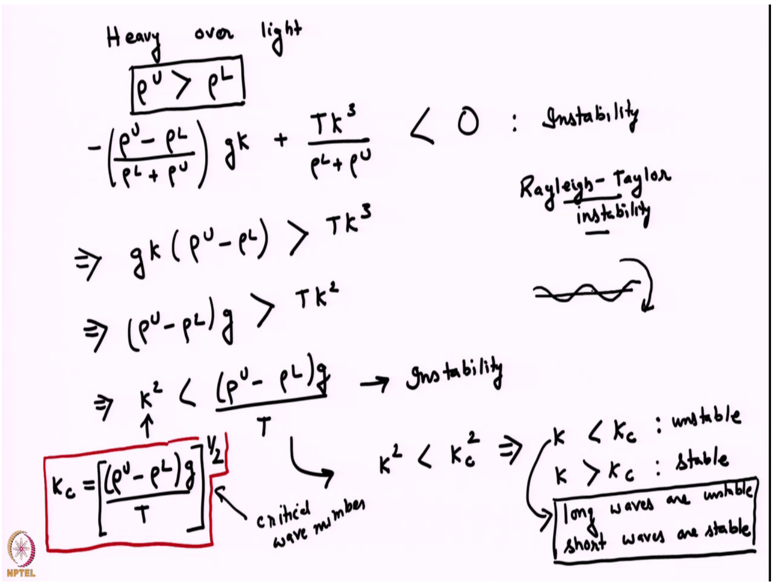
Introduction to interfacial waves
Prof. Ratul Dasgupta
Department of Chemical Engineering
Indian Institute of Technology, Bombay

Lecture - 57
KH dispersion relation: model of wind wave generation

We were looking at one of the limits of the dispersion relation, when there was no velocity in the upper as well as the lower fluid.

(Refer Slide Time: 00:28)



In particular we were looking at a heavy over light configuration where the density of the upper fluid was greater than the lower fluid and we had found expectedly, that there could be instability. More interesting was the fact that not all modes are unstable, it is only some modes which are unstable. We have also seen that long waves sufficiently long waves are unstable whereas, sufficiently short waves are stable.

We had found a critical wave number. The critical wave number such that K less than Kc was unstable and K greater than Kc was stable. So, now, let us try to understand why are long waves unstable and why are short waves stable. Instability is to be expected here because we have a statically unstable configuration. We have heavy over light.

So, sufficiently long waves follow the intuitive thing that we would expect that they are unstable. In more interesting at the sufficiently short waves which are stable by this argument. Let us try to understand why. Recall that I had told you earlier that waves with sufficiently small wavelength effectively behave as capillary waves.

Waves with sufficiently large wavelength effectively behave as gravity waves. It is only an intermediate regime where the effect of capillarity and gravity are both felt. You can see that for sufficiently short waves although the heavier fluid is above and the lighter fluid is below. So, it seems like a statically unstable configuration; however, the effect of surface tension is far more dominant than that of gravity.

There are two energies associated here. One is the gravitational potential energy and another is the surface energy. Whenever we have a flat interface, so, we have a flat interface like this and we are introducing a perturbation in the form of a Fourier mode, like that, whether the configuration will be stable, whether the base state will be stable or not depends on whether the net energy in the perturb state is more or less.

If the, if this perturbation increases the potential energy of the system then the system does not like to be in that configuration and it wants to return back to the base state where the potential energy is less. This is exactly what happens for sufficiently short waves. For these kinds of waves, it is the surface tension which is dominating the potential energy term compared to the gravitational potential energy.

So, for these waves the penalty in the form of excess surface area is way more than the increased the compensation in the through lowering the center of mass by going below. So, if it is a unstable configuration the heavier fluid wants to go below. This would effectively lower the center of mass. However, in go in doing this the amplitude has to grow and this would effectively put a penalty on the surface energy.

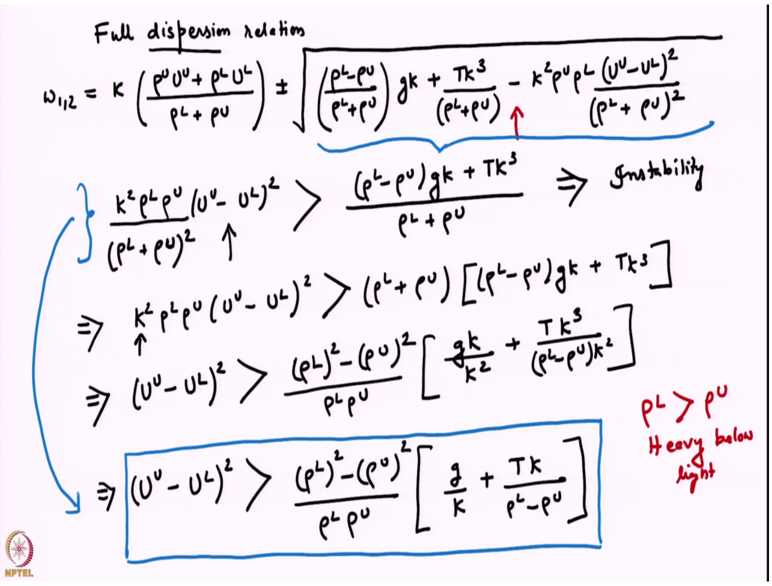
These waves within the confines of linear theory say that the theory says that for sufficiently short waves the penalty in the form of a surface tension term is much more than the potential energy to be minimized by lowering the center of mass. Consequently these waves are stable.

So, we have this very interesting situation where you can support a heavy fluid on top of a lighter fluid and you can have waves which are sufficiently small and these waves would basically oscillate. So, in this case these are just travelling modes. You can also look for standing waves. In this particular example where there is no base state velocity. So, these waves would just oscillate up and down in time.

The heavier fluid will not want to come down and the lighter fluid will not want to go up; however, one has to remember that this in general is not going to be a an observation for a long time because typically they will be other modes, long wavelength modes which will be bond in the system and those are unstable and they will typically tend to bring the heavier fluid down and the lighter fluid above. So, this is the Rayleigh Taylor Instability.

There are some mode in the heavy over light configuration. There are some modes which are stable which will oscillate and there are other modes which are unstable and which will bring the heavier fluid below and the lighter fluid above. In the reverse configuration which we have seen earlier where there is heavier fluid below and the lighter fluid above all modes are stable. So, now, let us now go to the full dispersion relation.

(Refer Slide Time: 05:06)



We have now until now looked at only limits of the dispersion relation. Let us now go to the full dispersion relation ok. Once again I would like to write down the dispersion relation just. We had already written this earlier this is just for our recall. We are basically interested in what is inside the square root.

So, we had seen earlier that what is inside the square root is this. So, once again we see that it is the this negative term which brings in the possibility of instability. Let us look at the case where we have a statically stable configuration. We have heavier fluid below light fluid, but now both our fluids are moving with some speed UU and UL.

This is typical of a air water scenario where we have a body of water over which there is air blowing. So, let us look at that configuration. So, what is of interest is the fact that if K square rho L L rho U. So, the last term inside the square root into UU minus UL whole square by rho L L plus rho U U whole square.

If this is greater than the first two terms, the sum of the first two terms, which is just rho L L minus rho U U then this implies. We are going to get instability. This is a different form of instability compared to the ones that we have seen until now. This instability requires a velocity UU minus UL whole square. Note that it the instability depends only on the difference of the two velocities.

So, it does not depend on the absolute value, it only depends on the difference and the square of the difference. So, it does not even matter whether the flow is going from left to right or right to left the sign does not matter. So, let us look at this instability. So, this implies, I can rewrite this as. So, this is k square rho L L rho U U into UU minus UL whole square. I am cancelling out a rho L L plus rho U U on both sides that is positive.

So, I will get a rho L L plus rho U U multiplying rho L L minus rho U U into gk plus Tk cube. If I shift all the other quantities except the difference of the two velocity squared on the right hand side then I will have UU minus UL whole square is greater than. So, I will have I am going to pull out a rho L L minus rho U. So, a plus b and if I pull out a rho L L minus rho U U from inside the square bracket then it will a plus b into a minus b.

That is a square minus b square and then in the denominator I will have k square I am going to push the k square inside. So, I will just haverho L L into rho U. So, I have taken out a rho L L minus rho U. So, there is a gk and there is a k square. Similarly, I have taken out a rho L L minus rho U.

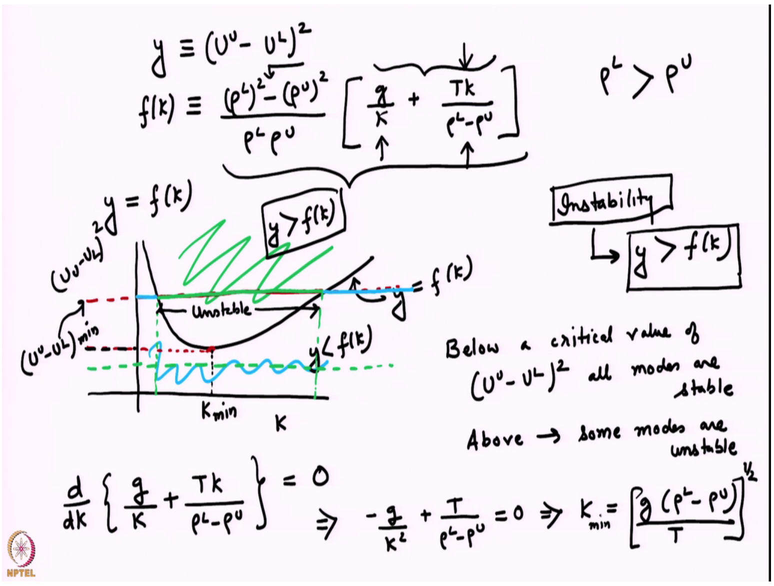
So, there is a rho L L minus rho U U in the denominator and there is also a k square. The k square is this k square which has been shifted to the right hand side and pushed inside the square bracket. Now, let me rewrite this like this UU minus UL whole square is greater than rho L minus rho U whole square by rho L rho U into g by k plus Tk divided by rho L minus rho U.

Note that we are operating under the assumption that rho L, we would like to do analysis assuming that rho L is greater than rho U. So, heavy below light, a statically stable configuration ok, and we are trying to see how the presence of the velocities in the base state can lead to instability.

You can see that this is what exactly it is predicting. It is predicting where if the difference between the square of the difference between the velocities. If it is greater than some threshold then we are going to get instabilities. Let us try to look at this prediction in a little bit more detail.

So, I will put this prediction in a box. This is basically coming from the discriminant of the root of the dispersion relation. So, I am just taking this quantity and I am asking when is this quantity negative. This is the analysis that we are doing here and this is leading me to this criteria. So, now, let us analyze this relation which is written in the box. I am going to rewrite that relation as.

(Refer Slide Time: 10:57)



So, I am going to call y as UU minus UL whole square ok and I am to going to call the right hand side as some function of k. So, f of k is defined as whatever we wrote on the right hand side. Note that because we are assuming rho L to be greater than rho U. So, these quantities are all positive.

The numerator here is positive and the denominator there is also positive. So, for positive k, f of k is going to be positive. Now what I am going to do is I am going to plot y as a function of k. So, essentially I am going to just plot this quantity. You can see what is the qualitative behavior, what is the qualitative nature for the coefficient of the square bracket is positive. So, the coefficient is not so important.

The qualitative behavior is that that for sufficiently small k this term will dominate and for, so, there is g here sorry, and for sufficiently large k that term will dominate. So, this is just another way of saying that for sufficiently small k the gravity term will dominate. Small k means large wave lengths.

For sufficiently large k, the surface tension term will dominate or small wave links will be dominated by surface tension. We have seen this before and the plot of y versus f of k is going to remain look something like this. We have seen such plots before. So this, so this axis is k and this is y is equal to f of k with a little bit of consideration you can convince yourself that this region and that region are represented by the inequalities y greater than f of k and this region is y less than f of k.

So, the curve is y is equal to f of k and it separates two regions, in one of which y is greater than f of k and in the other y is less than f of k. We have defined y to be UU minus UL whole square and so, you can see that there is a certain critical. So, if I choose for example, so, our instability criteria recall was just y greater than f of k let me write it properly. So, instability is y greater than f of k.

So, this is telling me that if I choose the value of UU minus UL to be let us say this value, UU minus UL whole square is just a constant ok, if I choose the value of the constant to be here then I lie in this blue region where y is less than f of k which essentially means all modes are stable.

If I however, choose it in this region, the value of UU minus UL whole square in this region, then some k's lie in the blue region and some k's lie the green region ok. So, I am going to label out the k's which are going to be. So, these k's lie in the green region. So, I am going to label them as green. So, any k between that limit lies in the green region. Any k, so, in these k's and all these k's going all the way to arbitrary large values lie in the blue region.

So, we can clearly see that if my velocity exceeds a certain critical velocity, then some of my wave numbers are going to be unstable. In particular for the for this particular choice, for this particular choice of UU minus UL whole square, I can see that these wave numbers are going to be unstable because they lie in the region which satisfies y greater than f of k. Remember that y greater than f of k is the criteria for instability. So, this is essentially telling us that just by having shear in the problem ok.

So, we have a base state velocity and there is a discontinuity in the velocity at the interface in the base state. We are able to get we are able to make a statically stable configuration unstable. So, we have heavy fluid under lighter fluid, air lying over water. If air starts blowing over water this is predicting an instability. This is predicting that some modes are going to grow, not all modes are going to grow, but some modes are going to grow that you can see.

You can also see what is the minimum velocity with which air needs to move. Remember that this analysis the velocities that is predicted from here depends only on the difference. So, I can for example, set the velocity of the water to 0 and keep velocity only in air or vice versa. So, if we want to examine, use this model for asking what is the minimum velocity at which wind need needs to blow in order to create waves on the surface of water, we can use this model as an example.

We will do this calculation and you can see that we will set the velocity of the lower fluid to 0. We will set velocity of water to 0. We will only have velocity in air. It does not care for which we set to 0 and which we do not. It only cares for the difference. The square of the difference and you can see that there is the minimum velocity is given by this.

If you are above this velocity then some modes are unstable. If you are below this critical velocity then all modes are stable. So, we have managed to find instability by putting in some base state velocity. These are examples of waves in shear flow. I had written that earlier when we started discussing this and this is what is known as the Kelvin Helmholtz Instability. I have mentioned this before. These are named after the two people who were among the first to study the problem.

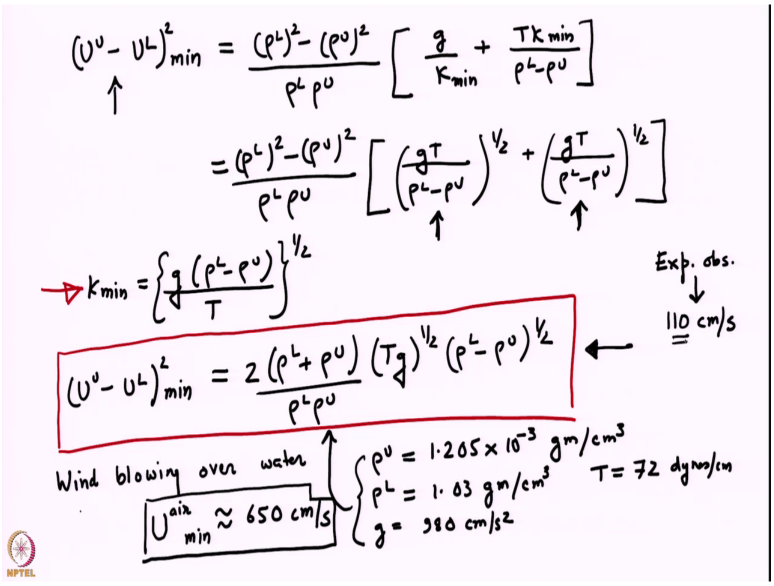
So, now so, below. So, let me write that. So, below a critical value of UU minus UL whole square, all modes are stable. Above some modes are unstable. Note that the presence of surface tension in this analysis is very important for this threshold. For example, if you set surface tension to 0, then you will not have this you will really you will lose the fact that this curve has a minimum and then you will find that all modes are unstable ok.

So, the threshold will go to 0. Let us analyze this in a little bit more detail. So, let us find this minimum value. So, this let us call this UU minus UL minimum ok. So, the minimum is basically coming from this part of the function f of k. The coefficient is irrelevant. So, the minimum is coming from there and. So, let us find its minimum ok. So, we will do d by dk of g by k plus Tk by rho L minus rho U is equal to 0.

And so, this will tell me that minus g by k squared plus T by rho L minus rho U is equal to 0 or in other words k is equal to g into rho L minus rho U divided by T. This whole thing to the power half. This is just giving us the coordinates of this k. I will call it k min ok. This is the first mode which is at the threshold of instability.

The moment the velocity starts the difference between the two velocities starts reaching its threshold value, this is the first mod which is just above to become unstable this is what this k implies. So, this is k min ok. So, the k which becomes unstable at the minimum possible velocity where it is just about the instability is just about to begin. At any velocity above this there is a range of k's all of which are unstable. Now, let us calculate the minimum velocity.

(Refer Slide Time: 20:59)



So, UU minus UL whole square and I am interested in the minimum. It is just the expression where I replace k with k min. So, it was the coefficient in this case is important. So, rho L square minus rho U square divided by rho L rho U and what was inside was g by k plus T k divided by rho L minus rho U.

So, all I have to do is evaluate this at k is equal to k min and that will give me the minimum value of UU minus UL whole square where we are just above the threshold of instability. So, this can be worked out as so, k min recall we had found was g into rho L minus rho U divided by T to the power half.

So, if I substitute this value of k min, then it just becomes gT divided by the rho L minus rho U to the power half plus gT to rho L minus rho U to the power half ok. So, using all this with a little bit more algebra you can show that the minimum value of UU minus UL square, where the instability just begins is just given by this expression, rho L plus rho U ok.

So, basically there is the rho L minus rho U here will cancel out the rho L minus rho U in the numerator ok. So, you will be left with just a rho L plus rho U divided by rho L into rho U into Tg to the power half rho L minus rho U to the power half. A rho L minus rho U to the power half will be left in the numerator because what is present here is just rho L minus rho U to the power half and in the numerator it is rho L minus rho U.

So, there will be a factor of rho L minus rho U to the power half. So, this is our expression for the minimum speed at which the instability happens, at which a single mode is just at the threshold of instability. This as a wind blowing over water. So, the water body is modeled as being infinitely deep.

The wind is blowing over it and we will because this depends only on UU minus UL whole square. So, we will just set the velocity only to the wind side and put 0 velocity on the water side. So, UL is 0 and we will just call it U wind square. So, let us do that calculation for wind blowing over water.

So, for saline water these are the air blowing over saline water. I am thinking of a typical condition over the ocean. So, for air in CGS units and for water the density is this. This is roughly saline water and surface tension is approximately 72. Gravity is the same and if you use these values and plug that in into this formula you will basically get a UU minus UL min which is approximately 650 centimeter per second.

You just need to take these values and plug them in here into this formula. This was one of the earliest models of wind wave generation ok using the Kelvin Helmholtz model. This is an example of shear instability and you can estimate what is the minimum wind speed. So, we will not use UU minus UL. We will just call it U air minimum this is the minimum speed at which air needs to blow.

In order to have at least one wavelength which is unstable and which will grow. So, this is the minimum wind speed. Unfortunately this is not a good model for modeling the formation of waves in the ocean. The minimum speed so, experimental observations both in the lab as well as in the field observations as well indicate that the minimum wind speed is much lower than this. It is about 110 centimeter per second.

There are more refined models which capture more physics which are necessary in order to understand where does this velocity scale come from. We will now in the next lecture, we will take another limit of this dispersion relation, of the dispersion relation that we have. We have now analyzed the full dispersion relation and we have looked at the Kelvin Helmholtz instability as a model for generation of waves on the ocean when the wind is blowing.

Now, we will take a limit of this dispersion relation where we will say that the two fluids are the same. So, we will say that it is just a single fluid with no interface. There is no surface tension and rho L is equal to rho U. We will see that this dispersion relation in that limit reduces to that of a governing the perturbations of a vortex sheet and we will find that all perturbations on a vortex sheet are unstable. We will also look at the physical meaning of that instability in the next video.