Introduction to Interfacial Waves
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Lecture - 51
Analysis of I=0 and I=1 modes for a spherical drop

We were looking at the surface tension driven oscillations of a spherical interface separating two immiscible fluids.

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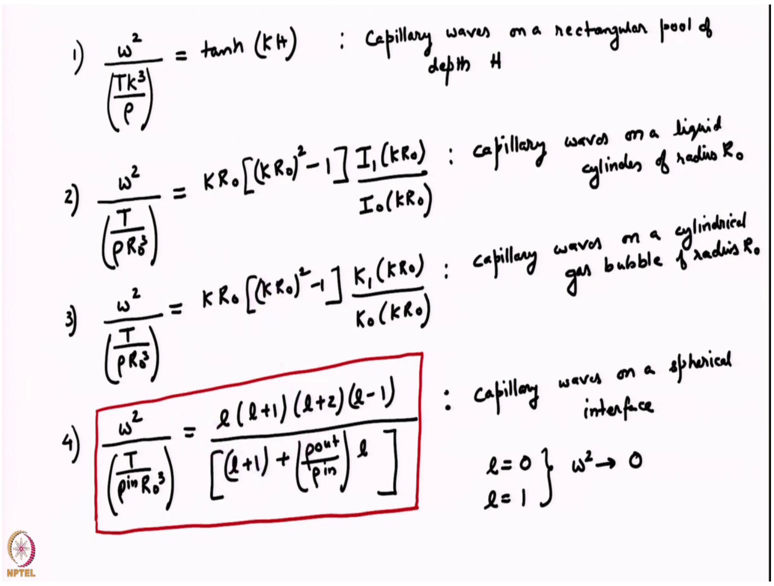

This was the general dispersion relation that we had found. The restoring force was purely due to surface tension and we had mentioned that this dispersion relation governs the oscillation frequencies for shape oscillations. This could be a parameter l. This is allowed to be integer values 0, 1, 2, 3 and so on.

It comes from the subscript of the legendary polynomial P of l. We had also mentioned that for l is equal to 0 and for l is equal to 1 the frequency evaluates to 0 and we were trying to understand the reason why this frequency is 0. Let us continue with that.

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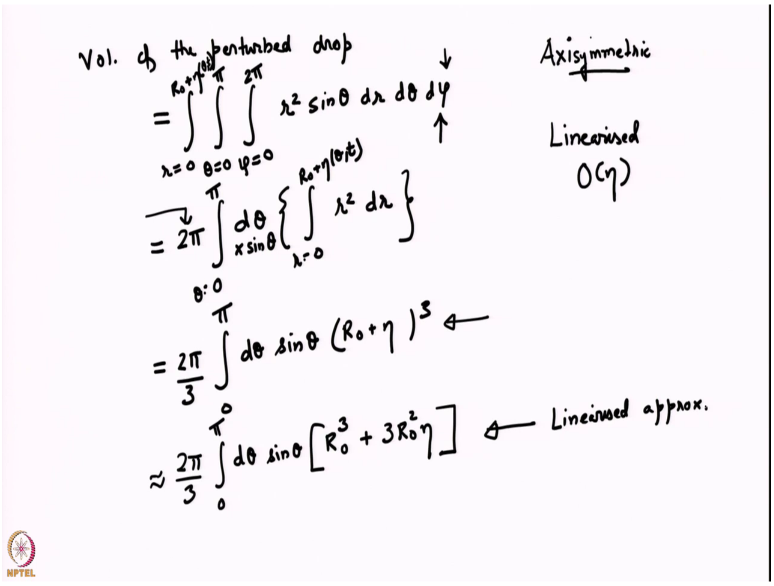

So, in order to understand this we had calculated the volume of the perturbed drop. So, we impose a single Legendre polynomial as deformation of the droplet surface or of the spherical surface and we calculate what is the volume of the perturbed drop. Note the limits of integration whatever we are doing here is under the axisymmetric approximation, so, axisymmetric approximation and so, the psi integral is the easiest to integrate because our quantities do not depend on psi.

So, the psi integral just comes out and it just gives us the factor of 2 pi here. We are left with the theta integral and the r integral. Note that the theta integral has a sin theta factor and the limits of the r integral contain an eta which is a function of theta. So, we did the r integral first. This gave us a small r cube by 3. I pulled the 3 out and wrote it as small r cube. The limits are 0 to R 0 plus eta. So, this gives me R 0 plus eta whole cube.

So, let us continue from here. So, this is 2 pi by 3 and now I am going to make an approximation here because we are in the linearized approximation or the linearized limit. So, we will only retain quantities which are linear in eta. So, we are not going to retain any quantity which is any higher power of eta other than 1. You can see that this if I open up this cubic it will give me a eta cube it will also give me an eta square. So, in a linear calculation I am going to ignore them as being small compared to what I retain.

So, this is approximated as 0 to pi d theta sine theta. I will just have two terms up to order eta. This is an order one term and then there will be 3 R 0 square into eta. So, this is my linearized approximation. This is not the exact volume, but this is the volume under the linearized approximation.

Note that I have ignored two terms here. I have ignored a term which would be 3 R 0 into eta square that is an order eta square term and then there will be an eta cube term which I have ignored because they are all higher powers of eta. So, now let us work on this.

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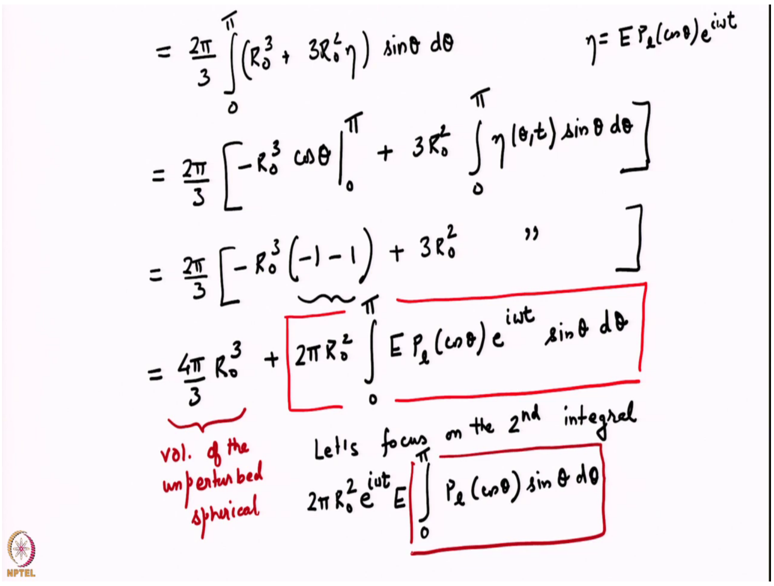

So, we have 2 pi by 3. I am just writing it again here R 0 cube plus 3 R 0 square eta into sin theta d theta. Let us work on this integral. So, you can see the first term is just 2 pi by 3 into minus R 0 cube integral sin theta d theta is cos theta minus cos theta, so, 0 to pi plus 3 R 0 square into 0 to pi eta which is a function of theta and t into sin theta into d theta. Let us continue this further. You can see that the first term is just minus R 0 cube into cos pi is minus 1 cos 0 is 1.

So, it just gives you a 2 R 0 cube and the second term will have this is 3 R 0 square and the same thing. If I open up the bracket then there is a factor of 2 here. So, this gives me 4 pi by 3 into R 0 cube and then the second term becomes plus 2 pi R 0 square into 0 to pi eta theta of t into sin theta d theta.

Now, eta theta of t we have seen earlier is some complex number E into P l of cos theta into e to the power i omega t into sin theta into d theta. So, I am basically using the fact that eta is equal to E times P l of cos theta e to the power i omega t, this we have seen earlier. Now, you can easily see that this is just the volume of the unperturbed sphere. This is just the volume of a sphere of radius R naught. So, when we do not impose any perturbations on the interface then we recover this volume.

So, the second contribution tells us at linear order what is the change in volume. Let us evaluate that integral. So, this is equal to; so, we will let us focus on the second integral. So, I am going to look at this integral. So, I am going to ignore the 2 pi factor of 2 pi R 0 square in this integral because that is just a constant.

You can see that I can pull the e to the power i omega t outside of the integral because it omega does not depend on theta and so, I can pull it outside. And so, I am just going to I can also pull the e outside and so, I am just going to have an integral of the form 0 to pi.

So, e to the power i omega t then there will be a 2 pi R naught square into E, all these will be out and then what is inside will be just P l of cos theta into sin theta d theta. So, if we know how to do this integral then we will be able to evaluate this term. So, just this in this integral. So, let us see how to evaluate this integral.

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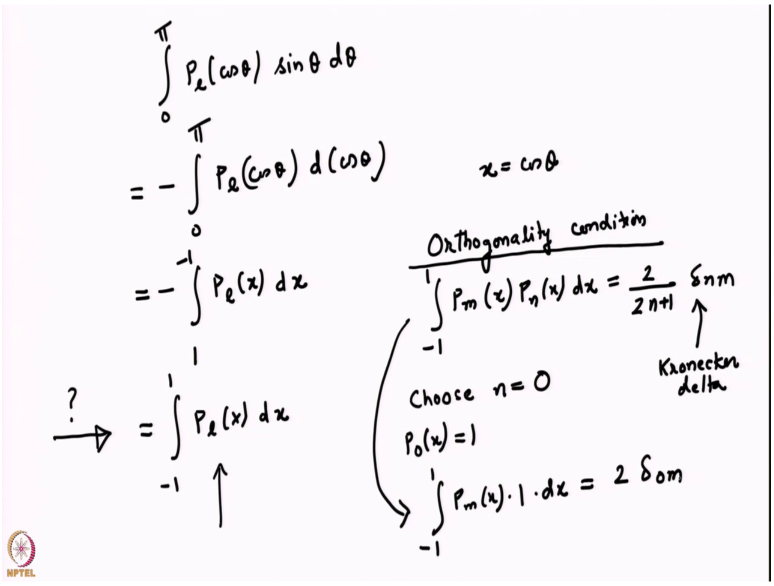

So, we are interested in 0 to pi P l of cos theta into sin theta d theta. This I can write it as minus 0 to pi P l cos theta d of cos theta. d of cos theta is minus sin theta d theta and so, there is a minus outside to make it overall plus. This if I use the substitution that x is equal to cos theta we have used the substitution before then this becomes 1, this becomes minus 1 P l of x d x. So, this simplifies to this integral and I can swap the limits and make it minus 1 to plus 1 P l of x d x that gets rid of my minus sign.

So, now how do we evaluate this integral? We are going to use an orthogonality condition for Legendre polynomials ok. So, I am going to write down a formula. I will provide you a reference where you can look up how this formula comes orthogonality. So, the orthogonality condition says minus 1 to 1 P m of x, m is some positive integer 0 1 2 3 4 like that again P n of x d x is equal to 2 by 2 n plus 1, this is the Kronecker delta.

It takes the value 1 when both the indices are equal and is 0 when both the indices are unequal to each other. So, using this orthogonality condition I want to determine the value of this integral. How do we do that? So, we will choose n equal to 0. Recall that P of 0 of x is 1. I have told you the formula for the first few Legendre polynomials in the last lecture, you can go and look up that and you will find that P 0 of x is 1.

This helps me reduce this integral to minus 1 to 1 P m of x P n of x is 1. So, this is just 1 into d x and if I substitute n is equal to 0 then it just becomes 2 delta 0 m. And it is clear from the property of the Kronecker delta that this integral is 0 for all values of m other than 0. So, it is only when it is delta 00 that this will give you 1; delta 01, delta 02, delta 03 and so on will all be 0 by the property of the Kronecker delta.

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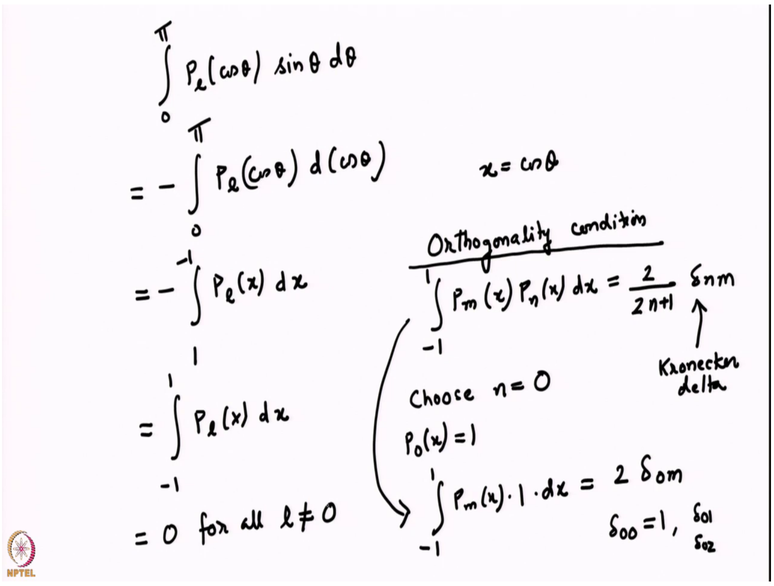

So, we find that this integral is 0 for all l not equal to 0. It is non-zero only when l is equal to 0, at l is equal to 0 it has some finite value. So, now, you can see what is going to happen.

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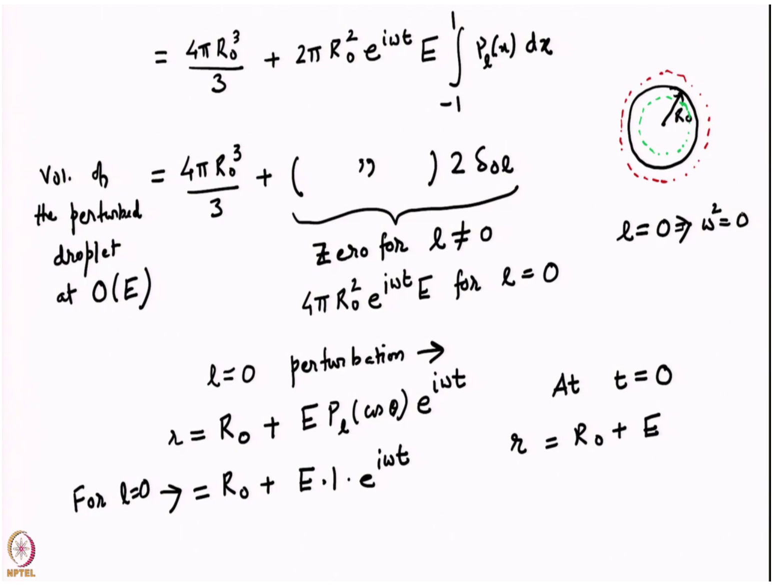

So, our volume integral was so, our volume of the perturbed droplet was of this form plus twice pi R naught square e to the power i omega t into E into the integral which is minus 1 to 1 P l of x d x. And this we have found plus the same quantity and this minus 1 to 1 P l of x d x is just 2 times delta 0 l. So, only for l equal to 0, so, this entire expression will be 0 for l not equal to 0 and it will be equal to 4 pi R 0 square e to the power i omega t into E for l is equal to 0.

Now, this is a very interesting conclusion. Recall that this is the volume of the perturbed droplet up to linear order at order eta or order E; eta was directly proportional to E. So, at order E we are finding that the volume of the perturbed droplet is equal to the volume of the unperturbed droplet plus a correction.

The correction is non zero only for l equal to 0. So, this implies that for all other perturbations P 1, P 2, P 3, P 4 and so on all of these are volume conserving perturbations up to linear order. It is only the l equal to 0 perturbation, which actually changes the volume of the droplet at for l equal to 0.

This is also easy to see if you say see that we have written r is equal to if you if I express the surface as spherical surface as E times e to the power or rather E times P l of cos theta into e to the power i omega t. You can see that this actually leads to for l is equal to 0 this leads to r is equal to R naught plus P P of 0 of cos theta that is just 1 into e to the power i omega t.

You can see that at time t equal to 0 this will be r is equal to R 0 plus some constant E and you can see that this actually represents a sphere. So, if this is my sphere of radius R naught, this actually represents a sphere. If I impose a l equal to 0 perturbation, this actually represents a bigger or a smaller sphere depending on whether E is positive or negative ok. So, you can have a bigger sphere or a smaller sphere.

So, you can see that the volume of the sphere is actually increased or decreased even at linear order by the perturbation l is equal to 0. Now, recall that we have done this calculation under the incompressible assumption. This implies in the incompressible limit my in this problem density is a constant and consequently the volume is conserved.

So, if I put a perturbation it changes the volume of the droplet. If I think of this as a droplet with a fixed amount of mass if I change if I deform the surface the volume of the deformed droplet under the incompressible limit is the same as the volume of the undeformed droplet.

So, this perturbation is not allowed under the incompressible limit that reflects as a 0 frequency in this calculation. So, that is why l is equal to 0 implies omega square is equal to 0. You can go back and check the formula for omega square that we have derived. There if you substitute l is equal to 0 you will get omega square is equal to 0 and that is because we are under the incompressible approximation and so, volume changes are not allowed.

This is a mode where volume changes happen and this is a pure radial expansion or a radial contraction. This mode of deformation is very relevant for oscillations of bubbles where the compressibility of the gaseous medium inside plays a role and the bubble oscillates in a pure radial mode.

However, those are volume oscillations here we are looking at shape oscillations. So, this explains why l is equal to 0 gives you a 0 frequency. What about l is equal to 1? Let us understand that.

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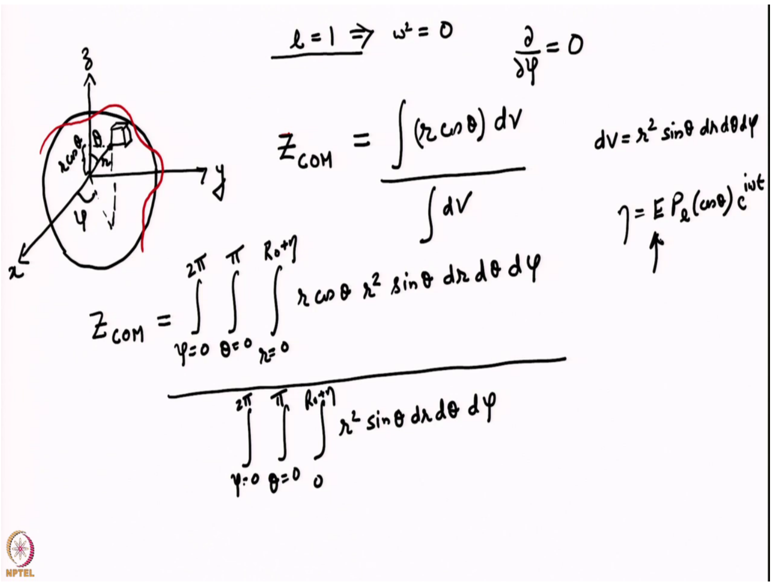

So, l is equal to 1, we are now trying to understand l is equal to 1 implies omega square is equal to 0. We have seen that before. Now, in order to understand the l is equal to 1 leading to omega square equal to 0, we will have to again do the same thing. We will have to put our single Legendre the Legendre polynomial on the surface of the droplet and this time we are going to calculate the center the coordinate of the centre of mass of the droplet ok.

So, let me draw the coordinate system once more. So, this is my z axis, this is x, this is y. We are going to do all of this in a spherical coordinate system. And suppose I have some this angle is theta, this angle is psi and all of this is under the axisymmetric approximation. So, del of del psi is 0.

And suppose I have a volume element here inside the sphere and I want to calculate the center of mass of the perturb sphere ok. So, I am going to put a perturbation. So, I am going to put some kind of a perturbation here on the sphere in the form of a Legendre polynomial and I am interested in calculating how much does that perturbation change the centre of mass of the resultant object.

In the unperturbed case we know if we set up our coordinate system at the center of the sphere we know that the center of mass coincides with the origin. The question that we are asking here is suppose I deform the sphere using a Legendre polynomial P l of cos theta, in this coordinate with respect to this coordinate system where is the center of mass of the deformed sphere.

We will in particular look at the Z component of the centre of mass. The Z component the Z component of the center of mass by definition is given by this integral r cos theta. So, you can see that r cos theta is just this distance this distance. This is r and so, this is r r cos theta into the small volume divided by the total volume.

Let us do this integral both the integrals in spherical coordinates. So, like before, so, I am just going to take some more space here. So, Z center of mass is equal to I will keep the psi integral first that is the easiest one to do then the theta integral and then the r integral. We then have r cos theta and then the dv in spherical coordinates is dv is just r square sin theta dr d theta d psi. So, is just going to be r square sin theta dr d theta d psi and similarly at the bottom and we can calculate this.

So, again the same thing psi is equal to 0 to 2 pi theta is equal to 0 to pi 0 to R 0 plus eta this is the small r integral and then the same thing is just dv. So, it is just r square sin theta dr d theta d psi. The lower integral we have already calculated. We have just calculated it in the last calculation. So, I can straight away replace it from here. Again we are doing this under the linearized approximation.

So, I am going to retain only terms up to order eta and when I replace eta as E times P l of cos theta into e to the power i omega t then I am only going to retain terms which are up to order E. So, I only have to work on my integral at the top. The integral at the bottom up to linear order has already been worked out in the last calculation.

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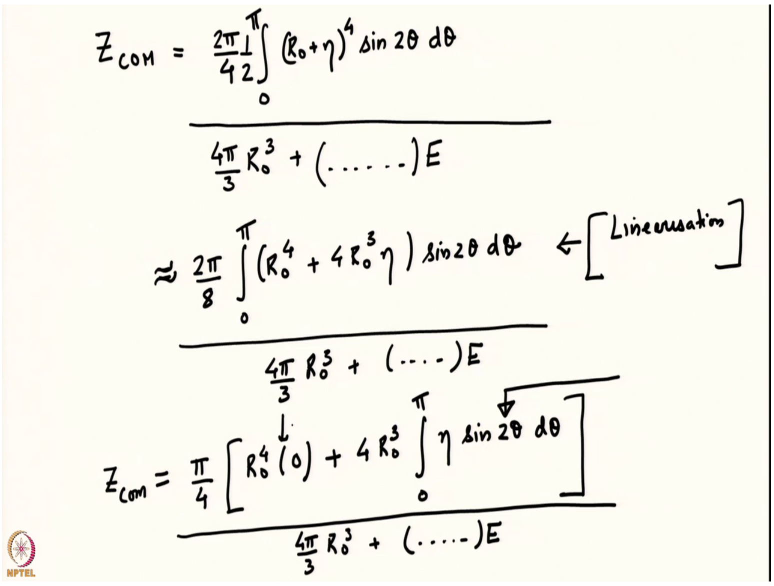

So, let us proceed. So, Z center of mass is equal to; so, I have a r cube in my integral in the numerator. So, the r cube will come will become r 4 by 4 when I do the integral over small r. The psi integral can be pulled out, it is just gives me a factor of 2 pi and so, I will have outside a 2 pi which is coming from the psi integral and divided by 4 in the denominator which is coming from integral r cube dr.

So, I have 2 pi by 4 and then I am left with just the theta integral which is 0 to pi R 0 plus eta. So, r 4 becomes R 0 plus eta to the power 4 into. So, I will put a factor of half here and then I will make this sin 2 theta d theta. Recall that there was a sin theta and a cos theta in the numerator.

So, if I multiply and divide by half if I multiply and divide by 2 then I get sin 2 theta divided by 2. I pull the 2 out. So, I have 2 pi by 4 into half into R 0 plus eta to the power 4 into sin 2 theta d theta. And the integral in the denominator we have already worked out up to linear order. I am not going to write this here because what I write here will not matter in my eventual calculation. You will see that this term does not appear in the eventual linearized calculation. So, I am just writing it as some something into E.

So, now this is 2 pi by 8 into 0 to 4 and this just gives me R 0 to the power 4 plus 4 R 0 cube into eta into sin 2 theta d theta. So, I am just going to write this as 4 pi by 3 R 0 cube plus and now let me work on the numerator. So, I have pi by 0 to pi. So, I have a pi by 4 here and then so, let me write this as R 0 to the power 4 integration 0 to pi sin 2 theta d theta which will just give me a 0 plus 4 R 0 cube into 0 to pi eta sin 2 theta d theta and then I have divided by 4 pi by 3 R 0 cube plus into.

So, as you can see the we are trying to calculate the center of mass and you can see that it is this integral that we have to basically work out. Again recall that I have done some linearization here in this step, which is why I have only two terms in the expansion of R 0 plus eta to the power 4. So, this should be actually an approximation. So, now, let us focus on this term.

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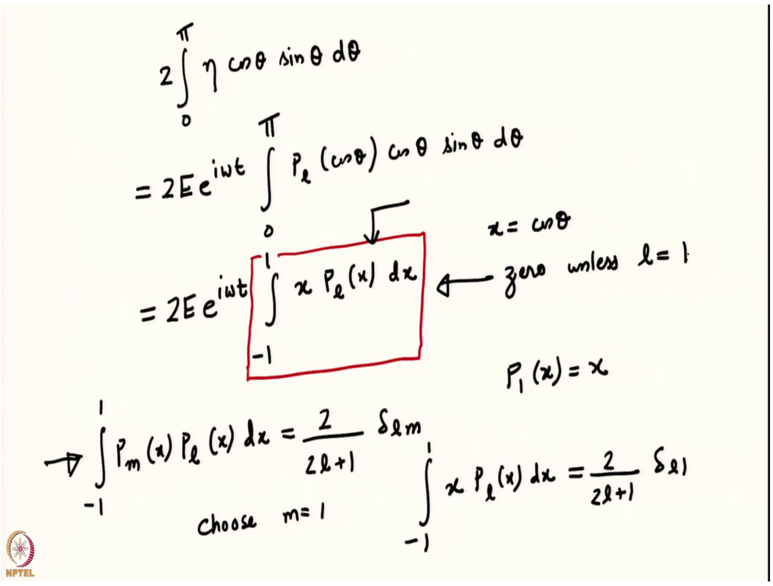

So, we are interested in a term of the form integration 0 to pi P l or yeah eta of sin 2 theta d theta ok. So, I will just put a factor of 2 out and I will still write it as eta into cos theta sin theta. It is easier to do it this way and so, this is just 2 E e to the power i omega t into 0 to pi P l of cos theta into cos theta into sin theta into d theta.

So, now these things do not matter and you can easily show that this can be converted into the standard form x times P l of x dx, x like before is once again cos theta just a simple substitution we will do. So, now we are interested now in a integral which is related to what we did earlier, but now it has an additional term. So, this is the integral which will decide what is the Z component of the center of mass.

So, let us work on this integral. So, we have seen earlier that minus 1 to 1 P m of x P l of x dx. I written it as P m P n, I am now replacing n by l. So, this just becomes 2 into 2 l plus 1 delta l m that is the Kronecker delta. We will use this integral to we will use this orthogonality relation to use this integral as well. How? You can see that for P 1 of x. Recall it was x. Earlier we had used P 0 of x, now we are using P 1 of x that will give me an x and so, I can use this orthogonality relation to calculate the value of this integral.

So, let us do that. So, choose m is equal to 1. This gives me the integral minus 1 to 1 x times P l of x dx and this is exactly what I want here. So, I can evaluate it from the right hand side of this. I have chosen m is equal to 1. So, this just becomes 2 by 2 l plus 1 delta l 1.

Once again we have the relation here now that this integral is 0 unless l is equal to 1. This is very interesting because recall the structure of what we had written. We had written it as this is the Z component and this is 0, this integral is 0, this term is 0. So, you can see that the second integral in the numerator will be 0. The second integral is proportional to E. So, let me write it like this.

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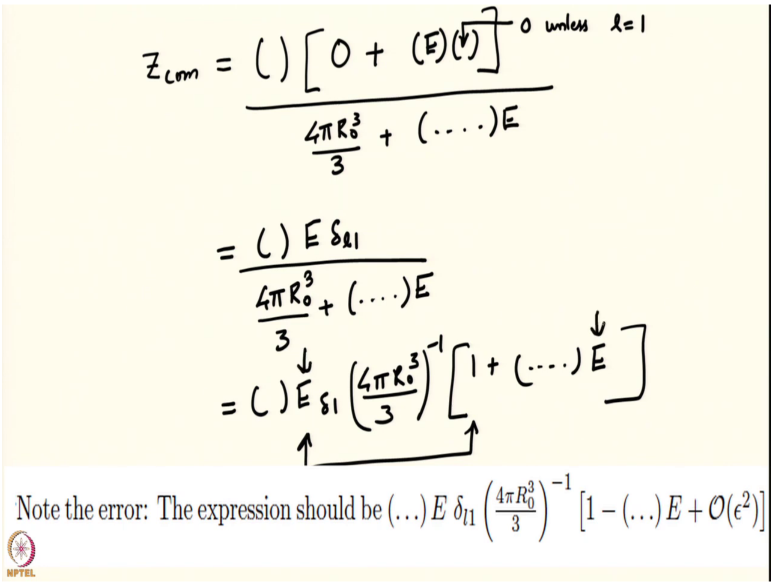

So, Z component of center of mass is equal to some pre factors here into a 0 plus something which is proportional to E, but which will be 0 whatever is here. So, there is a E here and then some other factors which do not matter and whatever is here will be 0 unless l is equal to 1 ok. And what we have in the denominator is 4 pi R 0 cube by 3 plus something into again something which is proportional to E.

So, you can see that the numerator is basically has the form some pre factors into E divided by 4 pi. So, again into something divided by. So, there what is here is basically delta l 1. The prefactors do not matter. The important thing is there is a delta l 1 here. So, this term is 0 unless l is equal to 1 and so, the denominator is this plus into E.

If I bring it at the top using the binomial theorem then you can see that this just becomes. So, 4 pi R 0 cube by 3 to the power minus 1 into 1 plus sum pre factor into E. If you multiply the first term with the first term then you get a linear term in E. The second term will be order E square because there is one E here and there is another E here. So, the second term will be neglected in a linear calculation.

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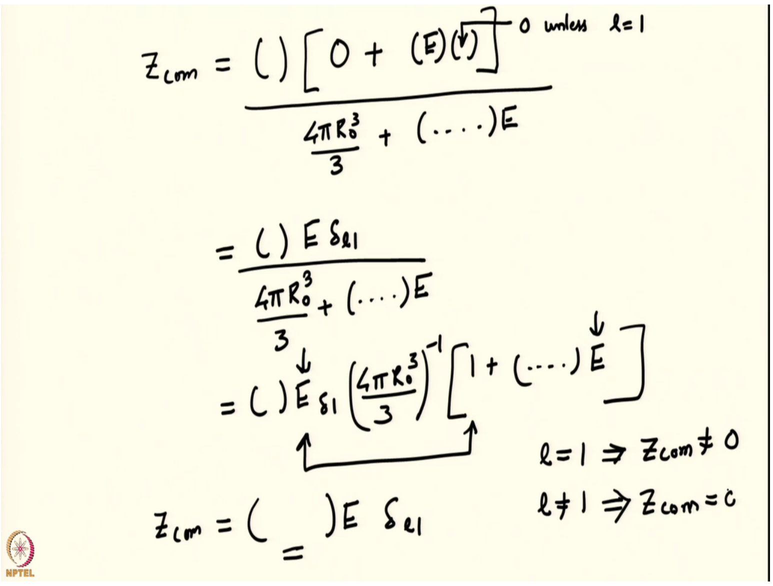

So, we will end up this entire calculation eventually boils down to some pre factors into E into delta l 1 that is what it was done. Whatever is here does not matter so much, we are only interested in is the center of mass displaced or not. And this is telling us that unless l is equal to 1, so, if l is equal to 1 this implies Z com is not equal to 0 in a linear calculation.

Whereas, if l is not equal to 1 for example, l is equal to 2, 3, 4 and so on then this implies Z com is equal to 0 for all other values of l. This is telling me something very interesting that if I put l equal to 1 perturbation it displaces the center of mass of the droplet in the vertical direction.

We have just calculated the Z component of the centre of mass. I encourage you to do the same for the x and the y component and find out whether there are any displacements along those directions, but the Z component definitely gets displaced. Recall, in this course earlier, on that we have done problems of spring mass systems, where one of the frequencies where one of the frequencies evaluate it to 0.

In those cases we found that the eigen vectors would be typically constant displacements. It will be like 111, if it was a 3 degree of freedom system. So, it was like equivalent to taking the system and shifting it. This is also the same reason here.

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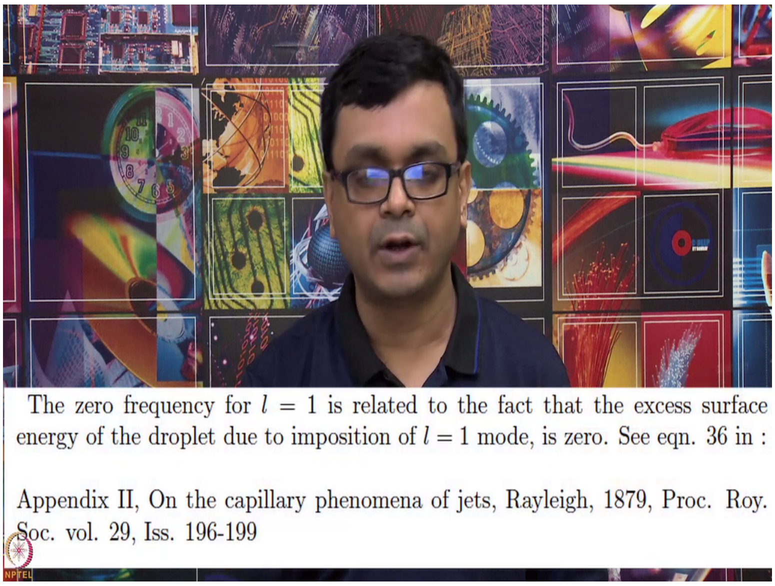

There we had found that the frequency of oscillation of that mode is 0. We have now understood why l is equal to 0 is 0, we have also understood why l is equal to 1 gives 0.

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