Introduction to interfacial waves
Prof. Ratul Dasgupta
Department of Chemical Engineering
Indian Institute of Technology, Bombay

Lecture - 56
Limits of KH dispersion relation: Rayleigh-Taylor instability

We were looking at equation 11 and we had linearized various terms in equation 11.

(Refer Slide Time: 00:22)

Linearise

$$T(\nabla\cdot\hat{n})_{z=\eta} \approx -T\frac{\partial^2\eta}{\partial x^2}$$

$$\phi^L = \phi^L_b + \hat{\phi}^L \qquad \hat{\phi}^L \text{ perturbation}$$

$$\phi^L_b = U^L x$$

$$|\nabla\phi^L|^2 = \left(\frac{\partial\phi^L_b}{\partial x} + \frac{\partial\hat{\phi}^L}{\partial x}\right)^2 + \left(\frac{\partial\hat{\phi}^L}{\partial z}\right)^2$$

$$= \left(U^L + \frac{\partial\hat{\phi}^L}{\partial x}\right)^2 + \text{''}$$

$$\approx (U^L)^2 + 2U^L\left(\frac{\partial\hat{\phi}^L}{\partial x}\right)$$

Substitute these in eq^n 11

$$|\nabla\phi^U|^2 \approx (U^U)^2 + 2U^U\left(\frac{\partial\hat{\phi}^U}{\partial x}\right)$$

$$-T\frac{\partial^2\eta}{\partial x^2} + \frac{1}{2}\rho^L\left[\cancel{(U^L)^2} + 2U^L\left(\frac{\partial\hat{\phi}^L}{\partial x}\right)\right] - \frac{1}{2}\rho^U\left[\cancel{(U^U)^2} + 2U^U\left(\frac{\partial\hat{\phi}^U}{\partial x}\right)\right]$$

$$+ \rho^L\left(\frac{\partial\hat{\phi}^L}{\partial t}\right) - \rho^U\left(\frac{\partial\hat{\phi}^U}{\partial t}\right) + (\rho^L - \rho^U)g\eta - \frac{1}{2}\left[\rho^L\cancel{(U^L)^2} - \rho^U\cancel{(U^U)^2}\right] = 0 \quad \text{at } z = 0$$

In particular we had found that the linearise approximation to the surface tension term this one and then there were contributions from the quadratic term in the Bernoulli equation at linear order also and that is because of the presence of a base state. And so, we have one contribution which is this and another contribution which is that.

Now, let us go and plug this back into equation 11 and see what form it takes. Recall that equation 11 is a boundary condition. So, if we substitute; so, substitute these in equation 11 by these I mean this this and this. So, when we substitute we find. So, the first term just becomes minus T del square eta by del x square.

(Refer Slide Time: 01:18)

$$\Rightarrow P_b^L = P_b^U \quad \text{at } \boxed{z=0}$$

$$\rho^L c^L - \frac{1}{2}\rho^L (U^L)^2 - \rho^L \cancelto{0}{gz} = \rho^U c^U - \frac{1}{2}\rho^U (U^U)^2 - \rho^U \cancelto{0}{gz}$$

$$\Rightarrow \rho^L c^L - \rho^U c^U = \frac{1}{2}\left[\rho^L (U^L)^2 - \rho^U (U^U)^2\right] \rightarrow (10)$$

We have $\boxed{p^L - p^U = T(\nabla\cdot\hat{n})}$ at $\boxed{z=\eta}$

(8)-(9)

$$(p^L - p^U) + \frac{1}{2}\rho^L|\nabla\phi^L|^2 - \frac{1}{2}\rho^U|\nabla\phi^U|^2 + \rho^L\frac{\partial\phi^L}{\partial t} - \rho^U\frac{\partial\phi^U}{\partial t}$$
$$+ (\rho^L - \rho^U)g\eta - \boxed{\rho^L c^L + \rho^U c^U} = 0 \quad \text{at } z=\eta$$

$$\Rightarrow T(\nabla\cdot\hat{n}) + \frac{1}{2}\rho^L|\nabla\phi^L|^2 - \frac{1}{2}\rho^U|\nabla\phi^U|^2 + \rho^L\frac{\partial\phi^L}{\partial t} - \rho^U\left(\frac{\partial\phi^U}{\partial t}\right)$$
$$+ (\rho^L - \rho^U)g\eta - \frac{1}{2}\left[\rho^L (U^L)^2 - \rho^U (U^U)^2\right] = 0 \quad \text{at } z=\eta$$

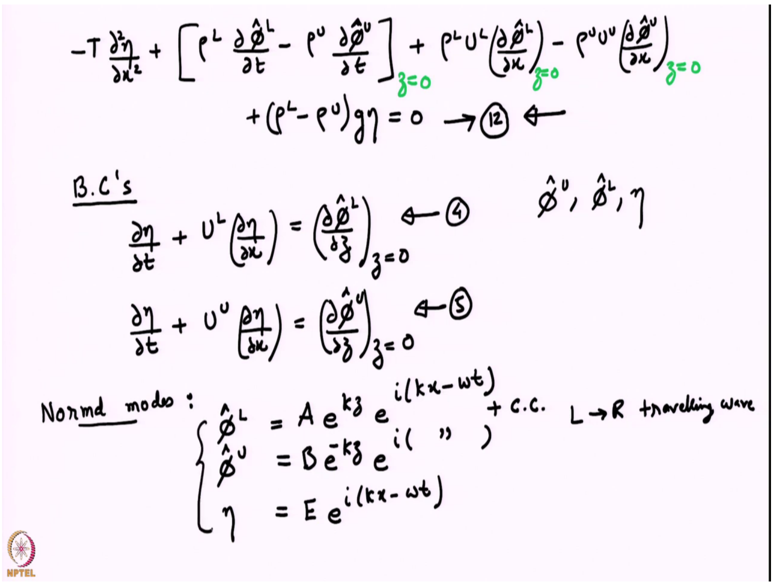
$\hookrightarrow$ (11)

The next term is half rho phi L square we have obtained an approximation to this a linear approximation. So, this just becomes U L square or half rho rho L and then for the upper fluid. And then we have our regular terms which is plus rho L del phi hat d t minus plus the gravity term minus and because this is our linearized boundary condition we have to apply this now at z is equal to 0.

This can be justified once again in the same manner that we have done until now any term that is the perturbation term which is applied at z is equal to 0 has to be expanded in a Taylor series and then you will find that if you go beyond the first term it will become an order epsilon square contribution.

So, all terms get applied at z is equal to 0 all terms which depend on z get applied at z is equal to 0. So, now, we can cancel out some of the things here as you can see. So, rho L U L square cancels out rho L U L square rho U U U square cancels out a rho U U U square. After this cancellation let us write the resultant equation.

(Refer Slide Time: 03:21)



So, we are left with minus T del square eta by del x square plus rho L by del t plus rho L U L and plus rho L minus rho U into g eta is equal to 0. And as I said earlier this has to be applied at z is equal to 0. The only terms which depend on z is this inside the square bracket and these terms.

The other terms depend on eta; eta by definition is not a function of z. So, we do not have to worry about the z dependence in those terms. So, these terms get applied at z is equal to 0 and you can see that this is one of the first boundary conditions that we have ok. So, this is the boundary condition ok. So, I will call this equation 12 this is the boundary condition.

And so now, we can we have now three boundary conditions ok. So, what are the boundary conditions? So, boundary conditions. We have already seen that del eta by del t plus U L del eta by del x is equal to del phi hat L by del z after linearization this will get applied at z is equal to 0.

Similarly, we have also seen a similar version of the kinematic boundary condition. Now, for the lower the upper fluid is this after linearization at z is equal to 0 and then we have equation 12. So, these are our three equations these are our three equations for the three perturbation quantities phi hat U, phi hat L and eta.

Once again we are going to do a normal mode analysis. In this case because the base flow is moving from left to right we are going to look for travelling wave kind of solutions. The domain is horizontally unbounded. So, I can take e to the power i k x minus omega t kind of solutions. Let us do that and let us work out the dispersion relation. So, we are going to now do a normal mode analysis.

Normal modes ok and our main equations will be equation 12 and the two kinematic boundary conditions. So, let us do that. So, we will say that phi hat L is equal to some complex constant A into e to the power k z exponential of i k x minus omega t, you can also try e e to the power i k x plus omega t this is the left to right travelling wave left to right travelling wave as we have seen before.

Similarly, phi hat U is equal to some complex constant B. So, of course, we have to add the complex conjugate I am not going to explicitly write this by now we are fairly familiar with this procedure and so, I am going to skip writing the plus c c every time e to the power. So, upper will be minus k z exponential of the same thing. And then eta which is some variable some complex constant into e to the power i k x minus omega t in all of them there has to be a complex conjugate added to it ok.

So, now, we have to go back and substitute this into the three boundary conditions. The procedure is quite straight forward each of the cases it will lead us to an algebraic homogeneous linear equation in the three unknowns A, B and E in the three complex unknowns A, B and E.

(Refer Slide Time: 07:54)

$$4 \Rightarrow -i\omega E + U^L(ik)E = kA$$

$$\Rightarrow i(kU^L - \omega)E - kA = 0 \longrightarrow A$$

$$5 \Rightarrow i(kU^U - \omega)E + kB = 0 \longrightarrow B$$

$$12 \Rightarrow \{Tk^2 + (\rho^L - \rho^U)g\}E + \rho^L i(kU^L - \omega)A - \rho^U i(kU^U - \omega)B = 0 \rightarrow C$$

$$\begin{vmatrix} i(kU^L - \omega) & -k & 0 \\ i(kU^U - \omega) & 0 & k \\ Tk^2 + (\rho^L - \rho^U)g & i\rho^L(kU^L - \omega) & -i\rho^U(kU^U - \omega) \end{vmatrix} = 0$$

$$\Rightarrow \boxed{(\rho^L + \rho^U)\omega^2 - 2k(\rho^U U^U + \rho^L U^L)\omega + k^2\{\rho^L (U^L)^2 + \rho^U (U^U)^2\} - \{(\rho^L - \rho^U)gk + Tk^3\} = 0}$$

So, equation 4 ok. So, I think I have called the first kinematic boundary condition as equation 4. So, this is I will call this equation 4. So, this is this was already written earlier. So, I am just rewriting and using the same numbers that was used earlier.

So, if I substitute these normal mode forms into equation 4, then we obtain an algebraic equation that is so, 4 implies minus omega E plus U L i k into E is equal to k times A. I can rewrite this as i times k U L minus omega into E minus k A is equal to 0. I will call this equation A the first algebraic equation that we obtain.

Similarly, if I substitute the normal mode forms in the second kinematic boundary condition I will get one more equation you can you can do it I will straight away write the equation that we obtain U U minus omega into E plus k B is equal to 0 this is B. And then equation 12 which is the third boundary condition at the top of this slide you can see that you can substitute the normal mode forms into this equation and once again get an algebraic equation in A, B and E.

So, I am just straight away writing the equation it is very easy you can try it yourself this is the coefficient of E. Note that we have done this procedure before except that earlier we did not have a velocity profile or a velocity in the base state. Now, we have a velocity in the base state. So, if you go back and replace U L is equal to U U is equal to 0 you should recover the expression that we have obtained earlier.

So, now, we have three equations three algebraic linear homogeneous equations in A, B and E. Once again the procedure remains the same we have to take a the determinant of the coefficients of A, B and E and set it equal to 0 which will give us the dispersion relation. Let us obtain the dispersion relation.

So, I am going to write down the determinant. So, the determinant is i k U L minus omega minus k and 0, i k U U minus omega 0 plus k and the third one is T k square plus rho L minus rho U into g and then it is i rho L k U L minus omega and then this is minus i rho U k U U minus omega. Let me shift this to here 0 ok this determinant is equal to 0.

Once again a three by three determinant you can easily work it out if you work it out with two or three lines of algebra you can recover the dispersion relation which I am going to write it here is equal to 0. This is my dispersion relation it is the quadratic in omega. So, this is my dispersion relation if I solve for omega from here I will get omega as a function of k.

Now, before we look at the roots of this dispersion relation let us first look at the this is the quadratic. So, let us look at the discriminant because that is what tells us whether there is any possibility of instability or not. Let us look at the discriminant.

(Refer Slide Time: 12:51)

Discriminant

$$B^2 - 4AC$$

$$= 4k^2(\rho^L U^L + \rho^U U^U)^2 + 4(\rho^L + \rho^U)\{(\rho^L - \rho^U)gk + Tk^3\}$$
$$- 4(\rho^L + \rho^U)k^2\{\rho^L (U^L)^2 + \rho^U (U^U)^2\}$$

$$= 4(\rho^L + \rho^U)\{(\rho^L - \rho^U)gk + Tk^3\} - 4k^2 \rho^U \rho^L (U^U - U^L)^2$$

$$\omega_{1,2} = \frac{2k(\rho^U U^U + \rho^L U^L) \pm \sqrt{B^2 - 4AC}}{2(\rho^L + \rho^U)\cancel{4}}$$

$$= k\left(\frac{\rho^U U^U + \rho^L U^L}{\rho^L + \rho^U}\right) \pm \sqrt{\left(\frac{\rho^L - \rho^U}{\rho^L + \rho^U}\right)gk + \frac{Tk^3}{\rho^L + \rho^U} - k^2 \rho^U \rho^L \frac{(U^U - U^L)^2}{(\rho^L + \rho^U)^2}}$$

Is of the form B square minus 4 A C. You can look at the form of this equation and you can see that B is given by this part with the minus sign. And A and C so, A is given by this and C is given by the entire term on the and the entire last term ok. So, this this entire term and that entire term with the minus sign.

So, let us do B square minus 4 A C, if we do that then this is rho L U L plus rho U U U whole square plus 4 rho L plus rho U rho L minus rho U into g k plus T k cube minus 4 rho L plus rho U k square into rho L U L square plus rho U square that is the expression for B square minus 4 A C.

We can simplify this a little bit you can open up the brackets and cancel out some of the terms if you do that then your final expression will reduce to this, your final expression will reduce to 4 times rho L plus rho U rho L minus rho U into g k plus T k cube minus 4 k square rho U rho L U U minus U L whole square.

The important point to note is that that there is a minus sign here that there is a minus sign here in the expression for B square minus 4 A C. So, consequently we will see we will soon see that even if we choose remember that we have two fluids now the upper fluid and the lower fluid. A statically stable configuration is where the heavier fluid is below and the lighter fluid is above.

We will see that even in a statically stable configuration just because of the presence of a base state velocity we can have instability or in other words we can have waves whose amplitudes grow as they propagate. This is the consequence of the negative sign here. Now, we are going to analyze this dispersion relation in quite a bit of detail, we are going to look at various limits of this dispersion relation ok.

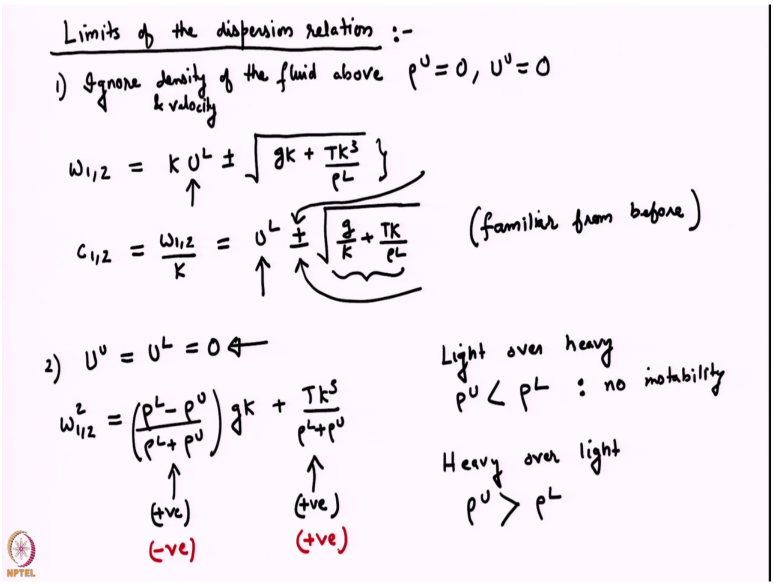
So, now, let us write down what are the roots of the dispersion relation recall that the our dispersion relation was a quadratic in omega. So, I can use the formula for a quadratic to write down the roots of the dispersion relation let us do that. So, omega of 1 comma 2. So, what I am doing is I am just writing down the root of the quadratic equation which is written in this rectangular box this is my dispersion relation.

So, omega of 1 comma 2 is minus B plus minus square root B square minus 4 A C which I have already written above divided by twice rho L plus rho U. And if you simplify this this basically becomes k times rho U U U plus rho L U L divided by rho L plus rho U plus minus if I substitute the formula for B square minus 4 A C inside and do some simplifications then I obtain rho L minus rho U divided by rho L plus rho U.

So, what I have done is I have just divided the numerator by the denominator term by term and I have pushed this two times rho L plus rho U inside the square root. So, it has gone inside as 4 times rho L plus rho U whole square and then I simplify the square root part.

Once again as you had mentioned earlier that there is a negative sign inside the square root ok and so there is a possibility of instability. Now, this is the dispersion relation this is the explicit form of the dispersion relation where I am writing omega as a function of k; there are going to be two roots two propagating waves with respect to the flow one propagating upwards and upstream and one propagating downstream.

(Refer Slide Time: 18:37)



Let us look at various limits of this dispersion relation. So, limits of the dispersion relation. So, what are the limits? So, the first limit it is a very simple limit we could ignore density of the fluid above this is what we have done in all the in in some of the earliest examples of waves that we have studied in this course those were surface waves ok.

So, now ignore density of the fluid above. So, we set rho U is equal to 0 and we also set ignore density and velocity. So, we are saying the upper fluid is not there or in other words its density is too small and it is not moving. So, rho U is 0 and rho U U is also 0 is a typical air water situation you can think of where water is much more denser than air and let us say it is only water which is moving, it is not air which is moving.

What happens to the dispersion relation that we just wrote in the last slide if you make these assumptions? So, in that limit what do we obtain we just obtain that omega of 1 comma 2 just becomes k times U L plus minus root over g k plus T k cube by rho L. It is easier to interpret things if we just write it in terms of a phase velocity. So, I have to just divide c omega 1 comma 2 divided by k.

So, C 1 comma 2 the phase velocity is omega 1 comma 2 divided by k and this is U L plus minus square root g by k plus T k by rho L, I have pushed the k inside it goes in as k square.

So, now you can see that this is nothing we have already encountered this except that this part was not there. This part is coming because of the velocity in the base state here we are considering only the lower fluid to be moving. So, the there are two components to any perturbation that there is an this component and there is another component which is like this.

You can think a little bit about this and you can see that if you go to the frame of reference in which the if you are traveling along with the base state along with the base flow that is with respect to the lab you are moving with the speed U L, then in that frame of reference you will see exactly the same dispersion relation that we had obtained earlier when the base flow was not there ok.

So, this is just a modified dispersion relation with this modification with this extra term k times U L this is basically a Doppler shift. Now, so this is this is something familiar to us from before familiar from before. We have seen this dispersion relation, we have seen this phase speed earlier, we have also seen this frequency earlier for capillary gravity waves.

You can see that these waves are completely stable there is no instability here the what is inside the square root for positive k is always positive ok. So, only travelling waves and it is a dispersive system every wave travels with its own speed we have analyzed this kind of systems in the absence of a base state before.

Now let us go to the next limit which is we say that U U is equal to U L is equal to 0. So, ignore all the velocities. Now, I am going to account for the density of the fluid above the density of the fluid below, but I am going to say that let us say that we are both of them are not moving it is a static configuration ok.

So, what happens to the dispersion relation again I am going to use these values in simplifying the dispersion relation the roots of the dispersion relation that I wrote in the previous slide if you do that then we will obtain omega 1 comma 2 let us write square is equal to rho L minus rho U divided by rho L plus rho U into g k plus T k cube by rho L plus rho U.

This basically generalizes what we have seen earlier. Now, there is one term like this and one term like that. In particular we know that if we have light over heavy. So, the lighter fluid overlies the heavier fluid it is a statically stable configuration heavier things go below lighter things go above or in other words rho L is or in other words rho U what is above is less than what is below the density.

If so, then you can readily see that rho because rho L is greater than rho U this term is positive this term is anyway positive and so, there is no instability under this case no instability this is intuitively expected. Now, we can go to the other limit wherein rho heavy over light this is the other limit here rho U the upper fluid is heavier compared to the lower fluid.

You can immediately see that in this approximation or in this case this is going to become negative and this will stay positive. So, there is a possibility that omega square can become negative. Let us look at that possibility.

(Refer Slide Time: 24:52)

Heavy over light

$$\boxed{\rho^U > \rho^L}$$

$$-\left(\frac{\rho^U-\rho^L}{\rho^L+\rho^U}\right) gk + \frac{Tk^3}{\rho^L+\rho^U} < 0 \quad : \text{Instability}$$

Rayleigh-Taylor instability

$$\Rightarrow gk(\rho^U-\rho^L) > Tk^3$$

$$\Rightarrow (\rho^U-\rho^L)g > Tk^2$$

$$\Rightarrow k^2 < \frac{(\rho^U-\rho^L)g}{T} \rightarrow \text{Instability}$$

$$k_c = \left[\frac{(\rho^U-\rho^L)g}{T}\right]^{1/2}$$

$$k^2 < k_c^2 \Rightarrow \begin{cases} k < k_c : \text{unstable} \\ k > k_c : \text{stable} \end{cases}$$

$$\boxed{\begin{array}{l}\text{long waves are unstable} \\ \text{short waves are stable}\end{array}}$$

So, we are looking at heavy over light; heavier fluid over lighter fluid or rho upper is greater than rho lower. In that case we it is clear from the dispersion relation that the first term is negative. So, let us reverse the sign of the first term and let us write it as minus rho U minus rho L earlier it was rho L minus rho U, I am just taking a minus common and writing it as into g k and then the second term is just T k cube by rho L plus rho U.

What do I gain by writing the first term like this it is clear that it is always negative because rho U what is inside the bracket is always positive and so the minus sign tells us that this is the negative sign this is the negative term. And so, this represents my frequency the square of my frequency.

And so, if this whole term the sum of this first term plus the second term. If this becomes less than 0 then I expect instability. This is also intuitively to be expected we know that if you place heavy things over lighter things, the heavier fluid will go down and the lighter fluid will rise to the top.

However, there is some interesting exceptions here let us look at that. So, this is instability because remember that this plus this less than 0 implies omega square is less than 0 omega square is negative it may implies that omega is purely imaginary. So, let us work out what is the criteria for instability.

So, we have g k into so, rho L plus rho U is there in the denominator it is a positive quantity I can cancel it out. So, g k into rho U minus rho L is greater than T k cube, I have cancelled out the denominator and so we have rho U minus rho L into g is greater than T k square k is again the positive quantity it is a wave number.

And so, this is telling me that for k square less than rho U minus rho L into g divided by T we get instability. So, only certain waves are unstable this is very interesting because we have a heavy over light configuration and this is telling us that some perturbations are actually stable while others are unstable. So, by this criteria we can define a critical wave number which is just related to the square root of the right hand side.

So, the critical wave number let us write it as rho U minus rho L into g by T. Notice that the critical wave number is positive because we are operating under this approximation rho U greater than rho L. So, rho U minus rho L is greater than 0. So, k c is a positive quantity and k c will have a square root sign here that is coming because there is a k square here. So, I have just taken a square root of the right hand side.

So, this can be rewritten as k square is less than k c square or in other words if k is less than k c some critical wave number which depends on the parameters of the system the surface tension, the two densities and the value of acceleration due to gravity.

So, all k's which are less than k c are unstable, all k's which are greater than k c are stable. So, this implies long waves are unstable, short waves are stable. This is coming from this analysis this instability is also known as the Rayleigh Taylor instability.

We will discuss this in slightly more detail and we will try to understand why short waves are stable, despite the fact that we have a heavy fluid overlying a lighter fluid. We will continue in the next lecture.