Introduction to interfacial waves
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Lecture - 50
Shape oscillations of a spherical interface (contd.)

We were doing normal mode analysis for perturbations about a spherical base state, we had a perturbation velocity potential outside as well as an inside and we had guessed the forms and we had arrived at all the boundary conditions. There were two boundary conditions coming from the kinematic boundary condition the linearized kinematic boundary condition and then there was a boundary condition which is coming from difference of pressure.

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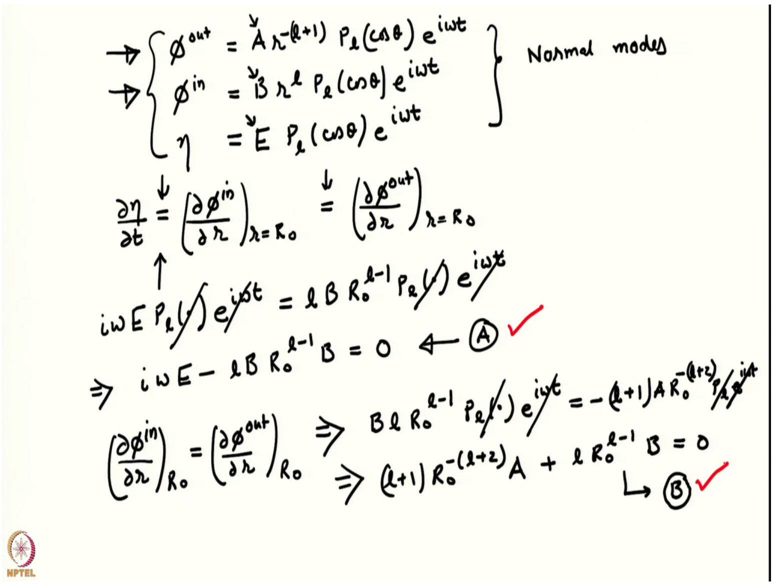

We have already used one of the equalities in the linearized kinematic boundary condition to achieve a homogeneous equation in E and B. So, let us call this equation A where we have utilized this equality let us utilize the second equality. So, I want to set del phi in by del r at R 0 to be equal to del phi out by del r at R 0 small r is equal to R 0 and I have to use those expressions for phi in and phi out and take the derivatives.

If I do that then you can see that for phi in I get l times. So, B l R 0 to the power l minus 1 p l e to the power i omega t this remain intact is equal to del phi out by del r. So, there will be a minus sign a R 0 to the power minus l plus 2 and then we will have a p l and e to the power i omega t. The p l and the p l will cancel, the e to the power i omega t will cancel because they are not 0 at all times and so the other side I will write a first. So, l plus 1 R 0 to the power minus l plus 2 into A plus l times R 0 to the power l minus 1 into B is equal to 0.

This is my second homogeneous equation in the 3 unknowns A, B and E, I will call this equation B. So, I have one equation which is homogeneous in the unknowns which is equation A and I have one more equation I need one more equation and that has to come from the pressure boundary condition that we have just written.

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$$\left(p_b^{in} - p_b^{out}\right)_{r=R_0} + \left(p^{in} - p^{out}\right)_{r=R_0+\eta} = T\left[\frac{2}{R_0}\left(1 - \frac{\eta}{R_0}\right) - \frac{1}{R_0^2}\cot\theta\frac{\partial \eta}{\partial \theta} - \frac{1}{R_0^2}\frac{\partial^2 \eta}{\partial \theta^2}\right]$$

$$p_b^{in} - p_b^{out} = \frac{2T}{R_0}$$

$$\left(p^{in} - p^{out}\right)_{r=R_0+\eta} = -T\left[\frac{2\eta}{R_0^2} + \frac{1}{R_0^2}\cot\theta\frac{\partial \eta}{\partial \theta} + \frac{1}{R_0^2}\frac{\partial^2 \eta}{\partial \theta^2}\right]$$

$$\Rightarrow \rho^{out}\left(\frac{\partial \phi^{out}}{\partial t}\right)_{r=R_0} - \rho^{in}\left(\frac{\partial \phi^{in}}{\partial t}\right)_{r=R_0} = -\frac{T}{R_0^2}\left[2\eta + \cot\theta\frac{\partial \eta}{\partial \theta} + \frac{\partial^2 \eta}{\partial \theta^2}\right] \rightarrow (5)$$

So, equation 5, because equation 5 is a lengthy equation. So, I am going to rewrite this and then use the expressions for phi in phi out and eta to obtain a homogeneous equation in the 3 unknowns from equation 5. So, let us write the pressure boundary condition once again.

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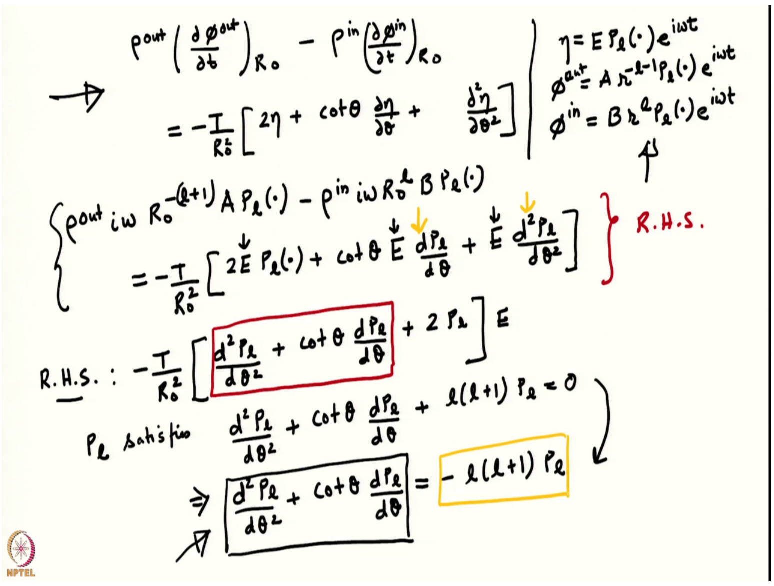

So, rho out into del phi out by del t at R 0 minus rho in del phi in by del t also at R 0 minus t by R naught square twice eta plus cot theta del eta by del theta plus 1 by R naught square del square eta by del theta square.

We have to remember that the expressions for phi in and phi out I will write it here, so that you can follow the algebra. So, eta was E times p l I not write the cos theta i just indicated it as a dot to save some space phi in was A times r to the power minus l minus 1 p l e to the power i omega t phi out sorry this was this will be out and this will be in, B times r to the power l or i omega t. So, we have to simplify this equation using these expressions.

If I substitute it then I get rho out there is a derivative with respect to time. So, i omega R 0 to the power minus l plus 1 because it has to be evaluated at capital R 0 into A into p l of dot that takes care of the first term. Then we will have minus rho in again i omega R 0 to the power l B p l and then there is a e to the power i omega t you know all the terms which I am going to cancel. So, I am not going to write it is equal to minus T by R naught square.

So, twice eta so, that is twice E into p l dot plus cot theta there will be a E and then there will be a d p l by d theta, again I am not writing the e to the power i omega t you can see that all the terms will have e to the power i omega t and that can be cancelled out. So, I am not writing that plus the last term. So, there should be no R naught square here yeah.

So, the R naught square has already been taken common outside the bracket. So, that was a mistake. So, del square eta by del theta square. So, there will be an E here and then this is just d square p l by d theta square.

Now, let us look at the right hand side of this equation, you can see that there are derivatives with respect to. So, let us look at the right hand side the R H S of this equation, we will have to do something about the right hand side without which we cannot just get our dispersion relation because you see what appears on the left hand side is p l of theta in both the terms.

Whereas, what appears on the right hand side. One term contains p l of theta, but the other two terms contain the first derivative and the second derivative of p l with respect to theta. So, if you want a dispersion relation which is independent of theta we will have to somehow convert the right hand side make all the terms proportional to p l of theta. Let us see how.

So, I will take the right hand side of this equation and I will see; I will write it as R 0 square I will write the highest derivative first. So, there is an E over all. So, I am writing I am going to write the E at the end. So, every term has a E. So, I am just going to write the highest derivative which is the last term above, then the next highest derivative which is cot theta times d p l by d theta and then the first term which is twice p l the whole thing get multiplied by E.

Recall that p l comes from the solution to an or second order linear differential equation that was my Legendre’s equation we have written that earlier.

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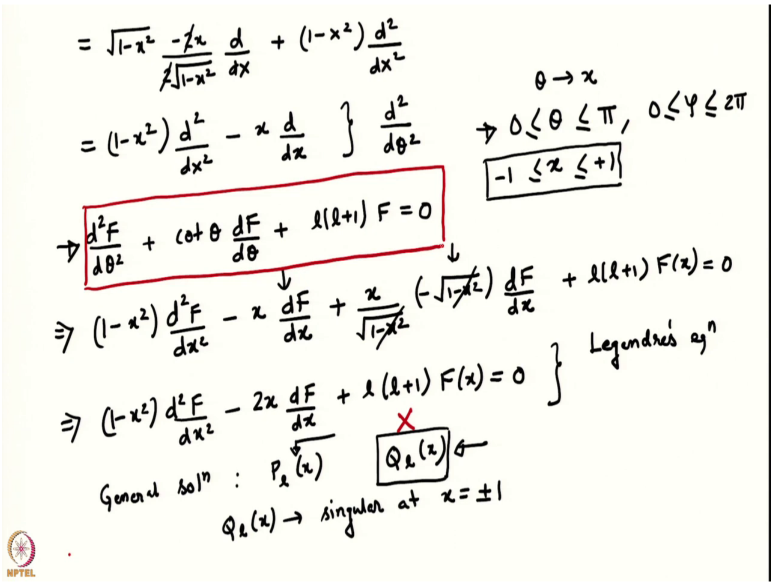

So, p l comes from the solution to this equation. So, you can see that p l satisfies the equation. So, p l satisfies d square p l by d theta square plus cot theta d p l by d theta plus l into l plus 1 p l is equal to 0 all I am doing is just taking p l and replacing it in this equation which I have put inside a red rectangle, you can see that all I have done is I have just replaced capital F by p l.

Why? Because I know that the solution to this equation one of the solutions to this equation is p l of cos theta. So, p l of cos theta must satisfy this equation. So, I know that this is the equation that p l of theta satisfies. So, if p l of theta satisfies this equation then I can use this equation to express d square p l by d theta square plus cot theta d p l by d theta as minus l into l plus 1 p l. This just comes from this equation I have just rearranged this equation.

What do I gain by this? I have seen this expression earlier we have made this expression earlier this is exactly this expression in my boundary condition. So, it allows me to replace the second derivative and the first derivative with just minus l into l plus 1 p l. So, the whole term on the right hand side of this equation becomes proportional to p l because on the left hand side both the terms already have a p l, I can cancel out the p l and what I will get is an equation in A, B and E whose coefficients do not depend on theta that will be my third equation in A B n. Let us do the algebra.

So, I am just going to take this identity that we just wrote and I am going to replace this term plus that term there is a E common. So, I can pull that out and I can replace the sum of those two terms using this, if I do that then this is what I obtain. So, we obtain.

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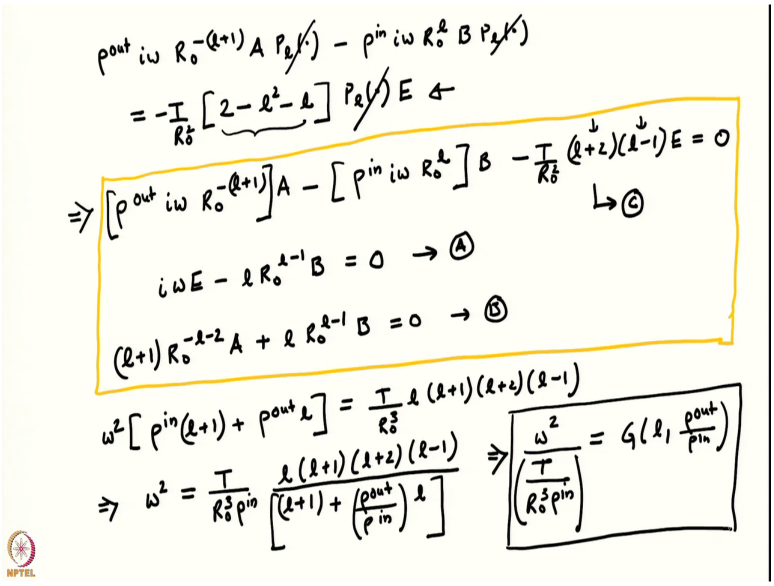

So, the left hand side remains the same. So, rho out i omega R 0 to the power minus l plus 1 I am just rewriting the left hand side A into p l of dot and I am not going to write the e to the power i omega t because that will get cancelled out minus rho in i omega R 0 to the power l B into p l of dot is equal to the right hand side.

The right hand side has three terms the first term is left as such because the first term is just two times E times p l of dot. So, the first term is just this we do not do anything with this and we replace the next two terms using the identity that we have just obtained. So, we obtain inside 2 minus l square minus l it is just l into l plus 1 with a minus sign into p l of dot into E. You see the Legendre's equation has helped us simplify the expression drastically on the right hand side.

Now, you can see that there is a p l everywhere I can cancel it out, once I have done that the coefficients of this equation which contains A, B and E become independent of theta. So, I can get an equation which looks like rho out into i omega R 0 to the power minus l plus 1 into A minus. So, I can keep this in a bracket to remind us that this is the coefficient of A.

Similarly, a coefficient of B rho in i omega R 0 to the power l into B minus I am shifting the term on the right hand side to the left, if I do that and I will keep it as minus because I am swapping I am pulling out a minus from here. So, you can see that I can write this term as T by R 0 square and I can factorize what is inside as l plus 2 into l minus 1, you can multiply and convince yourself that this is correct.

So, that is my third equation. So, I will call it C we had three equations. So, let me write it again. So, we had i omega E minus l R 0 to the power l minus 1 into B is equal to 0 this was my equation A. Then I had an equation again. So, this was obtained from the linearized kinematic boundary condition.

Then I had one more from the linearized kinematic boundary condition which was l plus 1 R 0 to the power minus l minus 2 into A plus l R 0 to the power l minus 1 into B is equal to 0 this is equation B. And then we have equation C. So, those are our three equations in three unknowns not all the unknowns appear in all the equations, but they are linear homogeneous equation algebraic equations in the three unknowns.

Once again we can write it as a matrix equation, as a homogeneous matrix equation and the unknown vector will be A, B and E. The determinant has to be equal to 0 in order to obtain non trivial solutions to A, B and E. If we do that I leave it to you to work out the determinant I will just tell you the dispersion relation, it is a little bit of algebra it is not difficult you can try it out for yourself.

If you work out the determinant of this you have to be careful you have to rewrite these equations such that in every equation. So, if you write it in the order A, B and E then you have to make sure that you are writing down the coefficients correctly. So, for example, equation A has to be written with B first and the term with E as next, similarly equation B equation B is already written in the correct form. So, equation A has to be the two terms have to be switched.

If we work out the algebra then it just turns out that the dispersion list, you can see that it will give us a quadratic equation in omega and that quadratic is the following. So, it is omega square into rho in into l plus 1 plus rho out into l is equal to T by R naught cube l l plus 1 l plus 2 into l minus 1. You can see where this is coming from the l plus 2 and the l minus 1 was already there. So, and then we are collecting more and l into l plus 1.

I can shift everything to the right hand side except omega square and I can write it as T by R naught cube and I will pull out a rho in because T by R naught cube into density has the dimensions of frequency squared. So, whatever is left must be a non-dimensional quantity. So, this is l into l plus 1 l plus 2 l minus 1 divided by l plus 1 plus rho out, it is a density ratio divided by rho in because I have taken rho in common into l.

Once again we achieve the same form these are surface tension driven oscillations. So, I can write a non-dimensional frequency as T by R 0 cube into rho in and this is some non-dimensional function of the density ratio and l. So, this is some non-dimensional function G of l and density ratio and we have determined the form for G.

So, this is our dispersion relation. Let us compare this dispersion relation with all the other pure capillary driven dispersion relations that we had obtained so far.

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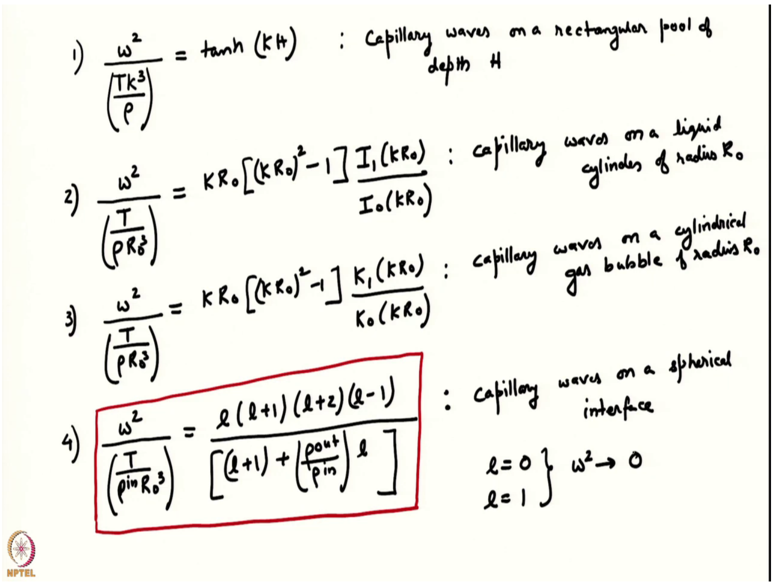

So, we had obtained earlier for pure capillary waves on a pool of finite depth tan hyperbolic K H, the right hand side is non-dimensional this was for capillary waves on a rectangular pool of depth H. We have seen this before.

Then we had waves on a cylinder on a liquid cylinder this was T by rho R 0 cube I am writing the non-dimensional frequency on the left hand side and whatever is left on the right hand side is a non-dimensional function of its arguments. So, this was K R 0 we had found an instability here K R 0 square minus 1 the modified Bessel functions I 1 and I 0, this was for capillary waves on a liquid cylinder ignoring the gas outside of radius R 0.

Then we have found in the same problem we have found T by rho R 0 cube is equal to the same formula K R 0 into K R 0 square minus 1, but now the other Bessel function modified Bessel function and these are capillary waves on a cylindrical gas bubble of radius R 0, here too there was instability. Here in the third formula we are solving for the liquid outside we are ignoring the effect of the gas inside, in the second formula we are solving for the liquid inside we are ignoring the gas outside.

Now, we have added one more to this list and that is omega square by T by rho in R 0 cube is equal to l l plus 1 l plus 2 l minus 1 divided by l plus 1 plus rho out by rho in the density ratio into l. And these are the dispersion relation for capillary waves on a spherical interface, this takes into account the density of both the fluids.

So, this should give you an overall sense of what we have looked at so far. The first problem that we looked at was a Cartesian geometry, the second and the third problem was a cylindrical geometry, the fourth problem was in spherical geometry, these three are the most common geometries that we encounter typically in engineering applications.

And we have looked at waves on (Refer Time: 20:39) base states in all the three geometries with the exception of the first we have ignored gravity in 2, 3 and 4. In the first problem we have included gravity as well and we have found what is the effect of gravity we have looked at capillary gravity waves in writing the formula 1 I have just said G is equal to 0. So, this is these are the various formulas that we have found until now.

Let us look a little bit in more detail at this dispersion relation, recall that l is a positive integer. So, 0 1 2 3 and so on this comes from the subscript of the Legendre function. I have told you the form of p 0 of x p 1 of x p 2 of x and so on. So, you can see that for l is equal to 0 and l is equal to 1 this frequency is 0 because there is a l and l minus 1. So, the frequency is 0 for l equal to 0 l equal to 1.

Apart from these two for everything else for all other positive values of l, l equal to 2, 3, 4, 5 and so on, you can see that this will never give you omega square negative omega square is always positive. So, the first conclusion is that that this base state the spherical base state unlike the cylindrical base state the spherical base state is stable to small amplitude perturbations we always get oscillations or waves we never get instability. So, this dispersion relation does not contain any instability.

Now we will understand the l equal to 0 the l equal to 0 and the l equal to 1 limit. So, because the both of these predict that omega square is 0, 0 frequency relation ok. In one of the assignments early on in the course we have seen examples where examples of mechanical systems where the one of the normal mode frequencies was 0, one of the cases here is related to that.

Let us look at this dispersion relation in slightly more detail now. So, we will write two limits of this dispersion relation.

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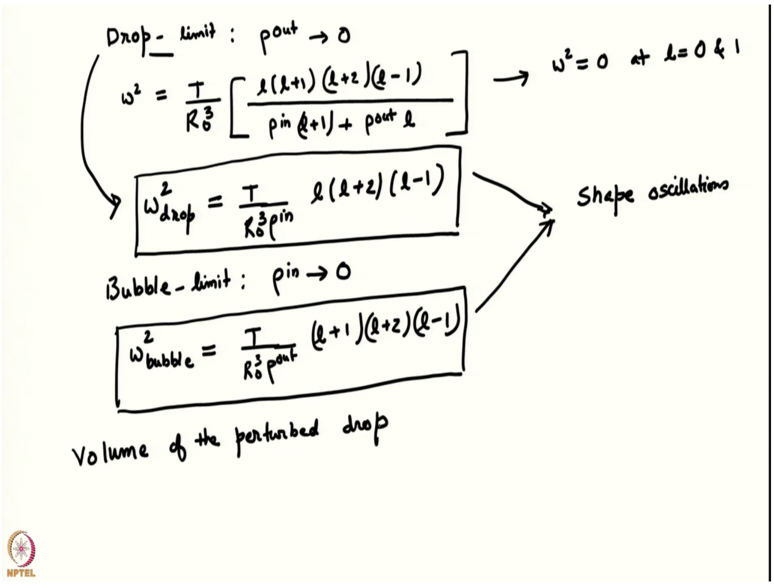

First, so, one is the drop limit, the drop limit is the limit where we are saying that the outer fluid is a low density fluid typically a gas and so, we can ignore the density of the outer fluid compared to the inner fluid. So, it is a liquid bubble surrounded by let us say air. So, a water bubble in air rain drop is a good example.

So, the drop limit we set row out to 0 retaining row in, if we do that then the dispersion relation simplifies. So, our dispersion relation is. So, I am just going to write it in dimensional form T by R naught cube and I am going to pull the push the row in inside. So, that is easier for me to set things to 0 l plus 2 minus 1 divided by there was a rho in. So, this is rho in into l plus 1 plus if I push the rho in inside then this ratio just becomes rho out into l.

So, now if I take the drop limit of this dispersion relations so, then omega square of drop is equal to. So, I just set rho out equal to 0. So, the l plus 1 and the l plus 1 get cancelled out, I get a rho in here and then I have l into l plus 2 into l minus 1. This is a very well-known dispersion relation it is used frequently in analyzing the natural oscillations of liquid droplets.

There is a bubble limit also bubble limit, here we set row in to 0 we are saying that there is a liquid outside and there is a gaseous medium inside. So, we are going to solve for the liquid outside ignoring the gas inside that is a that is the bubble limit, you can think of an air bubble in water in that case the density of air is negligible compared to water.

So, we will set the reverse we will set rho into 0. So, omega bubble square is equal to. So, now, we are going to set rho in to 0. So, I will get a T by R naught cube into rho out, the l and the l will cancel out and so, I will get l plus 1 l plus 2 into l minus 1 this is the dispersion relation for bubble.

Note that these govern what are known as these are all oscillations, these are frequencies which tell us the frequencies of small amplitude oscillations note that these are shape oscillations. What does that mean, this implies that there is a shape, but at linear order there is no change in volume.

So, these are shape oscillations. In particular in the case of bubbles volume oscillations are also possible when the air behaves as a compressible medium inside the gaseous medium behaves as a compressible medium inside, these dispersion relations govern shape oscillations and not volume oscillations.

So, let us understand the origin. So, as I said earlier the full dispersion relation gives omega square is equal to 0 at l is equal to 0 and 1, why is the case let us understand this in a little bit more detail, for this we will need to calculate the volume of the perturbed drop.

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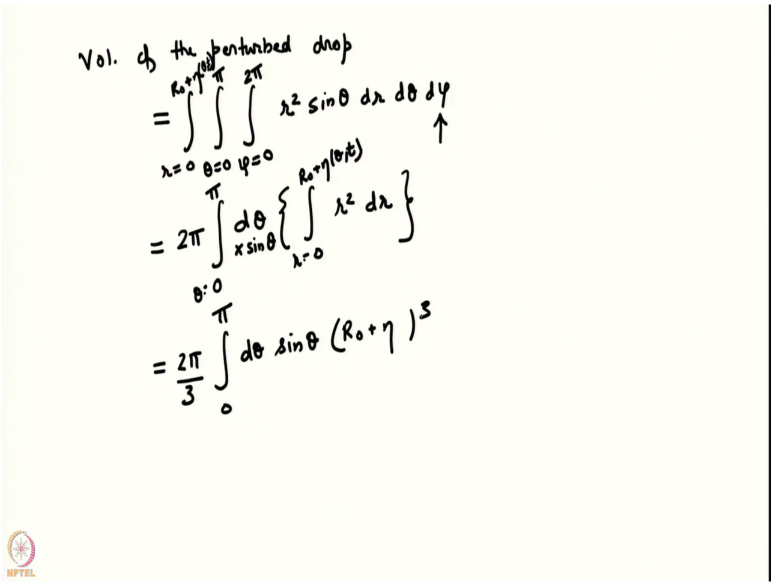

In spherical coordinates the volume of the perturbed drop the volume of the perturbed drop in spherical coordinates has to be computed through integration, in r theta phi coordinate system that is spherical coordinates the volume of the perturbed drop is given by this this integral.

So, r goes from 0 to R naught plus eta and eta itself is a function of theta and T and theta goes from 0 to pi and psi goes from 0 to 2 pi nothing really is a function of psi. So, the psi integral will be the easiest to evaluate. So, this is r square sin theta d r d theta d psi you can look this up in any text book on calculus, this is the volume of the perturbed drop we have put some surface perturbation at the drop and we are calculating what is the volume of this perturbed drop.

So, the psi integral is the easiest to do as I said earlier that is because none of these quantities depend on psi, the eta here is a function of theta and time. So, I can do this psi integral and that will just give me a 2 pi because the limits of psi is 0 to 2 pi.

So, the psi integral goes out and I am left with let me write the theta integral theta equal to 0 to pi because that will be the integral which will be done last into d theta and then we have the r integral which is 0 to R naught plus eta which is a function of theta. So, it will affect the theta integral and this one will be r square I should keep a sin theta here and then r square d r. If I do this then this just becomes 0 to pi d theta sin theta and r cube by 3 so, this will become R 0 plus eta whole cube.

We will continue this integral in the next video.