Introduction to interfacial waves

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Lecture - 59
Derivation of the Stokes travelling wave

Until now we have looked at linearized interfacial waves, now we will start with an example of non-linear surface waves. This is also the example that we are going to do is also known as Stokes waves. It is named after George Gabriel Stokes who was the first person to derive it analytically.

Now because the algebra tends to be lengthy because this is a non-linear wave, so, we are going to make a number of simplifying assumptions all of which will help make the reduce the length of the algebra.

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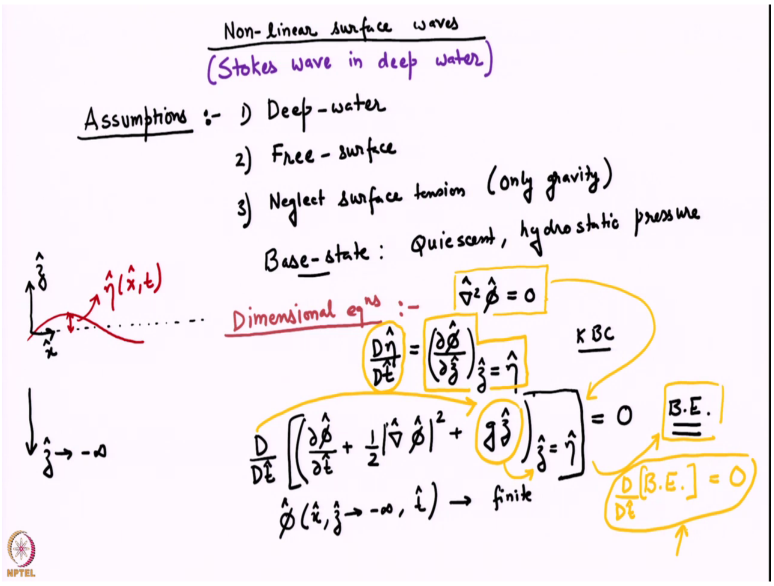

So, we are going to assume deepwater. We already know what is the deepwater approximation. So, we are going to pretend that in the base state the pool is infinitely deep. We are going to have a free surface which means that we are going to ignore the density of the fluid above.

So, if this is an air water situation we are going to ignore the density of air. We are also going to ignore surface tension and keep only gravity as the restoring force. As usual the base state is quiescent and the pressure is hydrostatic. We have already seen these approximations before. So, this is a pictorial depiction of the base state. So, the z is equal to 0 line represents the free surface in the base state and there is no velocity in the fluid below it.

Then we introduce a perturbation which are governed by the equations that we have encountered before. Let us go over that once again. So, the velocity field perturbations are governed by the Laplace equation governing the velocity potential the perturbation velocity potential.

We then have a kinematic boundary condition which is basically an equation of mass conservation. Note that we have written the full kinematic boundary condition now we are not linearizing it because we are going to do a non-linear calculation now. So, we need to write the full kinematic boundary condition. So, this is the full kinematic boundary condition. We have derived this earlier.

We also have the condition that pressure is equal to 0 at the free surface both in the base state as well as in the perturb state there is no surface tension. So, the pressure at the free surface is always 0. We can write the Bernoulli equation add the free surface set pressure equal to 0 and that gives us the Bernoulli equation applied at the perturb state. Note that this is applied at z is equal to eta now. In the linearized description we were earlier applying it a z is equal to 0.

In addition we also have the finiteness condition because this is a deep water calculation. So, the finiteness approximation says that the perturbation velocity potential remains finite as we go to deeper and deeper regions of the fluid. Like before we assumed the surface and the fluid is horizontally unbounded. So, the free surface extends from minus infinity in x to plus infinity and it goes from plus eta in z to minus infinity. So, with those approximations let us now analyze these non-linear equations.

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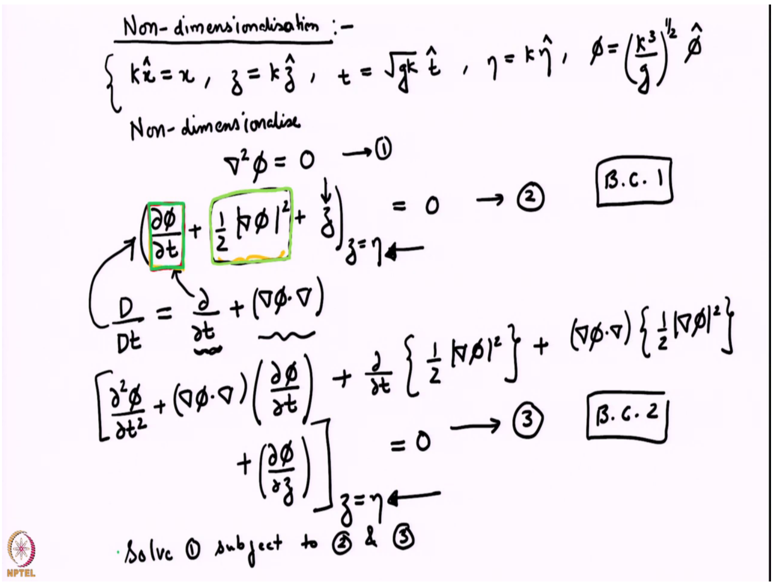

First let us so, non dimensionalization. We have done this before and the choice of scales have been known before. So, note that I have put a hat over all the variables because I want to introduce non dimensional variables without hat. So, this is a non dimensional x, this is a non dimensional z, this is a non dimensional time. These scales should be familiar to you know they come from the deepwater approximation and this is a non dimensional eta and this is a non dimensional phi.

If you put these scales into the equations that we wrote earlier then you will see that some things will scale out and you will get another set of equations. Now, I am going to do two steps in one exercise. So, I am going to non dimensionalize that is I am going to substitute these scales, but I am also going to do something additional. What I am going to do is I am going to take the equation for the this is Bernoulli equation applied at the free surface that allows me to eliminate pressure.

What I am going to do is I am going to take the D by D t hat of this equation derivative of this equation and. So, I am going to take the D by D t of this equation and you can see how that helps. The main help comes in this term the last term of the equation this term. This term is actually g into eta because this term is going to be evaluated at z hat is equal to eta. So, this is actually going to be g into eta hat and so, when the operator operates on this term g is anyway a constant.

So, this will just become g into D eta hat by D t hat that is exactly the left hand side of my kinematic boundary condition. Using the kinematic boundary condition I can replace that term in the Bernoulli equation with this term. How does that help? That helps because all the 3 terms now in my Bernoulli equation now become just depend on phi hat because I am going to replace D eta hat by D t hat as del phi hat by del z hat.

You will see that this has some advantages and it simplifies the algebra a little bit. So, we are going to what we are going to do is we are going to analyze the Laplace equation. This is the same equation that we have written before and now we will have two copies of the Bernoulli equation.

The Bernoulli equation without this operator acting D by D t the Bernoulli equation in its original form and then the Bernoulli equation with the D by D t applied on it in addition plus using the kinematic boundary condition to eliminate the D eta hat by D t hat term. This will give me a Bernoulli equation with a g into D eta hat by D t hat.

So, I will have two copies of the Bernoulli equation one written in the original Bernoulli equation and one Bernoulli equation D by D D by D t hat of the Bernoulli equation and this is also equal to 0. So, I am going to use instead of using the Bernoulli equation and the kinematic boundary condition I am going to use a Bernoulli equation and a slightly modified Bernoulli equation as my two boundary conditions.

Note that the kinematic boundary condition is not going to be explicitly used that is because in deriving the second copy of the Bernoulli equation, the kinematic boundary condition has already been used. So, using this these two boundary conditions I am going to solve the Laplace equation using a perturbative expansion and we will see what happens at non-linear order. This is essentially our goal.

So, let us continue further. So, once we non dimensionalize and do that step of applying D by D t hat operator on the Bernoulli equation we get another copy of the Bernoulli equation along with the kinematic boundary condition, it becomes a modified Bernoulli equation.

So, now I have a Laplace equation, I have two Bernoulli equation; one the original one the modified and then I have some finiteness conditions and I can use all of these scales to non dimensionalize all of those equations. Once we do that we obtain the following set of equations.

So, the Laplace equation remains the same. Now, it is a non dimensional Laplace equation. The Bernoulli equation the original Bernoulli equation, this is the boundary condition. g just get scaled out after non dimensionalization and now all variables do not have a cap on top.

So, I will call this equation 1, this is equation 2 and in addition as I told you earlier we have another Bernoulli equation that is obtained by taking the capital D by D t the total derivative operator on the Bernoulli equation. So, now, you can see I am going to just write it out in because recall that the D by D t operator now it is a non dimensional operator is basically del by del t plus u dot grad u is grad phi that grad, this is my operator.

So, when you do the second Bernoulli equation is, basically applying this operator on that equation and use modifying this term using the kinematic boundary condition. So, if I open up the D by D t operator then I will get del square phi by del t square this is because of del by del t operating on this term plus grad phi dot grad operating on del phi by del t plus. So, this is D by D t operating on just this term.

Now, D by D t also has to operate on this term. So, again I am opening up D by D t and writing it as del by del t plus grad phi dot grad operating on this term. So, that will give me a del by del t operating on half grad phi square plus grad phi dot grad operating on the same thing. And then the last term is D eta by D t which is just del phi by del z from the kinematic boundary condition which you are not using anymore and this whole thing has to be applied at z is equal to eta is equal to 0.

So, that is my 3rd boundary condition or rather 2nd boundary condition. So, this is boundary condition 1 and this is boundary condition 2. These are boundary conditions because these equations are true only at z is equal to eta they are not true in the bulk of the fluid ok. And so, I have to solve equation 1 subject to 2 and 3 and of course, we have this restriction that we have the familiar restriction that as z goes to minus infinity phi has to stay bounded.

So, I will not explicitly write that. We have to keep that in mind. So, now, I have to solve 1 subject to 2 and 3 2 and 3. Now, you can see that this is a complicated exercise because although our equation is linear our boundary conditions are non-linear. We have seen this before and so, we are going to use a perturbative expansion.

I am going to straight away right perturbative expansion with an expansion for time also just as we had expanded the frequency in the rather we had expanded the timescale for the non-linear pendulum. This should remind you of the Lindstedt Poincare technique and at the end of this calculation I will justify why we need to expand we need to do a perturbation expansion for time as well ok.

It is exactly the same reason unless we do that they will be secular terms in the expansion and unless we do that expansion for time we will not be able to eliminate those secular terms. We have seen this before in the context of ordinary differential equations, now we are seeing this in the context of partial differential equations. So, let us proceed.

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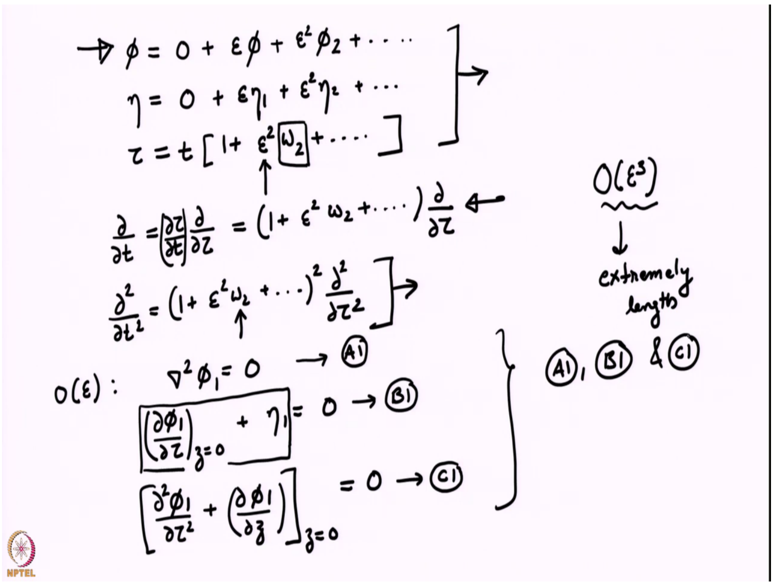

So, perturbative expansion. So, we will do a perturbative expansion the base state is 0 plus some small parameter which is related to the product of amplitude into a wave number plus now, we have to write at least up to order epsilon square because this we need to do a non-linear calculation.

This is also 0 plus epsilon eta 1 plus epsilon square eta 2. And as I said earlier I am introducing a stretch time variable which is basically my non dimensional time into 1 plus epsilon square into omega 2 plus dot dot dot. You will see that the first correction appears at order epsilon square this is familiar to us from the pendulum calculation.

We will see that at linear order we do not need any correction to time. So, the first correction appears at non-linear order and this will basically introduce an amplitude dependence in the dispersion relation. So, with those let us proceed. So, because now time is also expanded, so, we will have to express all partial derivatives with respect to time small t with respect to tau and that is just.

So, let me write it like this. And this is just 1 plus epsilon square omega 2 plus dot dot dot into del by del tau. We have second derivatives with respect to time and so, this just becomes 1 plus epsilon square omega 2 whole square into del square by del tau square. With that we have to go back and substitute these expansions and these expansions in our equations that we have written earlier, equation 1, 2 and 3.

At linear order we will get back exactly what we have got until now. There will be no surprises there. We are interested in what are the corrections at non-linear order, but as before first we need to solve the linear equations at order epsilon before we can go to the order epsilon square equations. Like before we will see that the first order solution appears as an inhomogeneous term in the second order and so, we will have to first solve for the linear order before we can proceed to the non-linear order.

So, if we substitute into this and then collect terms at order epsilon there is nothing at order 1 because order 1 is just the base state which is trivial here. So, this is grad square phi 1 is equal to 0. It is just putting that expansion into the Laplace equation and collecting terms at order epsilon. Then we have from the Bernoulli equation I will explain this, we have already seen this before, but now is just derivative with respect to tau.

And then the modified Bernoulli equation; so, these are our equations at order epsilon. The z is equal to 0 should be familiar to you from the Taylor series argument that we have already gone through. We just need to understand the origin of these terms. So, let us look at the equations. So, equation 2 has this term del phi by del tau or del t and we are going to replace del by del t with del by del tau.

Del by del tau is just this and you can see the del by del t and del by del tau or the same as far as linear order is concerned. The difference between them order appears only at order epsilon squared. So, we are right now at order epsilon and so, I can just replace del by del t with del by del tau which is what has been done here. And so, this term which I have put in a red box here is just becoming at order epsilon it is just becoming del phi 1 by del tau.

Recall that this was applied at z is equal to eta. So, this was applied at z is equal to eta. So, by that usual Taylor series argument we have to apply this at z is equal to 0. Now, this term does not contribute at linear order or order epsilon. So, the only contribution will be from this term and that is applied at eta and eta is itself expanded as epsilon eta 1. So, at order epsilon this will just become eta 1.

So, we will have del phi 1 by delta tau plus eta 1 and del phi 1 by del tau will be evaluated at z is equal to 0. So, this is our order epsilon equation from equation 2. Similarly, from equation 3 you can see what is the at order epsilon what will happen. At order epsilon del square by del t square is the same as del square by del tau square, again evaluated at z is equal to 0.

So, there will be a contribution from here and this will become del square phi 1 by del tau square. This is a non-linear term this is also a non-linear term. So, is this a non-linear term. The only other contribution from at linear order will be from here and this will again become del phi 1 by del z at z is equal to 0. So, this is how I have written those two equations.

So, let us call this equation A1, B1 and C1. Now, until now we have stopped at this level and then we have gone on to solve these equations. We will do the same here, but before we do this let us write down the equations or let us learn how to write down the equations at order epsilon square.

You will see that this leads to in homogeneous equations. The solution at order epsilon will appear as in homogeneities at order epsilon square. So, let us understand how to write down the equations governing phi 2 and eta 2 at order epsilon square.

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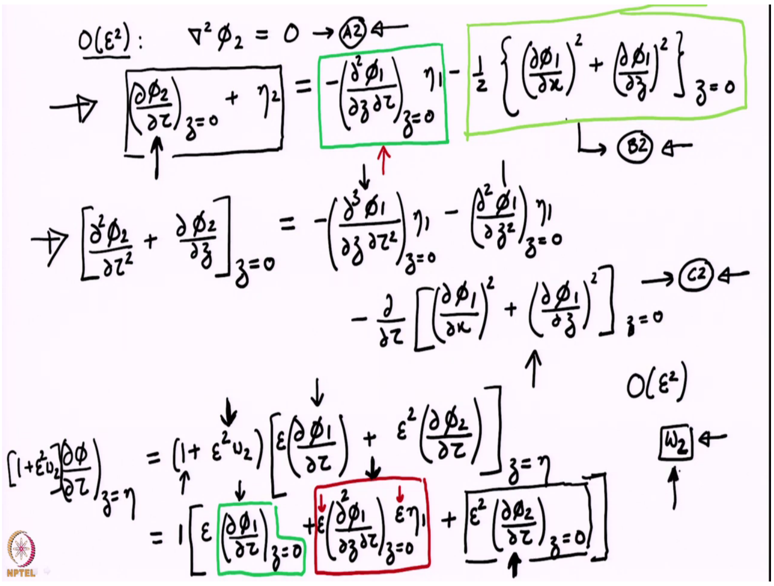

So, it order epsilon square. So, let me write down the equations and then I will explain. So, this is Laplace equation is easy it just becomes at order epsilon square grad square phi 2. The two boundary conditions have phi 2 in eta 2 appearing on the left hand side. Note that the form of the operators remain the same, but what changes is only on the right hand side.

So, on the right hand side we have and then on the right hand side we will only write those things which depend on the previous order. So, they will be only things with subscript 1. So, phi 1 and eta 1, only those things can appear on the right hand side. I will call this A2, this has B2 and there is one more.

Again the left hand side of the equation remains the same except that it operates on phi 2 and then the right hand side is slightly more complicated. So, this is at z is equal to 0. And I will call this B sorry B2 this is C2.

So, clearly in the right hand side has become complicated now. There are a whole bunch of terms and we will have to understand how to look at or how to derive these terms. So, let us look at them one by one. So, let us first understand the origin of this term, so, this term. The pattern remains the same.

We will basically have to take the terms. Substitute the perturbation expansions and then appropriately do a Taylor series expansion up to as far as possible until we ensure that there are no terms which appear at that particular order which have been left out. So, let us understand the origin of this term in the green box.

So, this term is basically coming from. So, let us write the term. The term is del phi by del tau into 1 plus epsilon square omega 2. So, this is the term and this is a prefactor and this is applied at z is equal to eta. So, let us now apply the perturbation expansions to this term.

So, the prefactor remains the same and we will write this as del. So, there will be an epsilon here del phi 1 by del tau plus epsilon square del phi 2 by del tau and this whole thing gets applied it z is equal to eta. Now, let us expand on this. This is 1 plus epsilon square. So, you can already see that even before we expand you can already see that the there is no need to retain this term in the prefactor because this term all the terms inside the square bracket is either epsilon or epsilon square.

So, if I have to retain at we are trying to obtain the expansion at order epsilon square. So, you can see that this is already an order epsilon square term and so it will multiply this with this and give you an order epsilon cube term or epsilon to the power 4 term. So, as far as at this order is concerned the prefactor is just 1, it is just this term 1 multiplied by what is inside the square bracket. Now, let us look at what is inside the square bracket.

Now note that what is inside the square bracket is del phi 1 by del tau the first term. Using a Taylor series approximation I can write this as del phi by del tau at z is equal to 0 this is what we had done at order epsilon. But now I will put one more term in the Taylor series and that will be del phi 1 del square phi 1 by del z del tau again at z is equal to 0 into eta. There is an epsilon here and eta itself is epsilon eta 1 plus epsilon square eta 2 and so on.

So, you can see that this is what? This is the term that we would have written at order epsilon, but this is the new term that is coming at order epsilon square. The product of this epsilon and that epsilon makes the whole term and order epsilon square term and this is exactly what we have written. So, this term is going to appear on the left hand side of the equation and we are going to shift it to the right hand side. So, this is the origin of the term that I have written in this green box.

Let us also understand what else would be there in this expansion. So, the first term will just give these two terms one at order epsilon and one at order epsilon square. The second term at order epsilon square would just give one term which is del phi 2 by del tau at z is equal to 0. I will not write one more term because you can see that if I had written one more term that would be a higher power of epsilon than 2.

It would be epsilon square into del square phi 2 by del z del tau this is the next term in the Taylor series approximation into eta an eta is itself epsilon eta 1 at least. So, this the product of these two would be an order epsilon cube. Since we are right now discussing order epsilon square, I do not need to write this term. So, that is why I have stopped the Taylor series expansion at this order here.

So, you can see that at this order epsilon square we are getting two terms. One of which is this and another of which is that. The term in the red box has been shifted to the right hand side because it depends on quantities which have already appeared at the previous order. They are depending on phi 1 and eta 1.

Whereas the term this term in the black box I have written it on the left hand side because this is depending on phi 2. So, this is basically this term in the expansion. I encourage you to try these expansions by hand. Unless you try this yourself it will not be clear. Clearly the algebra is a little bit complicated because you can see that in equation B2 there is one more term, in equation C2 there are many more terms.

In particular the complicated aspect of this calculation is that that we are basically trying to determine what is ok we called what is omega 2. Omega 2 is our expansion of time and as we will see later omega 2 is the amplitude correction in the dispersion relation. This will bring an order epsilon square correction in our dispersion relation that has not happened so far. So, far we have found that the frequency of the wave depend only on its wave number or other parameters on the problem, but not on epsilon itself.

Here we will see that the moment we go to order epsilon square epsilon itself will start appearing in the dispersion relation just had just as it had appeared for the non-linear pendulum. So, what we really want to know is what is the value of omega 2. Omega 2 is basically a number and we need to know what is its value because that will appear in the dispersion relation.

Here in this problem in order to determine omega 2, we will actually have to do the calculation up to order epsilon cube. At order epsilon cube the equations are extremely lengthy, but the process remains the same. One has to correctly do the Taylor series expansion and include as many terms are as necessary so that one does not miss out any contribution which would have appeared at the given order where one is working.

So, this is the same procedure. It is just that that as you go to higher and higher orders the Taylor series expansion keeps getting longer and longer and consequently you keep getting more and more terms on the right hand side which appear as inhomogeneities. So, this is the way in which I have written the first term. I encourage you to try the second term.

So, this term once again it is on the right hand side. So, it has to depend on the previous order. So, it must depend on phi 1. You can readily see looking at the structure of this term you can see where it must have come from. So, it has come from this term. It has come from this term. Write this term yourself, expand it out and see what it produces at the given order and convince yourself that it does not contribute anything to the left hand side. It just keeps contributing to the right hand side.

So, notice that the left hand side of the operator the left hand the operator which appears on the left hand side has the same structure at order epsilon square as it was at order epsilon. Here it is del phi 2 by del tau at z is equal to 0 plus eta 2. What was it at order epsilon? It was exactly the same thing except that it was now operating on del phi 1 by del tau at z is equal to 0 plus eta 1.

So, it is as if you have just replaced the phi 1 and eta 1 with phi 2 and eta 2. In all successive orders that you will do this calculation this feature will always there be there. The left hand side of the operator remains the same, the right hand keeps getting more and more complicated. We have also encountered the same thing when we did the doffing oscillator, the non-linear pendulum except that these are partial differential equations there we had encountered ordinary differential.

Carry this calculation first to order epsilon square. So, these are the equations at order epsilon square. So, this is the second set of equations. So, this is one set of boundary conditions this is another set of boundary conditions. You can try deriving these terms and convincing yourself where they come from.

Note that all the terms even an non-linear order are applied at z is equal to 0 that is because in a Taylor series expansion every successive derivative appears only at z is equal to 0 and you just multiply it by higher than higher powers of eta. So, you will have eta, eta square eta cube and so on depending on how far you go in the Taylor series approximation ok and then you will have to substitute eta itself as epsilon eta 1 plus epsilon square eta 2 plus so on and so forth.

Again I am mentioning that what appears on the right hand side should only depend on the previous order, what appears on the left hand side should have the subscripts relevant to the order at which you are working. So, now, our task at this order is to solve equation A2 which is just Laplace equation for phi 2 subject to equations B2 and C2; so, equations B2 and equations C2.

Our task at linear order was much simpler. The equations B1 and C1 had the same structure as B2 and C2, but B1 and C1 are homogeneous equations, the boundary conditions are homogeneous ok. So, these are homogeneous equations whereas, B2 and C2 have terms on the right hand side. So, we will first we will have to in order to solve these set of equations we will have to determine the solution to first A1, B1 and C1 that we have already done before and we are just going to repeated quickly.

Once we know phi 1 and eta 1 we are going to and this will also determine the dispersion relation at this order. We are going to go back to order epsilon square. Substituted into the right hand side of equation B2 and C2 because these right hand sides depend only on phi 1 and eta 1, which are now known. So, we will substitute we will determine what are those expressions explicitly and then we will have to solve equation A2, B2 and C2.

In order to determine the numerical value of omega 2, we will have to actually write down the equations up to order epsilon cube. We will not be able to we will not need to solve the equations are order of epsilon cube, we will only need to eliminate the secular terms at order epsilon cube and we will see that that process will generate. This is very similar to what we had done for the simple pendulum. We will continue this in the next video.