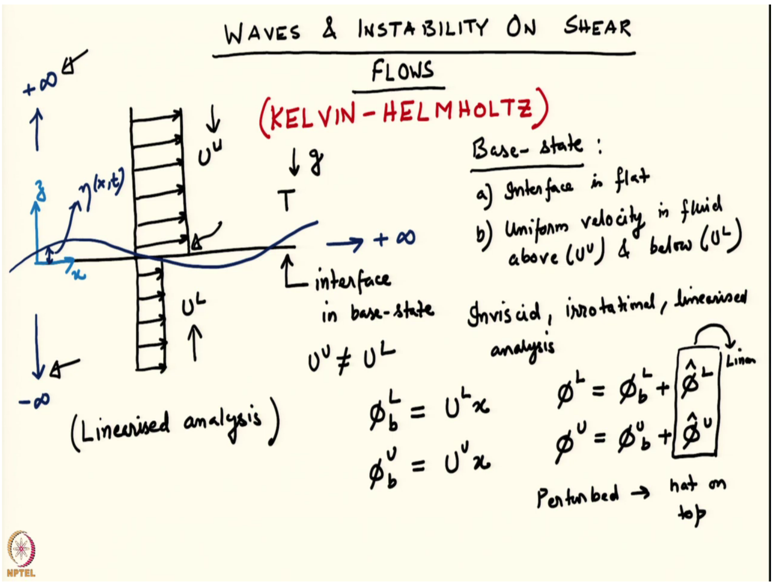
Introduction to Interfacial Waves
Prof. Ratul Dasgupta
Department of Chemical Engineering
Indian Institute of Technology, Bombay

Lecture - 55
Waves and instability on density stratified shear flows - the KH model

Up to now in this course we have looked at base states where there was no flow. Earlier in the course we have looked at base states which were time independent. So, there was the pressure dependence was either just purely hydrostatic or the pressure was just uniform as in the case of drops and bubbles and there was a pressure jump across the interface. Then in the Faraday wave case we looked at example where the base state was time dependent, but only for pressure the velocity was still 0 the interface was flat.

(Refer Slide Time: 00:59)



Now, we will go to the next problem which is waves and instability on shear flows and we will work out a model problem where the in the base state there is a velocity profile. What we have here on the left hand side as you can see is an interface. We are assuming we are going to do an inviscid analysis just like before an inviscid irrotational analysis and we will assume that there are two immiscible fluids in the base state. In the base state the interface which separates them is flat. So, this is the interface. So, this is the interface in the base state the flat line interface in base state.

However, the liquid above and the liquid below are not questioned. The liquid above flows with a constant velocity U U, so, U U. The U at the top indicates upper and the liquid below flows with the velocity U L. The capital L at the top indicates lower U upper and U lower. There is no gradient of velocity inside each of the liquids. The only gradient is at the interface and you can see that there is a if you take the derivative of this there is a delta function if you take the derivative of the velocity profile at the interface.

Now, this problem is a model problem and as you will see it contains as a special case a lot of the things that we have discussed until now. We will take into account gravity we will take into account surface tension. Although I have drawn U U to be greater than U L that is not necessary and so, in the final dispersion relation we can put either U U and U L to be 0 we can put U L to be more than U U or vice versa and so on and so forth.

We will assume for simplicity that the domain is vertically unbounded both for the upper fluid and the lower fluid. So, this goes to plus infinity and this goes to minus infinity and it is also horizontally unbounded as we have assumed until now. Gravity acts perpendicular and there is a surface tension T. And we will perturb the interface and we will ask the question what is the dispersion relation, which governs the interface for this flow.

The essential difference compared to what we have done earlier is the presence of this velocity profile for the upper the uniform velocity profile for the liquid above and the liquid below and the presence of a gradient between them at the interface. So, in my base state the interface is flat and uniform velocity up in fluid up and down, uniform but different velocities.

So, U U in general is different from U L which is greater which is smaller will not be used in the analysis. And so, the dispersion relation will be true for arbitrary U U and arbitrary U L. Uniform velocity in fluid above U U and below U L and the pressure corresponding to a uniform flow. So, this is our base state.

Now, as usual we will do an irrotational inviscid, irrotational analysis, linearized analysis and try to obtain the dispersion relation. In particular you will see that in this case because of the presence of a shear instabilities become possible which were absent earlier. Earlier we had seen that if you have we had done the problem for just one liquid and we had seen that we get surface gravity waves or capillary waves or capillary gravity waves.

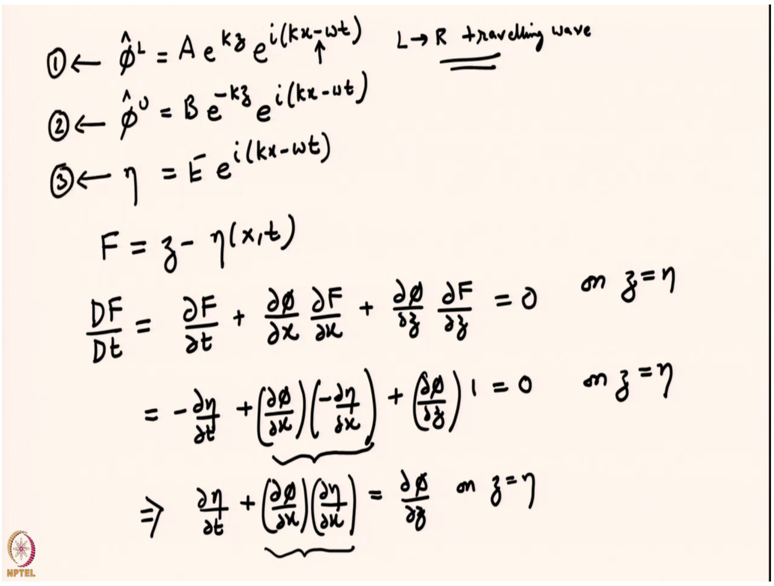
The most general case was the capillary gravity wave and depending on the length scales it could be either predominantly a gravity wave or a capillary wave. Now, you will see that because of the presence of this shear the velocity gradient it is possible for us to get instabilities on this system. So, let us proceed. So, our base state so, it is inviscid irrotational. So, in the base state I will indicate the base state with the subscript b.

So, in the lower fluid this is U L into x. If you take the derivative of this with respect to x you will get U L. The derivative of the velocity potential with respect to x gives us the x component of velocity and there is no y component. So, the derivative with respect to y is 0, similarly this is U U of x.

We will in general write our perturbations or our total velocity potential as a sum of base plus perturbation, perturbation quantities are indicated with a hat. Similarly, for the upper fluid we will write base plus perturbation. So, perturbation perturbed quantities have a hat on top. And when we say we are doing a linearize analysis we are going to retain only things which are linear in the perturbation variables.

Let us proceed. So, because this is a system where in the lab frame of reference my flow is moving from left to right. The way I have drawn it the flow both the upper fluid and the lower fluid is moving from left to right. So, we are not going to look for standing wave solutions, we are going to look for travelling wave solutions. In particular let me choose a left to right moving travelling wave.

(Refer Slide Time: 06:47)



So, for the perturbation I choose, so, the perturbation has A hat. So, I choose A e to the power k z. This we already know is coming from variable separation of the Laplace equation and then a left to right travelling wave. So, we will choose for our perturbation velocity potential a left to right travelling wave.

So, left to right left to right travelling wave indicated by the minus sign here. Similarly, for the perturbation velocity potential in the upper fluid we do the same thing: B e to the power minus k z e to the power i k x minus omega t. These exponentials we have already encountered before. They come from variable separation of the Laplace equation. We do not have to explain them once again.

You have a combination of A to the A e to the power k z plus B e to the power minus k z and one of them will diverge depending on which fluid we are in the upper or the lower and we have to set the corresponding coefficient to 0. A in general, A and B in general are complex constants and then we also have to put a perturbation on the interface and that is E exponential of i k x minus omega t. So, we have chosen a left to right travelling wave.

I encourage you to try this for a right to left travelling wave in which case you will do e i to the power k x plus omega t. You will find that the dispersion relation actually remains the same. The all the conclusions that we will derive from the dispersion relation will not depend on whether you impose the left to right travelling wave or a right to left travelling wave.

So, let us now proceed. So, our kinematic boundary condition as is usual is obtained by taking the total derivative of a quantity whose value is constant on the interface on the perturbed interface as well. So, this is Df by Dt. We have done this before and so, this leads to; so, del F by del t plus del phi by del x into del F by del x plus del phi by del z del F by del z is equal to 0 and this is true on z is equal to eta on F is equal to 0.

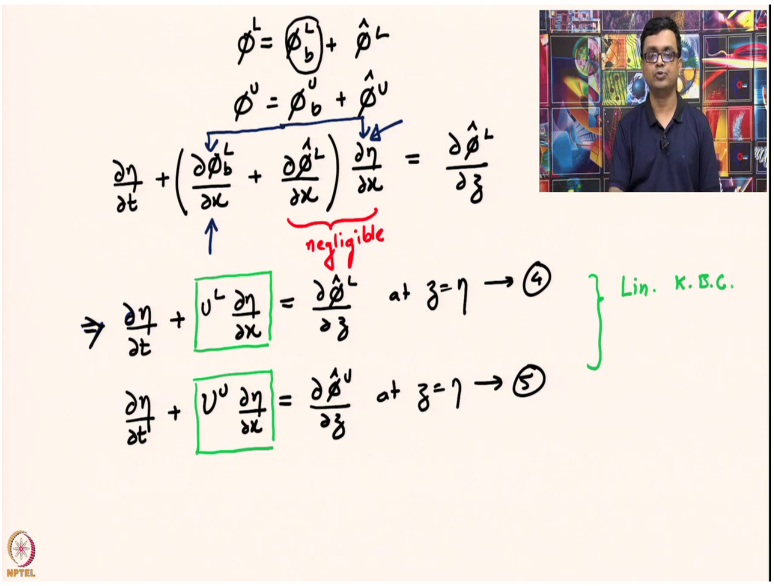
So, this just gives me minus del eta by del t plus del phi by del x into minus del eta by del x plus del phi by del z into 1 is equal to 0 on z is equal to eta. We have to be careful here with this term. Earlier when our base state did not contain any velocity, this term was set equal to 0. Why?.

Because this phi there represented the perturbation velocity potential, it was an order epsilon quantity and this term has a product of two order epsilon quantity. So, it was an actually an order epsilon square quantity. So, we ignored this and we recovered the kinematic boundary condition earlier from just this and that. Now, we have a order one term in the expansion. Why?.

Because our base state has a velocity profile or rather has a velocity. So, in these quantities del phi by del x, we will have to do an expansion and make sure that we retain only up to order epsilon and not beyond that. You will see that this quantity actually contributes makes a contribution at order epsilon.

So, let us proceed. So, this we are not going to ignore and so, this implies del eta by del t plus del phi by del x del eta by del x is equal to del phi by del z on z is equal to eta. Same as before except that now I have an additional term and I will have a order epsilon contribution from this term as well although it appears to be a quadratic term.

(Refer Slide Time: 11:01)



We are going to expand. So, we are going to write phi as before. So, we have written already. So, phi L is equal to I am just repeating this b plus phi hat L and then we have phi U is equal to phi U b plus phi hat U. If we substitute this in the boundary condition that we obtain, this boundary condition can be used for two values of phi at z is equal to eta.

One can come from above in which case it will be phi corresponding to the upper fluid or we can go from below in which case it will be phi corresponding to the lower fluid. So, there will be two sets of kinematic boundary conditions obtained from this equation. So, we will obtain del eta by del t plus del phi b L by del x plus del phi hat L by del x. So, the sum of base plus perturbation into del eta by del x is equal to del phi L by del z.

In the right hand side I am not writing the contribution from the base state because the base state is not a function of z. This is only within at the interface coming from below. So, we are in the lower fluid and so, del phi this quantity is not a function of z, it is just a function of x.

Now, you can see from this equation that this contribution is negligible in a linear theory that is because it is a product of two order epsilon quantities phi hat and eta. However, this quantity is not negligible, the product of this and that. This is a contribution coming from the base state, this is the contribution coming from the perturbed state.

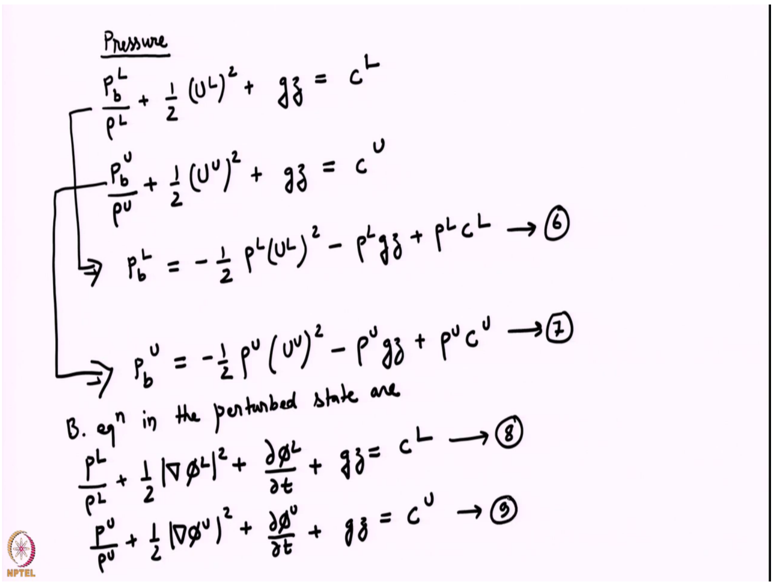
So, this overall is an order epsilon quantity. This is the first one is an order one quantity del eta by del x is an order epsilon quantity. So, we cannot throw away this term. We have to include it in a linear calculation. So, we obtain del eta by del t plus we will just keep the product of the first two terms and del phi L b by del x we know is just U L. So, this is U L into del eta by del x is equal to del phi hat L by del z at z is equal to eta.

So, I am going to label my equations. So, I am going to call this and this as 1, 2, and 3 and then I will call this as equation 4. This equation was obtained by approaching the interface from below. We can approach the interface from above in which case I will just have a similar equation with U L and phi hat L being replaced by the corresponding quantities for the fluid above.

So, I will have another equation which is del eta by del t plus U U del eta by del x is equal to del phi hat U by del z at z is equal to eta, this is equation 5. So, these are my equations these are my kinematic boundary conditions. You can see that now because of the presence of a velocity in the base state we have picked up two additional contributions in both the equations.

Earlier we had earlier we did not have this term. So, I am just going to point out the additional term that we are getting. So, this is an additional contribution and this is an additional contribution. Both of them were 0 earlier. If you set U L is equal to U U equal to 0 you will recover the kinematic boundary condition that we have obtained earlier, the linearized kinematic boundary condition. So, this is the linearized K.B.C. with a U U and a U L. Let us proceed further.

(Refer Slide Time: 15:19)



Let us look at pressure in the base state. So, in the base state the two fluids satisfy the Bernoulli equation. So, I am writing down the Bernoulli equation in the base state. Let us call the Bernoulli constant for the lower fluid as C l. So, this is the Bernoulli constant. Similarly, we can write the Bernoulli equation for the upper fluid also in the base state and here the Bernoulli constant is let say C U.

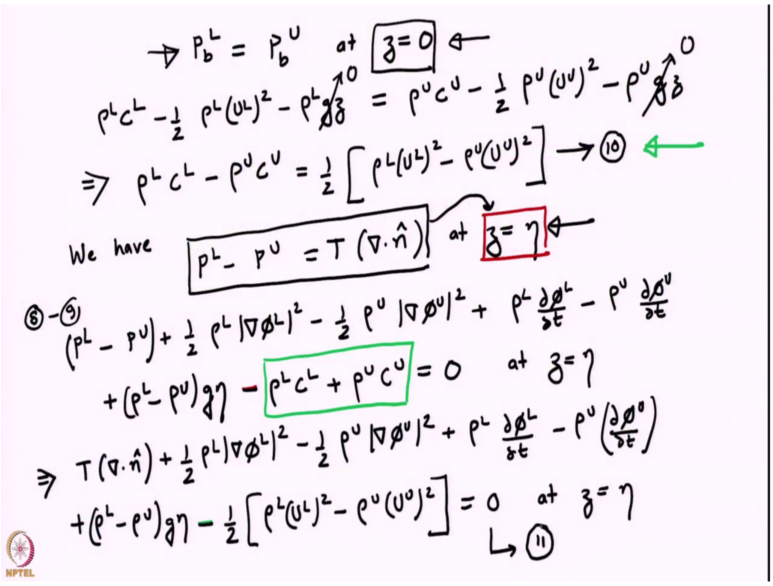
Note that in the base state the flow is steady. So, there are no derivatives with respect to time. Also note that this Bernoulli equation is true at any value of z. This is not just at the interface, but at any value of z in both the lower as well as the upper fluid. So, now, we can use these equations to obtain an expression for pressure in the base state.

Let me call this equation 6 and similarly I can use this equation to obtain another equation for pressure in the base state for the upper fluid, equation 7. Now, the Bernoulli equation is also true in the perturbed state. So, let us now write down the Bernoulli equation in the perturbed state.

Note that the main difference will be that there will be a time derivative because in the perturbed state we are putting a perturbation which in general will be a function of time. So, the Bernoulli equation for the two fluids in the perturbed state are; so, I will not put a subscript b now because this is my perturbed variable or this is my total pressure which is a sum of base plus perturbation. This is my time dependent term, similarly in the lower fluid.

Now, we have to use these equations along with boundary conditions. Note that in the base state the interface is flat. Consequently the pressure is going to be continuous. We are going to include surface tension in our analysis. However, we will find that because the interface is flat in the base state the there is no pressure jump in the base state or in other words the pressure is continuous at the interface in the flat state.

(Refer Slide Time: 18:50)



So, that basically leads us to; so, we have P b of L is equal to P b of U at z is equal to 0. Once again I would like to repeat that this does not imply that there is no surface tension. Recall the surface tension requires curvature of the interface in order to produce the pressure jump because in the base state there is no curvature of the interface. The interface is flat. So, despite having curvature despite having surface tension we are going to have a continuous pressure in the base state.

So, now let us use this condition that the pressure in the base state at z is equal to 0 is continuous. We have already written down the Bernoulli equation in the base state. Let us use this and get an expression using those two Bernoulli equations. So, using equation 6 and 7, so, equation 6 I had written earlier and 7. So, now, I am just going to set z is equal to 0 in both the equations and then equate them. This ensures that this condition gets satisfied.

So, let us do that. So, this leads to rho L C L. So, I am multiplying the Bernoulli equation throughout by the respective densities. So, rho L C L minus half rho L U L square minus rho L g z and I have to apply this at z is equal to 0. So, this term will go to 0 is equal to rho U C U minus rho U g z and once again this is also 0 because this condition is true only at z is equal to 0.

This tells us that rho L C L minus half rho L U L square or let me write the difference of these two. So, rho L C L minus rho U C U is equal to half rho L U L square minus rho U U U square. Let me call this. So, I will call this equation 8 and equation 9, the Bernoulli equation in the perturbed state. So, I am going to call this equation 10.

Now, let us proceed further. In the perturbed state we know that now there is a pressure jump at the perturbed interface because of surface tension. Note that now because of the perturbation the interface will be developing some curvature. So, we have like before that P L minus P U is equal to surface tension times the divergence of the unit normal to the perturbed interface that is at z is equal to eta.

So, now it is the usual procedure. So, what we do is let us go back to equation 8 and 9 that we wrote in the last slide and we take the difference of these two equations. We apply these equations at z is equal to eta and then we take the difference. If we do that then we obtain the following equation.

We obtain P L minus P U plus half; this term arises because we are applying the equation at z is equal to eta. Remember that this boundary condition holds good at z is equal to eta, this is at the perturbed state and z is equal to eta. So, we have to apply equation 8 and 9 at the perturbed interface eta and then take the difference of those two equations.

So, we have this term and then we will have one more minus rho L C L plus rho U C U is equal to 0. So, basically we are doing the difference of we are doing 8 minus 9 and this whole equation is true only at z is equal to eta. We have applied equation 8 and 9 at z is equal to eta and then taken the difference of those two equations. Why are we doing this?.

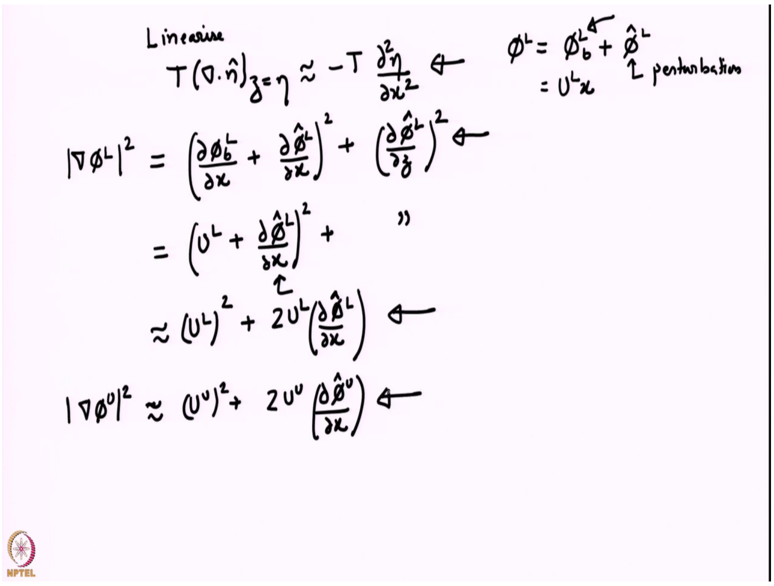
Because we want to apply this boundary condition, we want to use this boundary condition and this boundary condition is true only at z is equal to eta. Now, we have an equation where P L minus P U this combination is at z is equal to eta. So, I can use the boundary condition at the top and replace this term with T times divergence of n and this is of course, applied at z is equal to eta and so, the rest of the terms remains the same.

So, let me reorganize this square. So, those terms remain intact and I am going to use equation 10 in replacing this combination. I am going to use equation 10 in replacing this combination. You can see that equation 10 tells me the difference between the difference of rho L C L minus rho U C U and this is what I have here. So, if I take a minus common out then I get minus rho L C L minus rho U C U. So, I am going to replace that difference

So, if I take a minus out then I am left with will be a factor of half and then it is just whatever is there on the left hand side. So, I have used now equation 10 at z and this is true at z is equal to eta. I will call this my equation 11. So, now, we have to work on this equation.

In particular we will have to write the variables as a sum of base plus perturbation. We have to do this with the variable phi and you will see that the base state contributions will get cancelled out then we will have to do some linearization in the perturbation variables. Let us do that. So, we will now let us first linearize.

(Refer Slide Time: 26:09)



So, the first term; that means, linearization is T times divergence of n and recall that this has to be evaluated at z is equal to eta. We have done this before and if you go and look back into the previous lectures you will see that we have done this linearization before. This is just a Cartesian geometry and so, this linearization is easy and in your linear approximation this is just equal to minus T times del square eta by del x square.

Then we also have to linearize the quadratic term in the equations which are coming from the Bernoulli equation. Note that these quadratic terms earlier were always 0 because we did not have any base state velocity. Now, these quadratic terms will not be 0 there will actually be a contribution from this quadratic term because now we have a velocity in the base state.

So, let see what are the contributions. So, a typical term like grad phi L square, recall that we are writing phi L as phi L b plus sum phi hat L. This is base and this is perturbation and we have seen earlier that this is U L into x, the velocity the uniform velocity in the lower fluid into the distance x. Similarly, there is a phi L in the upper phi b in the upper fluid and which has a corresponding expression which is U U into x and we will express phi U as a function as the sum of base plus perturbation.

Now, let us express this sum. So, we will have note that this quantity is just a function of x, it is not a function of z. So, I am going to write this as del phi b L by del x plus del phi L hat by del x whole square plus del phi L by del z whole square. Note that note that there is no derivative with respect to z in the second term as far as the base state is concerned that is because my base state is independent of z.

This is I am doing it for the lower fluid the velocity profile is uniform both in the upper fluid and the lower fluid. The only discontinuity exists at z is equal to 0, everywhere else it is there is the velocity is continuous. So, the velocity potential is not a function of z. So, this I can write it as this is U L plus del phi hat L by del x whole square plus the same thing.

Now, you can see that all these quantities with hats they are perturbation quantities they are order epsilon quantities. So, in a linear calculation I am only going to have two terms here. One will come from the base state. So, it will be U L square and another will be the product of the base state into perturbation.

I am neglecting two terms here, one is the square of this and one is the square of that. You can see that they are both squares of perturbation quantities order epsilon square. So, we will not consider them in a linear calculation. Similarly, you can also argue that grad phi U square would be in a linear theory would be U U square plus twice U U del phi hat by del x.

So, we have these two quantities. Now, our task is to plug in this linearization this linearization and this linearization and go back to equation 11 and plug these approximations back. And then remember that this as a part of linearization we also have to Taylor expand all the quantities and when we Taylor expand we will find that instead of getting applied at z is equal to eta they will all get applied at z is equal to 0. We will continue this in the next lecture.