice one thing that we do not have a plus minus and that is because I have chosen not to put a plus minus here. Remember that sin m pi is 0 for plus minus 1 plus minus 2 and so on.

But I will remember that and when I write down the most general solution. I will add e to the power i omega 1 t and I will also add e to the power minus i omega 1 t. So, if I remember that, then I really do not have to put a plus minus here, but one can put that for the sake of correctness.

So, now, we have found what are the eigen frequencies or rather what are the frequencies of oscillation. There are an in countable infinite number of frequencies which are given by this relation. What do the eigen functions look like? Because the eigen function is dependent on m, so I am going to put a discrete index m also on the eigen function corresponding to the eigen value omega m.

And this is given by some constant into sin remember that we only have the sin part in the eigen function the cos part dropped out because of the boundary condition. And so we will have and omega m is given by the formula above. So, if I just substitute it, I just obtain m pi x by L. So, the argument of sin has to be non-dimensional.

So, this is a m of x the mth eigen mode. Once again m goes from 1 2 3 up to infinity. So, accountably infinite sequence of frequencies and for each such frequency, there is a particular shape of oscillation. Now, can we write the most general solution to our linear equation as a linear combination of these eigen functions multiplied by e to the power i omega m t? Yes, we can.

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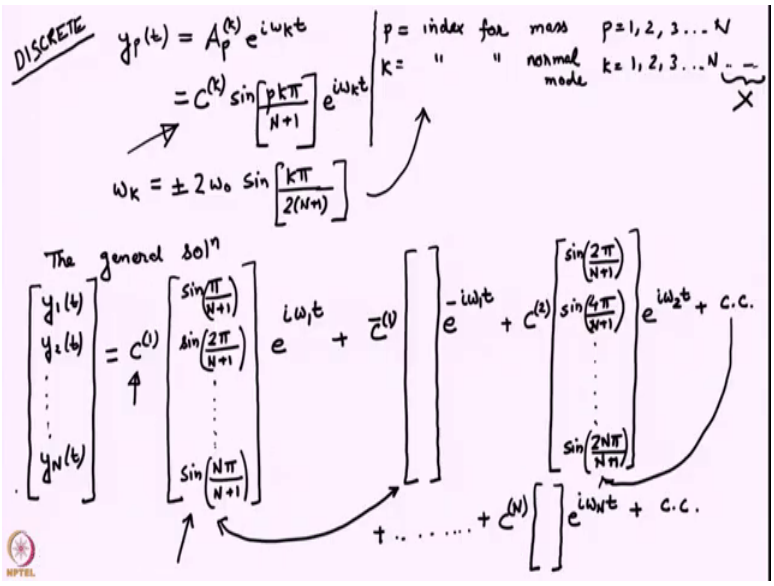

We will follow exactly the same structure that we have done earlier. So, we will follow exactly this. This structure that I have written where I have written it as a sum of n column vectors multiplied by c m into e to the power i omega n t, except that now our summation will not run from 1 to n, but it will run from 1 to infinity because it is a continuous system.

So, I can anticipate that the solution to my wave equation, I will write the wave equation here for completeness. Let me write T by rho this is the equation. We have done a normal mode analysis of this equation subject to the boundary conditions fixed. And we have found the eigen frequencies and the eigen functions. Now, we are going to write down the most general solution of this equation for those boundary conditions as, so some constant into the first eigen mode m is equal to 1.

So, sin pi x by L into e to the power i omega 1 t. Omega 1 is given by the formula in the square bracket. You can substitute omega 1 and get square root T by rho into pi by L.

But this you have to add the complex conjugate part of this. So, C 1 bar again the same eigen function that is just a real function; similarly, C 2 sin 2 pi x by L – the second eigen function into the second plus its complex conjugate. And you can keep writing this all the way up to infinity.

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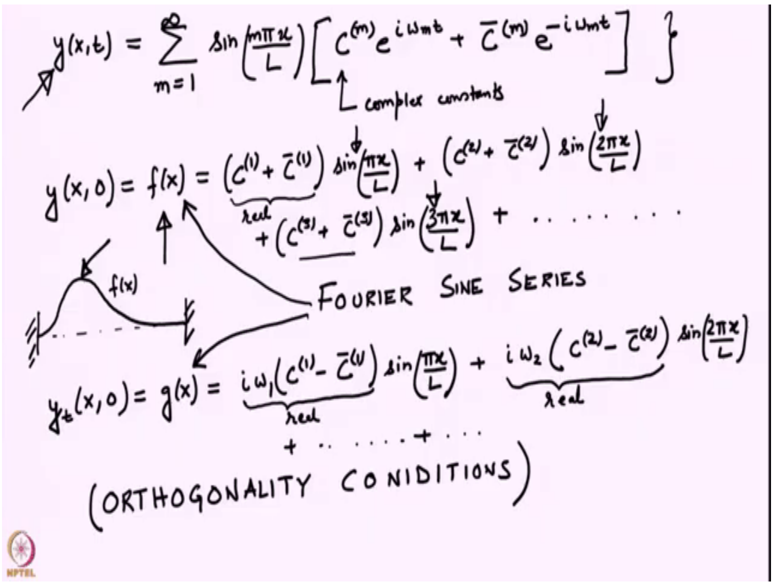

So, in general if you write if you want to express this in a more compact notation, you could write this as I am taking into account the negative frequencies also to make the expression real. And so you can write it like this because for every term every pair the eigen function is common.

So, I can pull the eigen function out, and write it as a sum of a complex number plus its conjugate. So, that this whole the summation is real. And there are infinite such number of such terms in the expression. It is an infinite series.

So, now the question is like before we can ask the question how do we determine these C ms? C ms are in general complex constants. And like in the finite degree finite number of degrees of freedom case, the C ms have to be determined from initial conditions. Now, how do we specify initial conditions here? You can specify the initial displacement of the string.

Remember that in the finite number of degrees of freedom, we were specifying the initial displacement of every mass. If I had two masses, then I had to specify the initial displacement and the initial velocities of both the masses. Here I have an infinite number of masses or in other words I have a mass at every point in space or I have a mass at every x.

So, I will have to specify the displacement – the initial displacement of this string at every x. So, I will have to specify what is the value of y X comma 0. This in general is going to be a function of X let me call it f of X. The general solution which we have anticipated from the finite degree of freedom case has this structure. If I put t equal to 0, then the entire term in the square bracket the exponentials just go to unity and I recover.

What have we found? We have found that whatever is the initial shape of the string. So, you can give it an initial condition. So, this is my string in the base state or equilibrium state. So, suppose I give it an initial condition respecting the boundary conditions, you know this is not a very symmetric perturbation. But it respects the boundary conditions at both ends.

It peaks at some value of x and then goes to 0. If I give it that kind of a initial condition, and I call this f of X this function f of X. Then this just tells me that f of X can be expressed as a Fourier sine series. This is quite interesting because through this process of doing eigen modes and normal mode analysis, we have discovered the Fourier sine series.

You can see that the coefficients of all the sin terms are real, because each bracket has a complex number plus its conjugate. So, each of this is real. So, you can call this another constant. You can use whatever value you want, whatever symbol you want, and then you can write it as a Fourier sine series. Is this enough? Recall that we are looking at a second order system second order in time.

So, it is not enough to just specify the initial displacement of the string at every x. In the finite degree of freedom, we had to specify the position as well as the velocity of every mass at time t equal to 0. We will have to do the same here. So, we also need to specify what is the initial velocity of every point on the string at time t equal to 0.

So, we need to specify if I call the velocity as del y by del t, I will indicate that with the subscript I also need to specify what is y t of x comma 0. The initial velocity of every point in the string that let me call it some function this is you can see that this is also a function of x and I will call it some function g of X.

Now, if you write it out, you can see that this would have the form i omega 1 C 1 minus C 1 bar into sin pi x by L. I am just taking a del y by del t of this expression and then putting t equal to 0, all the way to infinity. You can once again see that I times C 1 minus C 1 bar is a real number, omega 1 is any way real, omega 2 is also real, so i times C 2 minus C 2 bar is again real.

So, once again you are getting a Fourier sine series once for f of X, and once for g of x. How do we determine the coefficients of this series? If you are familiar with Fourier analysis you will know that if I give you a function f of X and g of X, if I tell you that it can be expanded in a Fourier sine series – both of them can be expanded in a Fourier sine series with unknown coefficients.

You can use what is known as the orthogonality conditions to take the inner product of these functions f of X and g of X with the respective Eigen modes one at a time, and that will tell you these inner products will be 0 for all terms in the expansion except one.

And that taking step by step such inner products will give you the coefficients of these Fourier sine series for both f X and g of X ok. So, that formally completes our discussion of normal mode analysis of the one-dimensional wave equation.

It would also be interesting to plot to visualize in space just like we had done for the discrete degrees of freedom case or our shapes of oscillation. We had plotted what does mode one look like, what does mode two look like. One can also do the same here. Let us do that.

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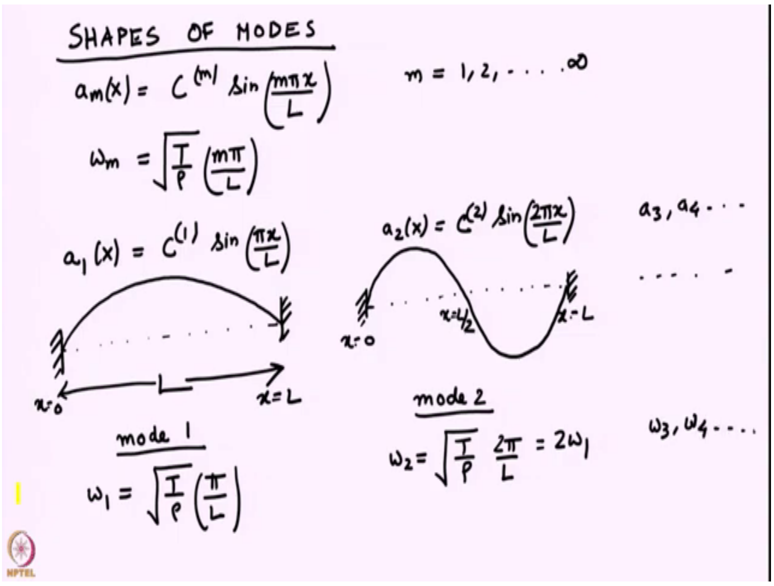

So, shapes of modes. And so we have seen that our eigen mode a m of X is C m sin and with the corresponding frequency alright. So, now I can visualize the first mode a 1 of X is C sin pi x by L. Let us plot it. So, these are the two walls separated by a distance L. And how would it look like? This is just a sin at x is equal to L, x is equal to 0 is here.

And if x is equal to L is there, then at x equal to L this goes to pi. So, this is a half sin wave starts from here goes like this. The C m is not relevant because whatever C m I can choose C m to be plus 1 or minus 1. So, if you multiply this by minus 1, this will get inverted and you will have a negative sign that is also an eigen mode just as we have seen earlier. So, this is mode 1.

What is the frequency of this mode? It is square root T by rho pi by L. If you initialize the string by giving f of X to be this shape or some scalar multiple of this shape with 0 velocity everywhere. So, I am choosing in these in these conditions I am choosing f of X to be some scalar multiple of sin pi X by L, and I am choosing g of X to be 0.

So, a 0 velocity everywhere and a displacement which is just proportional to the first mode, then the string will oscillate purely in the first mode of oscillation, and its frequency will be given by the formula that we have written down there. Similarly, what is the second mode? So, a 2 of X C 2 sin 2 pi x by L. So, again I draw the wall separated by distance L, the base state at X is equal to L this goes to 2 pi. So, this is a full sin wave. This is x is equal to L by 2. So, you can see that if you initialize the string like this, it will oscillate this is mode 2.

So, if you initialize the give the initial shape f of X to be some scalar multiple of this, this function sin 2 pi x by L with 0 velocity everywhere it will set up an oscillation. These are all standing modes and the frequency would be given by which is just twice of the first mode.

Similarly, you can work out all the other modes a 3, a 4 and so on. And you can also work out what is omega 3, omega 4 and so on. There is no reason why the system should vibrate only in a pure normal mode ok. So, in general, if you choose your f of X to be an arbitrary, but well-defined function like something like what I have drawn here, then f X will have projections along each eigen mode ok.

The eigen modes are this, this, this and so on. And so the coefficients by projection I mean the coefficients are nonzero. The coefficients C 3 plus C bar 3 is nonzero. If f X has a projection along the eigen mode sin 3 pi x by L. So, if you choose an arbitrary shape like the one that I have drawn in general you will end up exciting many normal modes simultaneously.

And the system will vibrate in a superposition of many normal modes. The resultant motion may look extremely complicated. And it may not even be periodic in time just as we had seen earlier even in the discrete case. The most important difference is here that we are dealing with an infinite series.

Similarly, you can do many things. You can start with 0 displacement, but nonzero velocity. So, you can give f X is equal to 0, but you can choose g X not to be equal to 0. This will also have an effect. And depending on which modes you excite, the system will vibrate in a linear superposition of each of those modes. An important point to remember is that that because these are linear systems, each mode behaves independent of the other.

The amount of energy that is contained in a given mode is purely determined by initial conditions. If initially if your initial condition does not have projections along a certain mode, that mode cannot appear in the system at any later instant of time. This is a property of linear systems.

So, now having done our 1D linear wave equation let us go into slightly more complicated examples. We will stretch this discussion of partial differential equations and doing normal mode analysis slightly further because this will introduce us to some of the essential mathematical tools that we will require when we study waves on fluid interfaces. Because those are all of them are continuous systems, and we will be dealing with partial differential equations there.

So, in our next example, we are going to take the same wave equation, but we are going to take our 2D wave equation. So, because now I want to have two horizontal directions, until now we only had del square by del x square now I want to have del square by del x square and del square by del y square.

So, I will choose the dependent variable to be eta instead of y, it represents the displacement. In this particular case, we will take a membrane.

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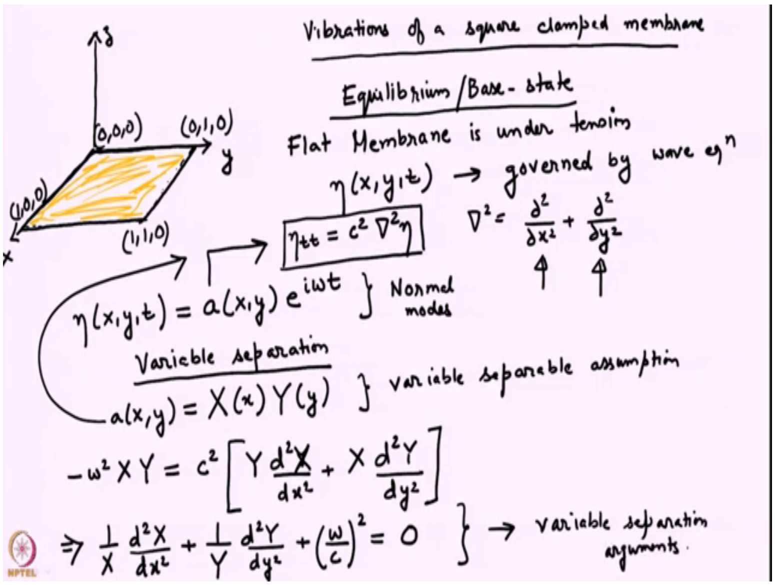

So, let us say we have a membrane. Once again we are going to look at the simplest boundary condition which is the membrane is clamped at all sides. So, we are let us say we are taking a vibrations of a square clamped membrane. So, by clamped I mean it is attached on all four sides. So, let us say this is the x-axis, the y-axis and the z-axis. And the membrane is something like this.

So, this is my membrane. And let us say the coordinates of this point is 1, x is equal to 1, y equal to 0, z is equal to 0. If this is x equal to 1, y equal to 1, z is equal to 0; this is x equal to

0, y equal to 1, z is equal to 0. And this is the origin. So, my membrane is clamped at all sides at its edges. And it is under tension.

So, in the equilibrium state or the base state, this is an elastic membrane you can think of a tabula membrane. We will after this we will soon do the actual case of a tabula membrane where the membrane is circular. So, membrane is under the membrane is flat in the base state, and it is under tension. So, you can intuitively see that if I give it a kick, it will start vibrating.

The equation one can derive it, but I am just going to use the same equation that we have derived, and I will call the displacement in the vertical direction of this membrane as eta.

So, eta by definition can only depend on x, y and t it cannot depend on z because eta itself is the displacement in the z-direction. So, eta is a function of x, y and t. And eta is governed one can also show this by doing a force balance of a certain part of the membrane applying Newton's second law and then doing a small displacement approximation.

We are going to avoid that process, and we are just going to use the same equation written in coordinate free notation. So, I have told earlier that the wave equation is governed by eta t t is equal to C square eta x x. Now, in this case eta is a function of x comma y.

So, it will be grad square eta, where grad square is the scalar Laplacian. So, grad square in this case would be del square by del x square plus del square by del y square – the 2D scalar Laplacian, so that is my governing equation.

Once again the wave equation the only difference being that I have two spatial dimensions x and y, and so this is a partial differential equation with three independent variables x, y and t. Once again we can do a normal mode analysis on the base state because we expect oscillatory solutions. So, I am going to write eta of x, y, t like before is going to be my normal modes a of x comma y into e to the power i omega t this is my normal modes.

Now, what do I do? I substitute as usual I substitute this into the governing equation. Now, before I do that, I would like to do one process which is called variable separation. This comes because of the coordinate system and the boundary conditions which have been specified. The membrane is clamped from all sides. And so the nature of the boundary condition as well as this particular coordinate system allows me to do variable separation.

I hope all of you have seen variable separable solutions on the Laplace equation ok. This will be in the same spirit. If a so with variable separation in mind we are going to say that a x, y is a variable separable can be written in a variable separable form. So, it is some function capital X of small x, and capital Y of small y. So, this is my variable separable assumption.

And so my problem of determining my eigen modes or the eigen functions. So, instead of having you can see that because I my eigen function here was a of x comma y. If I did not do this variable separable separation, my equation governing a of x comma y would continue to be a partial differential equation.

This variable separation technique allows me to split it into X which is a function of small x alone and capital Y which is a function of small y alone. And derive separate equations for capital X and capital Y, those will be ordinary differential equations and that is one of the advantages.

Now, if we substitute this into the governing equation, we would obtain. So, if we substitute this form of a of x comma y into e to the power i omega t in into the governing equation. We obtain minus omega square capital X capital Y into C square Y sorry this is X.

And if I divide throughout by X Y, then I can write it as 1 by X d square X by d x square plus 1 by Y d square Y by d y square plus omega by C whole square is equal to 0. We are going to analyze this equation using variable separation arguments in my next video.