Introduction to Interfacial Waves
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Lecture - 52
Faraday waves on an interface - stability of time dependent base states

We were looking at shape oscillations of a spherical interface due to perturbations imposed at the surface. We have just finished calculating the dispersion relation for linearized perturbations. Let us look at some applications of this dispersion relation.

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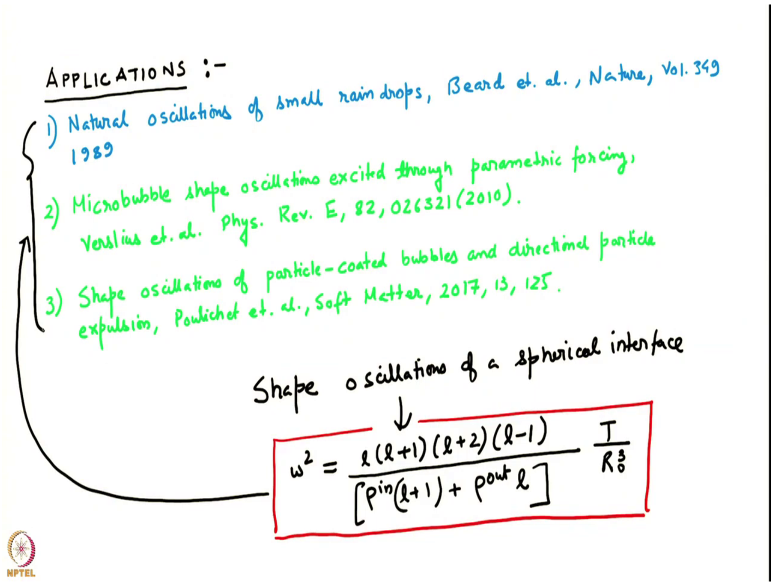

The dispersion relation finds widespread applications in many areas of engineering. I have just mentioned two of the important applications. One of them is raindrops when they fall undergo shape oscillations as they fall. So, under the droplet limit of this dispersion relation this dispersion relation is frequently used to calculate the frequency of oscillation of rain drops. This dispersion relation is very important in this analysis.

So, this particular paper which is pointed out in blue here you can go through it if you are interested to learn more about it. The second application the second important application is in the case of shape oscillations of bubbles. Micro bubbles are frequently used as tracers in ultrasound related applications. Here these bubbles undergo shape oscillations and in the bubble limit this dispersion relation is used to understand those shape oscillations.

Again the two papers that I have indicated here in green, you can go through them if you are interested and you will find applications of these dispersion relations as well as applications of this of the knowledge and understanding generated from understanding these shape applications.

So, with that we complete our study of shape oscillations of droplets and bubbles. Until now, we have mostly looked at problems where the base state was quiescent. In fact, in problems where gravity was treated the there was a pressure gradient in the base state and that was generated due to the hydrostatic pressure in the base state where we looked at pure capillary waves there was no gravity and So, the base state velocity was 0, the base state pressure was uniform and independent of time..

Now, let us look at a slightly more complicated base state, where the base state itself is dependent on time. We have seen one example where the base state was time dependent early in this course when we were looking at oscillations of mechanical systems. There we had looked at the example of a Kapitza pendulum where the point of support of the pendulum is moved up and down with a certain amplitude and a certain frequency.

We had found that the fixed points of the pendulum continue to remain the same independent of whether the point of support is moving or not or in other words the fixed points are 0 velocity of the pendulum and the pendulum either at theta equal to 0 or theta equal to pi. So, the bottom most point or the top most point.

And we had in particular looked at the stability of the base state and there we had found an interesting conclusion where we had found that the lower fixed point can actually become unstable if the point of oscillation is moved at a sufficiently large amplitude. In particular recall that we had encountered the Mathieu equation there whose analysis we had done using Floquet theory.

And then we had found that the Mathieu equation, the stability chart of the Mathieu equation had these tongue shaped curves, which were alternatively harmonic and sub harmonic. Harmonic implies that the frequency of the oscillation is the same as the forcing frequency. Sub harmonic implies the frequency of the oscillation is half of the forcing frequency.

We had found regions where the pendulum is stable and we had found regions where the pendulum is unstable and if you give it a perturbation the perturbation will grow if one is in the unstable region. We are going to find the fluid equivalent of that problem and we are going to find something called faraday waves. This problem was first mentioned in the literature by Michael Faraday and hence is named after him.

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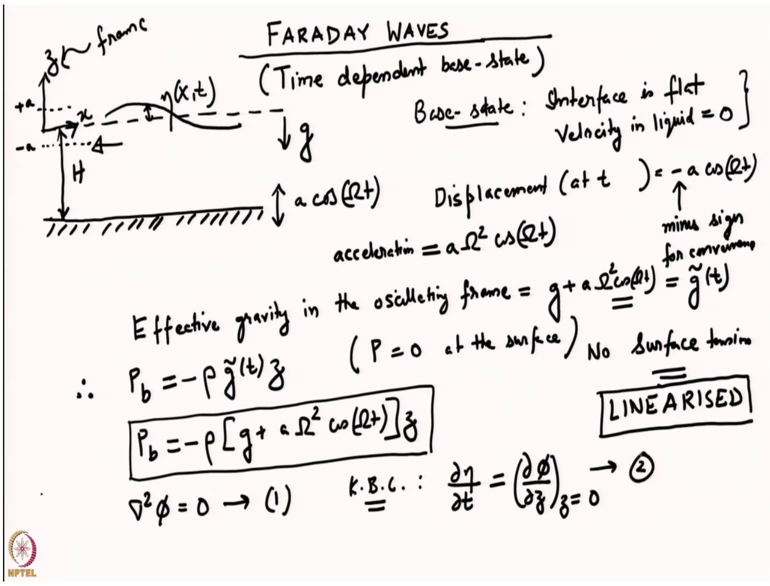

So, we are going to move to faraday waves. Now, these also will be waves on the interface, but the main difference between the waves that we have studied until now and this problem will be that the base state will be time dependent here time dependent base state.

In particular you will find that the pressure field in the liquid. The interface will still remain flattened in the base state, but the pressure field will become a sinusoidal function of time because we are oscillating the container up and down with a certain frequency. So, let us draw the geometry.

So, we have some kind of a container. For simplicity we are going to ignore the confining lateral boundaries. So, there will always be some kind of a boundary of the container, for simplicity we are going to ignore it. It is important and has to be taken into analysis, but for as a first step let us ignore the presence of confining boundaries. So, this is a horizontally unbounded system.

So, in the base state my interface remains flat. The coordinate system is say is at the undisturbed interface in the base state. This is the x direction and this is the z direction, g as usual is acts in the negative z direction and this depth undisturbed depth is H and we are shaking the this container. So, the wall of the bottom of the container is being moved up and down with a certain amplitude a and a certain frequency capital omega. This should remind you of what we had done earlier for the pendulum.

We had taken the point of support of the pendulum and we had moved up and down. We are now doing the same thing to a fluid problem. We are taking a container where there is fluid. As a first step we are ignoring the presence of the confining boundaries and we are shaking the bottom of the container. This will cause the fluid in the container to also move up and down you can go and think a little bit for yourself that moving the container up and down.

So, in the base state the interface is flat, the velocity in the liquid is 0 and it is being oscillated up and down. Now, you can convince yourself that this oscillation is not incompatible with the interface remaining flat. The oscillation does not cause the interface to become wavy. Think a little bit about it and go back and check whether this is a solution to the equations.

The interface being flat, the base being moved up and down and the velocity being 0, you will find that this is a solution to the equations. So, this is a valid base state. So, this is our frame of reference and this frame of reference will move along with the flat interface. So, in that frame of reference that is a accelerating frame of reference because it moves up and down along with the fluid. And so, in that frame of reference I am going to see a non inertial body force.

So, it is as if gravity will become time independent. This is exactly the same as our analysis earlier where we had analyzed the pendulum in a we had mentioned there that you could do the analysis in a oscillating frame of reference and that would also lead you to the equation governing the pendulum. So, in this frame of reference in addition to gravity we will also see a body force which is due to the oscillatory motion up and down. So, in particular we are going to choose the displacement.

So, the initial displacement of the free surface let say at time t equal to 0 will be chosen to be equal to minus a. So, the displacement at any time rather is given by minus a cos omega t. This minus sign is purely for convenience. You can put plus a and it will not make any difference, your final equation instead of having a plus sign will have a minus sign. It will be the same Mathieu equation and its stability properties will remain exactly the same.

This minus sign is equivalent to saying that at time t equal to 0 my interface. So, because this is being done in an oscillating frame of reference, in the lab frame we will see the interface moving between plus a and minus a. So, this is minus a and plus a. So, the interface is going to go up and down in the base state between these two limits.

So, choosing minus a cos omega t as my displacement at any time is equivalent to saying that at time t equal to 0 the interface was at location z is equal to minus a in the lab frame of reference. So, it started from here and so, because it starts from the lower extreme it can only go upwards. And as it goes upwards it will cause a body force in the opposite direction which will be the same direction as the acceleration due to gravity.

So, you can see that this is the expression for displacement the z displacement. If I take its second derivative with respect to time that will give me an acceleration and the acceleration will be. So, it will just be a omega square cos omega t. So, this is the acceleration at any time. So, I have to add this acceleration to gravity and the effective value of gravity is g plus a omega square cos omega .t.

So, in the oscillating frame of reference the effective gravity in the oscillating frame is g plus a omega square cos omega t they are both in the same direction. So, therefore, my pressure in the base state, recall that even without oscillation the pressure in the base state in the presence of gravity is just a hydrostatic profile.

Now, it is as if your gravity your effective gravity has become a function of time. So, this is some g tilde let say and g tilde is a function of time because of the second term. So, my pressure in the base state will still be a hydrostatic pressure gradient. However, it will be decided by the instantaneous effective value of gravity. So, it will be minus rho g tilde of t into z.

So, instead of just g it will be g tilde of t the effective value of gravity. I have assumed here that the pressure is 0 at the surface at all times. We are ignoring surface tension in this calculation as of now, but surface tension can be very easily included. Just to keep it simple we are not putting in surface tension, but and when we have finished this analysis you will see that surface tension can be very easily added into the analysis.

So, let us proceed. So, g tilde of t is basically g plus a omega square cos omega t into z. So, this is my pressure in the base state. So, like before our perturbation we will do this in the inviscid irrotational framework except that we are now in the oscillating frame of reference where my base state pressure is given by this expression. So, like before we will write down the equations governing the perturbation velocity potential and some perturbation at the interface and then we will ask how do we analyze these things ok.

So, the perturbation velocity potential is basically just grad square phi is equal to 0. I will call this equation 1. The kinematic boundary condition remains the same. If I impose a perturbation if I impose a perturbation eta all of this is being done in the oscillating frame of reference. So, in that frame of reference in the base state I do not see any motion, I only feel an effective gravity whose value oscillates in time.

So, this is in that frame of reference. So, eta of x comma t and the kinematic boundary condition is exactly the same as before in the linearized approximation is equal to del phi by del z at z is equal to 0. So, all this is linearized. So, that is why instead of applying it at z is equal to eta, I am applying it at z is equal to 0. So, this is my kinematic boundary condition, let me call this equation 2.

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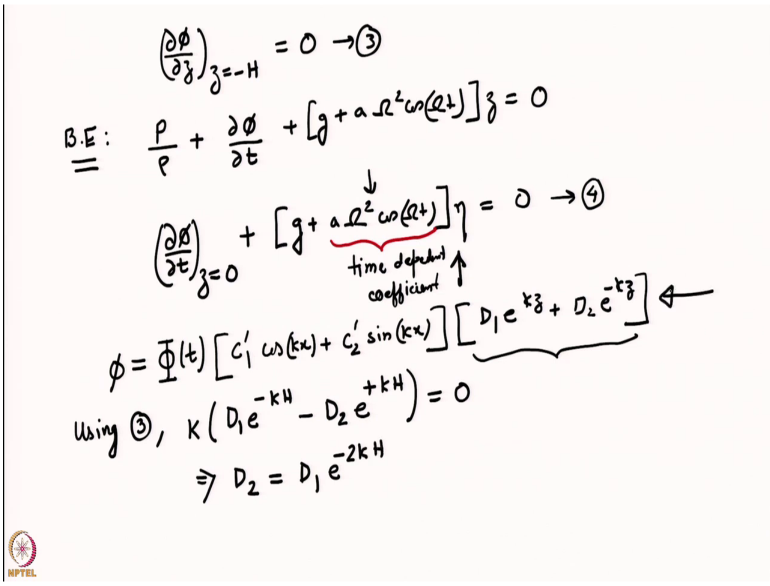

Then I have a no penetration boundary condition which is del phi by del z is z is equal to minus H is equal to 0, phi is the perturbation velocity potential. In the base state there is no velocity and so, the velocity potential in the base state is 0, then the Bernoulli equation. So, this is 3 and the Bernoulli equation is just P by rho plus del phi by del t plus g plus a omega square cos omega t into z is equal to 0, this is the Bernoulli equation at any point in the fluid.

We are interested in getting a Bernoulli equation at the surface. This will give us a boundary condition at the surface. At the surface the pressure is 0 and so, at the interface we will have del phi by del t and I am going to do a linearization we have seen this earlier.

So, instead of applying it at z is equal to eta this will be at z equal to 0 from the Taylor series argument and this would be plus g plus a omega square cos omega t into eta. This term has to be applied at eta we have discussed this before when this second term was not present.

This term is an order eta term and so, it has to be applied at eta. If you apply it at z is equal to 0 then we will miss out one order eta contribution. So, this is actually equation 4. So, now you can see that your equations are equation 1, 2, 3 and 4 and so, the first thing that you note about the equations is that that these all of them all of them are the same as before except equation 4 which actually has a coefficient which is time dependent.

Recall that there is this extra term now there is this extra term now which was not present earlier and this extra term makes a very large quality difference to the conclusions that we are going to draw in this problem. In particular we will find that this problem actually admits an instability. In the absence of this term we do not have an instability in this problem in the linearized approximation.

So, this is a time dependent or an oscillatory term time dependent coefficient. So, I have mentioned to you before that we cannot do e to the power i omega t when the coefficients of our differential equation are time dependent. So, we have to generalize this. We have learnt how to generalize this. This is a periodic this is a coefficient which is time periodic and so, we will have to do something like Floquet theory, but let us first keep this argument general and let us pose the form for phi and eta.

So, phi we all know. So, I am not going to do e to the power i omega t because there is a time dependent coefficient. So, I will say that I do not know what is the functional dependence on time. So, I will keep it arbitrary, some capital phi of t. We will have to determine it by substituting it into the governing equations.

And then the space part remains the same as before because the governing equation which determines the space part is still the Laplace equation. So, from variable separability it is exactly the same as before. So, I will put some prime here we will see shortly why.

So, linear combination of sin k x and cos k x into a linear combination of e to the power k z and e to the power minus k z, this is a finite depth problem. So, none of the so, both the e to the power k z and e to the power minus k z term stay because it is not an infinitely deep pool. So, now if I substitute using 3 you can find that the only part which is affected by 3 is this because this is the only z dependent part and the derivative with respect to z.

So, you can see that if I take a derivative with respect to z then it just tells me D 1 e to the power minus k H minus D 2 e to the power plus k H note the sign is equal to 0, this implies k is not equal to 0. So, D 2 is equal to D 1 e to the power minus 2 k H. We can use this to simplify the expression for my velocity potential. Let us do that.

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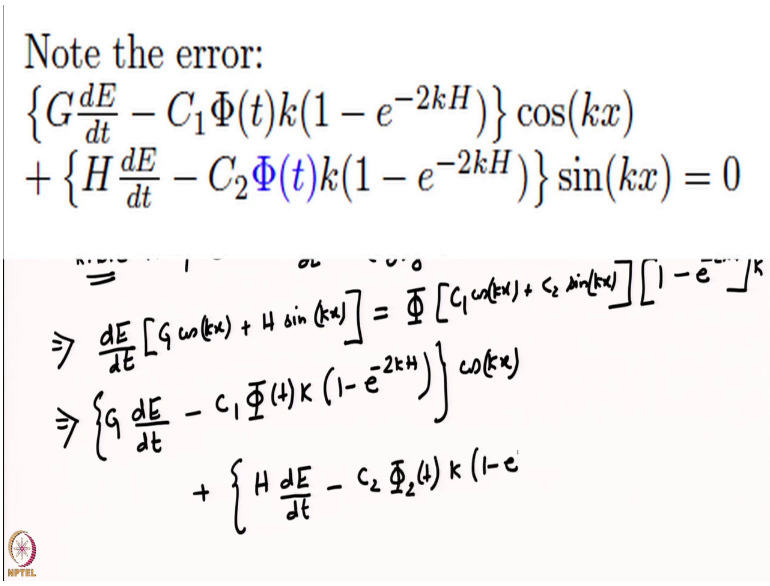

So, we will say let D 1 C 1 prime is equal to C 1. So, that is why I had put a prime. So, that I could use the variable C 1 elsewhere is equal to C 2. If you do that then it just becomes phi is equal to some capital phi of t which we do not yet know into C 1 cos k x plus C 2 sin k x and then the z part becomes e to the power k z plus e to the power minus k z minus 2 k H. This is my expression for the perturbation velocity potential. What about eta?

So, eta also I will do the same thing I will do e to the power or e as some function of time some unknown function into the same space part. So, I will just call it G cos k x plus H sin k x. Note that we are not doing e to the power i omega t ok. So, we are not doing e to the power i omega t because the coefficients of the equation are functions of time, e to the power i omega t will not work here, it has to be more general than that ok.

So, now let us use the kinematic boundary condition. Kinematic boundary condition was equation 2. So, that was the linearized kinematic boundary condition was del eta by del t is equal to del phi by del z at z is equal to 0. If I do that then I get d E by d t the rest of it remains the same G cos k x is equal to phi times the same thing C 1 cos k x plus C 2 sin k x and the last part just gets applied at z is equal to 0. So, this just becomes 1 plus e to the power minus 2 k H.

Now, there is a k missing here. So, I will put a k here there is a derivative with respect to z. So, there will be a k and this is minus. Let us collect all the coefficients of cos k x just as we do usually.

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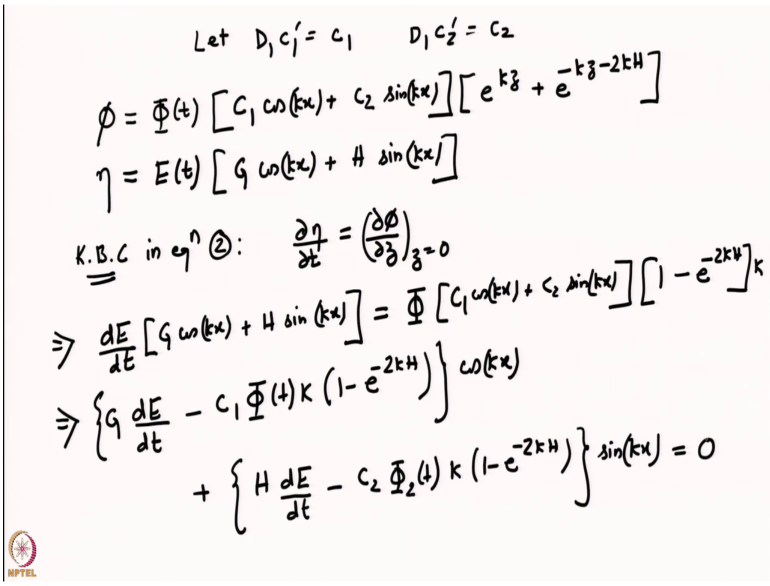

So, G into d E by d t plus a similar thing in H and C 2. So, this is our equation that we obtained from the kinematic boundary condition. By our usual arguments I can set the coefficient of cos k x to 0 and sin k x to 0. This will give me an equation which is as follows.

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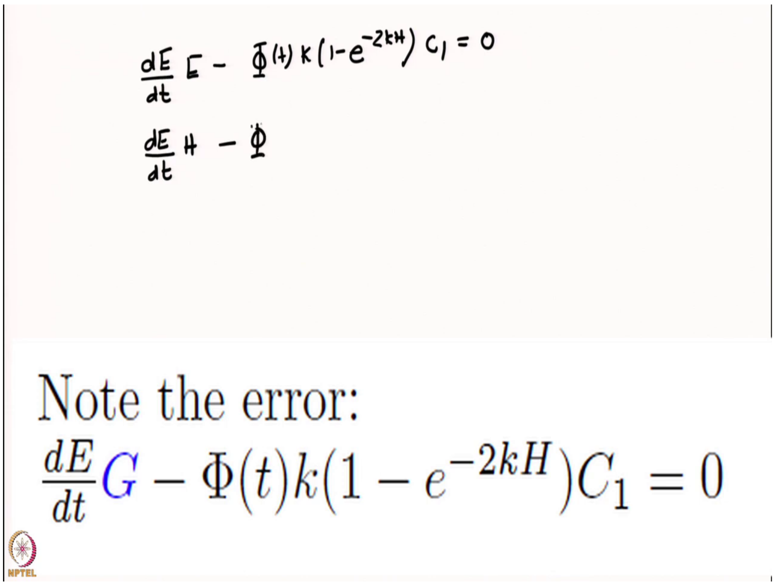

So, this will give me d E by d t into E minus phi of t into k into 1 minus e to the power minus 2 k H into C 1 is equal to 0.

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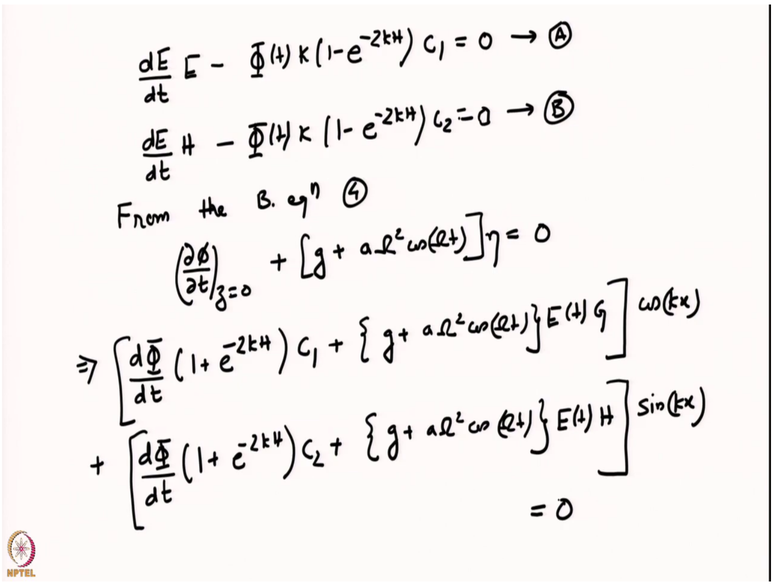

Similarly, I will get d E by d t into H minus phi of t k 1 minus e to the power minus 2 k H into C 2 is equal to 0. I will call this A and this one as B. Note one important difference. Earlier the equations the homogeneous equations that we had for these complex constants, the coefficients of them were constant.

Now, the coefficients are functions of time this is just a reflection that the base state is a function of time. So, we will have to come up with some choice of or some relation between d E by d t and phi and then an equation governing d E by d t itself which will make these equations as really constant coefficient equations ok. So, we will see that.

So, now we have to use the Bernoulli equation. From the Bernoulli equation, so, I think that was equation 4. So, we have del phi by del t at z is equal to 0 plus g plus a omega square cos omega t into eta is equal to 0. If we substitute the expressions for phi and eta and do the algebra, it just leads to the same thing.

A coefficient of cos k x which is given by this into C 1 plus g plus a omega square cos omega t into E of t into G this whole thing into cos k x plus the same thing for phi C 2 and H with the same coefficients is equal to 0. So, we now need to set the coefficients of cos k x and sin k x to 0 and this will give us two more equations in C 1, g, C 2 and H. So, we have a total of four equations. We will analyze these equations in the next video.