Introduction to Interfacial Waves
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Lecture - 53
Mathieu equation for Faraday waves

We were looking at interfacial waves on time dependent base states. Here the container bottom was being oscillated up and down with an amplitude a and a frequency capital omega, we had analyze this problem in the oscillating frame of reference where we had found that the effective value of gravity would be g plus a omega square into cos omega t.

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$$\left(\frac{\partial \phi}{\partial z}\right)_{z=-H} = 0 \rightarrow (3)$$

B.E.:
$$\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \left[g + a\Omega^2 \cos(\Omega t)\right] z = 0$$

$$\left(\frac{\partial \phi}{\partial t}\right)_{z=0} + \left[g + \underbrace{a\Omega^2 \cos(\Omega t)}_{\text{time dependent coefficient}}\right] \eta = 0 \rightarrow (4)$$

$$\phi = \Phi(t)\left[C_1' \cos(kx) + C_2' \sin(kx)\right]\left[D_1 e^{kz} + D_2 e^{-kz}\right]$$

Using (3), $k\left(D_1 e^{-kH} - D_2 e^{+kH}\right) = 0$

$$\Rightarrow D_2 = D_1 e^{-2kH}$$

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$$\frac{dE}{dt} G - \Phi(t)\, k \left(1 - e^{-2kH}\right) C_1 = 0 \rightarrow (A)$$

$$\frac{dE}{dt} H - \Phi(t)\, k \left(1 - e^{-2kH}\right) C_2 = 0 \rightarrow (B)$$

From the B. eq^n (4)

$$\left(\frac{\partial \phi}{\partial t}\right)_{z=0} + \left[g + a\Omega^2 \cos(\Omega t)\right]\eta = 0$$

$$\Rightarrow \left[\frac{d\Phi}{dt}\left(1 + e^{-2kH}\right) C_1 + \left\{g + a\Omega^2 \cos(\Omega t)\right\} E(t)\, G\right] \cos(kx)$$
$$+ \left[\frac{d\Phi}{dt}\left(1 + e^{-2kH}\right) C_2 + \left\{g + a\Omega^2 \cos(\Omega t)\right\} E(t)\, H\right] \sin(kx) = 0$$

This led us to equations for the perturbation velocity potential and boundary conditions and we found that the essential difference was that, that in one of the boundary conditions we had a time dependent coefficient, the coefficient was time periodic. Consequently when we wrote down our forms for phi and eta we kept the time dependence arbitrary, we did not set them equal to e to the power i omega t I small omega t as we have done until now.

This is because the equations have a time periodic coefficient or a time dependent coefficient and so e to the power i omega t is not going to work you can substitute and see that. So, we have phi of t and E of t and we had set that we are going to determine what function is phi of time and what function is E of time. Now, we have started with substituting these forms into the kinematic boundary condition, this had led us to two equations A and B.

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$$\frac{dE}{dt} G - \Phi(t)\, k \left(1 - e^{-2kH}\right) C_1 = 0 \rightarrow (A)$$

$$\frac{dE}{dt} H - \Phi(t)\, k \left(1 - e^{-2kH}\right) C_2 = 0 \rightarrow (B)$$

From the B. eq^n (4)

$$\left(\frac{\partial \phi}{\partial t}\right)_{z=0} + \left[g + a\Omega^2 \cos(\Omega t)\right]\eta = 0$$

$$\Rightarrow \left[\frac{d\Phi}{dt}\left(1 + e^{-2kH}\right) C_1 + \left\{g + a\Omega^2 \cos(\Omega t)\right\} E(t)\, G\right] \cos(kx)$$

$$+ \left[\frac{d\Phi}{dt}\left(1 + e^{-2kH}\right) C_2 + \left\{g + a\Omega^2 \cos(\Omega t)\right\} E(t)\, H\right] \sin(kx)$$

$$= 0$$

Again notice that the important difference is that that these equations these homogeneous algebraic equations have time dependent coefficients. Similarly, the Bernoulli equation gave us an equation coefficient of cos kx has to be set equal to 0 here and the coefficient of sin kx also has to be set equal to 0. This will give me two more equations in C 1 G C 2 and H. So, we have a total of 4 equations and 4 unknowns, so by setting the coefficient of cos kx and sin kx to be 0.

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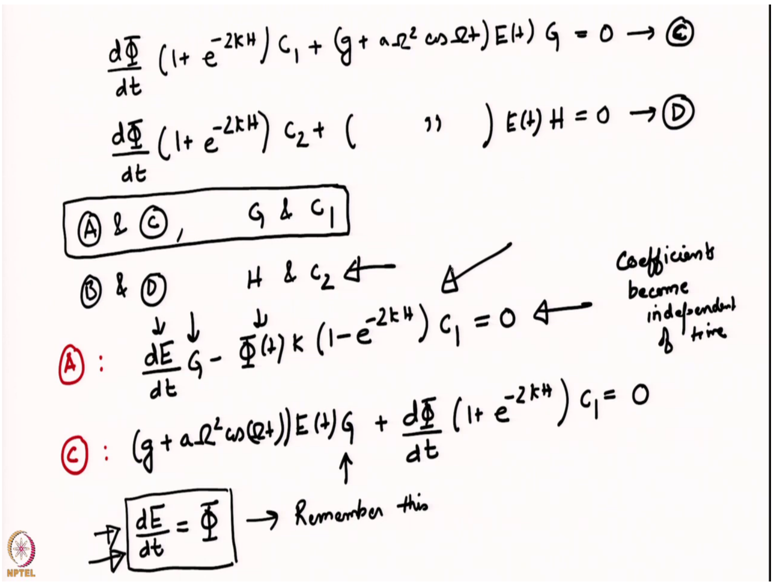

We obtain d phi by dt 1 plus exponential minus 2 KH into C 1 plus g it is call to 0. Let me call this equation C and you can write down a similar equation for C 2 and H the coefficients remain exactly the same plus exactly the same thing. I will call this equation d. So, you can notice a pattern here that either I can treat this as a 4 by 4 matrix multiplying C 1 C 2 G and H or I can take this as a 2 by 2 matrix.

The same pattern exists earlier also although we have not explicitly utilized it. You can see that either I can take equations A and C and equations A and C are just in the unknowns G and C 1 or I could take equation B and D and they would be in a unknown H and C 2. I am just going to use these two equations and work on them and you will find that if I had used instead this I would have been led to exactly the same conclusions.

Let us use equations A and C. So, the equations for A and C let me write it down properly. So, we have dE by dt into G minus phi of t into K into 1 minus e to the power minus 2 KH into C 1 is equal to 0 this is equation A, I am just rewriting it and then we have equation C equation C I am just going to rewrite it. So, that g comes first. So, capital G comes first.

So, I am going to exchange the two terms E into G. So, there is a g here first and a G here next and then plus d phi by dt into 1 plus e to the power minus 2 KH into C 1 is equal to 0. So, now you see we are going to analyze these two equation these are two homogeneous equations in 2 unknowns G and C 1. The only important thing to note is that they are coefficient are actually unknown functions of time.

This is unlike what we have done until now where in all the problems that we have done until now, the coefficients have been constants and the determinant has led us to a dispersion relation. Here we will find that we will have to establish a certain relation between E and phi capital E of time and phi of time and this will render the coefficients of these equations to be independent of time ok.

In particular note that, if I set dE by dt to be equal to phi then this equation the coefficients of this equation become independent of time you can see that very easily that there is a dE by dt here and a phi here, if I set them to be equal then I can write them as either phi of t into G minus phi of t into the rest and then pull the phi of t out that phi of t can be taken out as common phi of t is in general not 0.

So, it is just the rest of the equation whose coefficients are time independent. So, if I make this substitution then the coefficient of the first equation becomes time independent we will remember this and we will use this. Let us work on the matrix for this it will be a time dependent matrix and we will work out it is the usual criteria that the determinant has to be equal to 0, this will ensure this will give us some equation where there will be both dE by dt and phi and then we will plug this assumption in into that equation.

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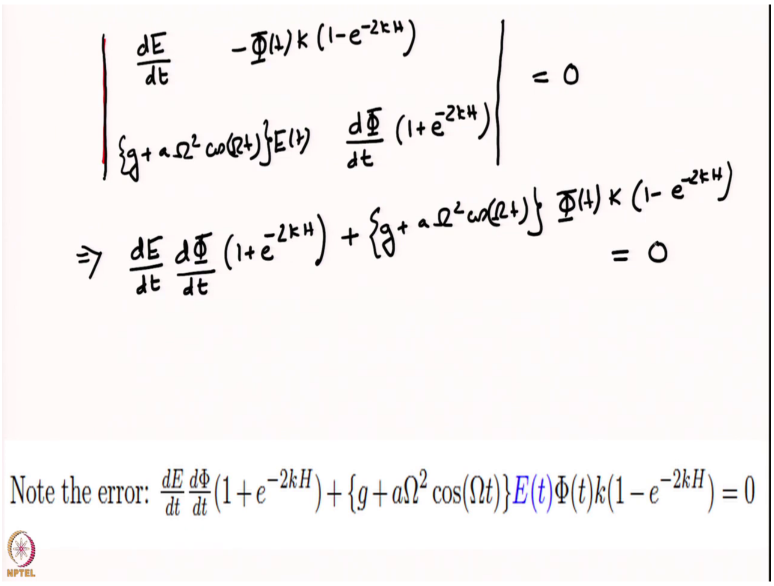

Let us work out the matrix first. So, our determinant will have dE by dt then minus phi of t K 1 minus e to the power minus 2 KH and then here there will be g plus a omega square cos omega t into E of t and then the next term is d phi by dt in to 1 plus e to the power minus twice KH is equal to 0.

This gives us the equation dE by dt in to d phi by dt 1 plus exponential minus 2 KH plus g plus a omega square cos omega t into phi is equal to 0. This is our equation clearly you can see that there are two unknowns here E and phi both of which are unknown functions of time.

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Note the error: $\frac{dE}{dt}\left[\frac{d^2E}{dt^2}(1+e^{-2kH})+\{g+a\Omega^2\cos(\Omega t)\}E(t)k(1-e^{-2kH})\right]=0$

$$\{g+a\Omega^2\cos(\Omega t)\}E(t) \quad \frac{\ }{dt}$$

$$\Rightarrow \frac{dE}{dt}\frac{d\Phi}{dt}(1+e^{-2kH})+\{g+a\Omega^2\cos(\Omega t)\}\Phi(t)k(1-e^{-2kH})=0$$

$$\Phi=\frac{dE}{dt}$$

$$\Rightarrow \frac{dE}{dt}\left[\frac{d^2E}{dt^2}(1+e^{-2kH})+\{g+a\Omega^2\cos(\Omega t)\}k(1-e^{-2kH})\right]=0$$

$$\frac{dE}{dt}\neq 0$$

As I argued earlier if I set phi is equal to dE by dt I have said this earlier that we are going to set phi is equal to dE by dt and this first of all it renders the first equation it is coefficients become independent of time. So, if I set phi is equal to dE by dt then you can see immediately that you have a dE by dt here this becomes d square phi by dt square into 1 plus e to the power minus 2 KH and there is a phi here and phi is dE by dt. So, I can pull out a dE by dt and keep it as common and then the phi has already gone out as dE by dt and then what is left I am just writing is equal to 0. In general dE by dt is not equal to 0 e of t is not equal to 0 dE by dt is not equal to 0 at all times.

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$$\begin{vmatrix} \dfrac{dE}{dt} & -\Phi(t)\, k\,(1-e^{-2kH}) \\ \{g + a\Omega^2 \cos(\Omega t)\}E(t) & \dfrac{d\Phi}{dt}(1+e^{-2kH}) \end{vmatrix} = 0$$

$$\Rightarrow \frac{dE}{dt}\frac{d\Phi}{dt}(1+e^{-2kH}) + \{g + a\Omega^2\cos(\Omega t)\}\, \Phi(t)\, k\,(1-e^{-2kH}) = 0$$

$$\Phi = \frac{dE}{dt}$$

$$\Rightarrow \underset{\neq 0}{\frac{dE}{dt}}\left[\frac{d^2E}{dt^2}(1+e^{-2kH}) + \{g + a\Omega^2\cos(\Omega t)\}\, k\,(1-e^{-2kH})\right] = 0$$

$$\frac{dE}{dt} \neq 0$$

So, this is not equal to 0. So, the only other option to satisfy this equation is to put the quantity in square brackets to be 0, but this has the effect that it gives us an equation governing E as a function of time. Let us see what kind of in equation are we getting. So, I am just setting the coefficient, I am setting the quantity inside the square bracket is equal to 0.

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$$\frac{d^2E}{dt^2}\left(1+e^{-2KH}\right)$$

And this gives me d square E by dt square there was a coefficient 1 plus e to the power minus 2 KH, I am dividing both sides by this quantity. So, this quantity is no longer there.

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$$\frac{d^2E}{dt^2} + \{gk + ak\Omega^2\cos(\Omega t)\}\left(\frac{1-e^{-2kH}}{1+e^{-2kH}}\right)E(t) = 0$$

$$\Rightarrow \frac{d^2E}{dt^2} + \tanh(kH)\{gk + ak\Omega^2\cos(\Omega t)\}E(t) = 0$$

$$\frac{e^{kH}-e^{-kH}}{e^{kH}+e^{-kH}} \rightarrow \tanh(kH)$$

$$\Rightarrow \boxed{\frac{d^2E}{dt^2} + \omega^2\left[1 + \frac{a\Omega^2}{g}\cos(\Omega t)\right]E(t) = 0} \rightarrow \text{Mathieu eq}^n$$

$$\omega^2 \equiv gk\tanh(kH)$$

Without oscillatory f. $\frac{d^2E}{dt^2} + \omega^2 E = 0$

So, this quantity will shift to the second term and the second term will become gk plus aK omega square cos omega t there was a 1 minus e to the power minus 2 KH here and I have divided out by 2 plus e to the power minus 2 KH. So, this put it in a bracket and then this is e of t is equal to 0. I can simplify this further and write this in a slightly more compact form d square E by dt square plus.

You can see that this is nothing but if I multiply up and down by e to the power plus KH then this is e to the power plus KH minus e to the power minus KH divided by e to the power KH minus plus e to the power minus KH. And this is nothing but tan hyperbolic KH this is the familiar tan hyperbolic KH which has appeared before when we studied waves where there was no oscillation of the bottom, now there is a oscillation of the bottom, but that time hyperbolic is still appearing.

So, I am going to write the tan hyperbolic, so this is basically tan hyperbolic. So, I am just going to write tan hyperbolic before the curly braces and then just put it inside. I can write this in a even more compact form once I recognize that gk into tan hyperbolic KH is just my dispersion relation for free oscillations when there is no vibration of the container bottom. So, if I call this omega square then what I have pulled out is gk because gk tan hyperbolic KH, we have seen earlier is the dispersion relation for surface gravity waves when there is no oscillatory motion of the bottom.

So, I will call this omega square and I have pulled out a gk. So, then there is a 1 here and if I pull out a gk then what I am left here is a omega square by g that is a non dimensional number into cos omega t into E of t is equal to 0 where I have set omega square is equal to gk this is defined as tan hyperbolic KH. You can immediately recognize what equation is this, this is our familiar Mathieu equation. We had studied this when we had looked at oscillations of the pendulum.

Now, we are taking our container containing a liquid and a free surface in the base state and we are shaking it up and down and we are asking that if we put a perturbation eta on the surface does the perturb what is the equation that governs the amplitude of the perturbation. Notice that all whatever forms we have chosen for phi and eta are in the standing wave form.

The space and the time part are separate. So, this is the equation that governs the coefficient E of t for eta and this is the Mathieu equation. We have looked at this equation in some amount of detail before and now we are again finding in, you can now go back and check what is the implication that E is governed by this equation, on these two homogeneous equations whose coefficients gave us the equation governing E.

The choice dE by dt is equal to phi already made the first equation have coefficients which are not dependent on time. You will see that this second differential equation that we have got that basically ensures that the second equation here equations becomes the same as the first equation which is equation A.

So, in other words these two choices dE by dt is equal to phi and the fact that D, E itself is governed by a Mathieu equation makes both of these equations time independent equations both equations A and C the coefficients it makes it time independent. So, we are essentially finding that the amplitude of the interface which is governed by E of t is actually governed by a Mathieu equation.

Let us see some limits of this and let us first convince ourselves that this is familiar and this reduces to the correct thing that we have already derived. So, first you can see that in the absence of forcing in the absence. So, without oscillatory forcing this equation just becomes d squared E by dt square plus omega square into E is equal to 0.

This is just telling us that E is a sine or a cosine function of time with a frequency which is given by the dispersion relation, this is exactly what we would have concluded if we had done a normal mode analysis on the problem setting the oscillatory part to 0. We have done that before and it is recovering our older results.

Note also in particular that setting equation A you can immediately see that setting d phi by dt or dE by dt is equal to phi actually makes this equation A as I have said before it makes it independent of time it makes the coefficients independent of time. Let us write their resultant equations this will give us a relation between G and C 1.

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$$\text{(A)} \Rightarrow \left(\frac{dE}{dt}\right)\left[G - K\left(1-e^{-2KH}\right)C_1\right] = 0$$

$$\downarrow$$

$$G - K\left(1-e^{-2KH}\right)C_1 = 0$$

$$\Rightarrow C_1 = \frac{1}{K\left(1-e^{-2KH}\right)}\, G \leftarrow$$

$$\text{(B)} \Rightarrow C_2 = \frac{1}{K\left(1-e^{-2KH}\right)}\, H \leftarrow$$

$$\phi = \Phi(t)\left[C_1 \cos(kx) + C_2 \sin(kx)\right]\left[\underbrace{e^{kz} + e^{-kz-2kH}}\right]$$

$$= \frac{dE}{dt}\,\frac{1}{K\left(1-e^{-2kH}\right)}\left[G \cos(kx) + H \sin(kx)\right]\left[\quad '' \quad\right]$$

So, I have dE by dt, I am writing 5 as dE by dt. So, now both the terms contain dE by dt G minus K into 1 minus e to the power minus twice KH into C 1 is equal to 0, dE by dt is in general not 0. And so my equation a just becomes a homogeneous equation for G and C 1 with coefficients which do not depend on time. This is consistent with what I said earlier this equation also gives me a relation between G and C 1.

So, I can write C 1 is equal to 1 by K 1 minus e to the power minus twice KH into G. Similarly you can use we have used equations A and C, I encourage you to go and do the same for equations B and D and you will not find anything new you will find exactly the same thing. That you have to set phi is equal to dE by dt and once you set that you will be led to the same Mathieu equation as we have obtained here.

So, either you can write it as a 4 by 4 matrix or you can write it as a 2 by 2 matrix we could have done the same in the unforced problems also earlier, so now if you use equations B. So, if you use equation B, so this is from A we have not use B, but if you did that then you would find that C 2 is equal to the same thing you would have found that phi is equal to phi has to be set to be dE by dt and then C 2 is equal to the same coefficient into H.

Let us now use these things recall that form for phi was phi of t into C 1 cos kx plus C 2 sin kx into e to the power kz plus e to the power minus kz minus twice KH, this was the form that we had written earlier. Now, we have determined C 1 and C 2 in terms of G and H and we know that phi is equal to dE by d t.

So, I am going to replace this as dE by dt and then I will have I am going to replace express C 1 in terms of G using this and C 2 in terms of H using that. So, I will have 1 by K into 1 minus e to the power minus 2 KH that is a common factor and then I will have G times cos kx plus H times sin kx into the same thing this part. So, I am just going to write this part here.

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$$\phi = \frac{dE}{dt}\frac{g}{gk(1-e^{-2kH})}\left[G\cos(kx)+H\sin(kx)\right]\left[e^{kz}+e^{-kz-2kH}\right]$$

$$= \frac{dE}{dt}\frac{g}{gk\dfrac{(1-e^{-2kH})}{e^{kH}+e^{-kH}}(e^{kH}+e^{-kH})} \qquad ''$$

$$= \frac{dE}{dt}\frac{g}{gk\left(\dfrac{e^{kH}-e^{-kH}}{e^{kH}+e^{-kH}}\right)e^{-kH}(e^{kH}+e^{-kH})} \qquad ''$$

$$= \frac{dE}{dt}\frac{e^{kH}g}{\omega^2}\left[G\cos(kx)+H\sin(kx)\right]\frac{e^{kz}+e^{-kz-2kH}}{(e^{kH}+e^{-kH})}$$

Let us simplify this a little bit more. So, phi is equal to dE by dt and I want to multiply and divide numerator and denominator by g. So, I will have a gk here into 1 minus e to the power minus 2 KH. And then I have my usual space part which is G cos kx plus x sin kx into e to the power kz plus e to power minus kz minus twice KH, I am doing this in order to write this problem in a manner which is as close as possible to the unforced problem that we had earlier seen.

So, that you clearly see what is the connection between the forced problem and the unforced problem. So, I will write this as dE by dt this is all this is happening in the denominator. So, this 1 minus e to the power minus 2 KH remains untouched I multiply by e to the power plus KH into e to the power minus KH because we have divided by this I have to multiply by this and the rest of the things remain the same not doing anything to them.

Now, I will just multiply the this becomes g, this becomes gk I will just multiply this term I will just multiply this term 1 minus e to the power minus 2 KH with e to the power plus KH. If I do that then it becomes e to the power plus KH minus e to the power minus KH. I am trying to get a tan hyperbolic KH which is why I am doing this because gk times tan hyperbolic KH is my omega square.

So, I have got this I multiplied e to the power plus KH. So, I have to put a e to the power minus KH. And then there will be e to the power KH plus e to the power minus KH and then the rest of the part remains the same. So, this just becomes dE by dt the e to the power KH here can be shifted to the top and then there is a g.

And then I can write this as omega square you can see that this part is now tan hyperbolic H. The entire thing inside the bracket is just tan hyperbolic KH. So, that is gk times tan hyperbole KH, we know is the dispersion relation for free oscillations I have written that as omega square.

So, and I have shifted the e to the power minus KH to the numerator. So, what am I left with I am left with the remaining which is G cos kx plus h sin kx into e to the power kz plus e to the power minus kz minus twice KH and I have left out this factor here. So, I am going to put this here, now you can notice that I can take this and insert it into this, I can multiply both the terms by e to the power KH and you will see that the resultant form of the numerator if I divide numerator and denominator by 2.

So, the numerator becomes cos hyperbolic of K times z plus H and the denominator becomes cos hyperbolic of just KH. I encourage you to try this on your own it is a very simple step take this e to the power KH multiply the numerator and then divide the numerator and denominator by a factor of 2. So, multiply the numerator and denominator by half you will get cos hyperbolic kz plus H.

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$$\phi = \frac{dE}{dt}\frac{g}{\omega^2}\left[G\cos(kx) + H\sin(kx)\right]\frac{\cosh[k(z+H)]}{\cosh(kH)}$$

$$\cosh(x) = \frac{e^{x}+e^{-x}}{2}$$

$$\eta = E(t)\left[G\cos(kx) + H\sin(kx)\right]$$

where $E(t)$ satisfies

$$\text{Mathieu eq}\left\{\frac{d^2E}{dt^2} + \omega^2\left[1 + \frac{a\Omega^2}{g}\cos(\Omega t)\right]E(t) = 0\right.$$

$$\omega^2 = gk\tanh(kH)$$

So, this resultant expression for phi can be written as phi is equal to dE by dt into g by omega square. Now, there is no exponential because I have shifted the exponential to the cos hyperbolic part the x part remains the same into the x part. Now, the numerator now becomes cos hyperbolic of k into z plus h divided by cos hyperbolic of KH for reference cos hyperbolic of x is defined as e to the power X plus e to the power minus X by 2.

Similarly eta, eta does not change eta is just we are writing everything in terms of E of t and that is just G cos kx plus H sin kx. And we have seen that where, E of t satisfies d square E by dt square plus omega square 1 plus a omega square by g cos omega t into E of t is equal to 0 this is our solution to the problem this is our solution to the problem.

What is left behind G H and the initial value of E has to be determined from initial conditions. The main thing that we find now is that unlike earlier is that that now, the coefficient E of t is not governed by a simple harmonic oscillator equation, it would have been gone by simple harmonic oscillator equation if we set small a to 0, here.

Then this just becomes a simple harmonic oscillator equation whose frequency is just the dispersion relation the dispersion relation also I will write omega square is equal to gk tan hyperbolic KH. So, we are finding that phi and eta have very similar expressions, we have derived these very similar looking expressions we have derived and we have found. So, this factor of course, hyperbolic kz plus H divided by cos hyperbolic KH this we have seen earlier in the case of unforced waves.

We are seeing the same factor and I have written it deliberately. So, that it is as close as possible to the previous problem, the only difference that we are finding now is that, that earlier there was just a dispersion relation. Now, there is an actual equation governing the coefficient E of t there is only 1 coefficient. Now E of t the coefficient the time dependent part of phi is just dE by dt. So, if we know E is a function of t then we also know dE by dt as a function of t.

So, this is the Mathieu equation and we have looked at its stability chart in the in one of the previous videos early on in this course. In the next video I am going to analyze this Mathieu question little bit more and explain what it means for the fluid problem what does instability mean for the fluid problem? And we will also discuss some practical applications of this it turns out that there are quite a few practical applications of this forced vibration problem particularly for atomization applications.