Introduction to interfacial waves
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Lecture – 05
Coupled, linear, spring-mass systems: continuum limit

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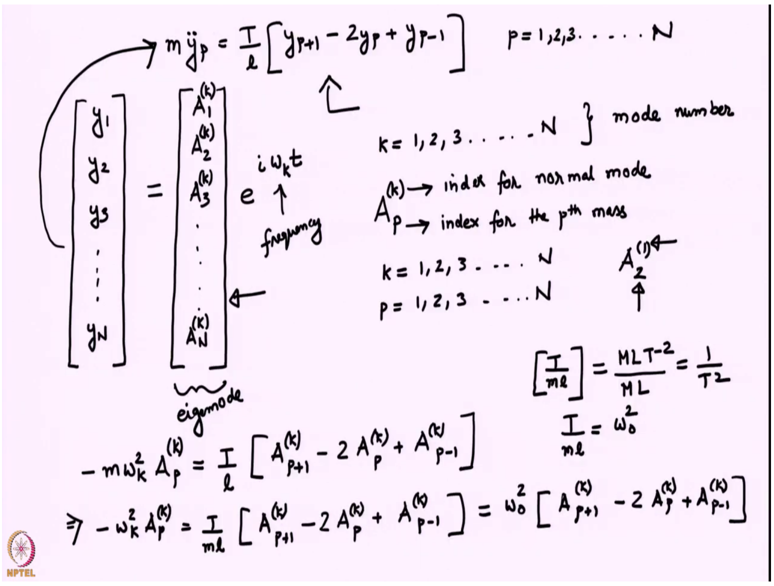

Recall that in the last class we had discussed the equations of motion of N coupled masses. Now this was inherently a non-linear system and we had linearized about the base state or the equilibrium state for small angles and we had got a coupled set of ordinary differential equations.

In this particular way of doing things we had bypassed the matrix method and we had found out the solution namely the eigenvectors and the frequencies of oscillation without explicitly writing down matrices and keeping the number of masses in the system arbitrary, but finite.

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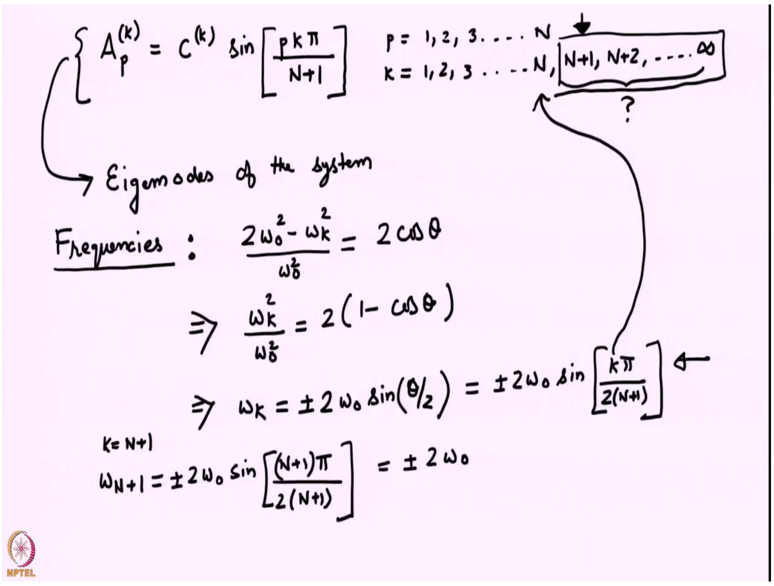

Now we had also found we had got a single formula for the eigen modes of the system, we also had a formula for the eigen frequencies of the system.

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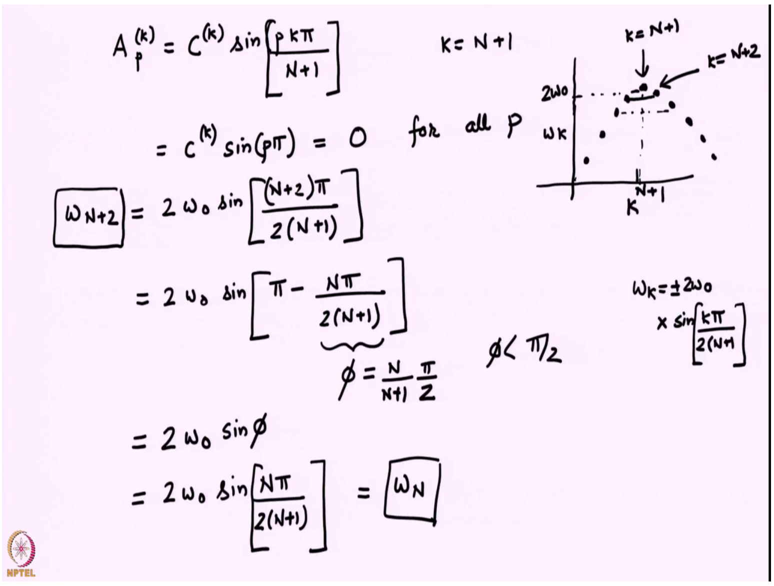

Now, this could we had tried this for, we had also shown that there are certain other extra values which appear and we had argued that these values do not contain anything new for the N plus 1th case it actually gives you a 0 eigen vector and from here onwards N plus 2 all the way up to infinity it is actually a repeated eigen value and a repeated eigen vector.

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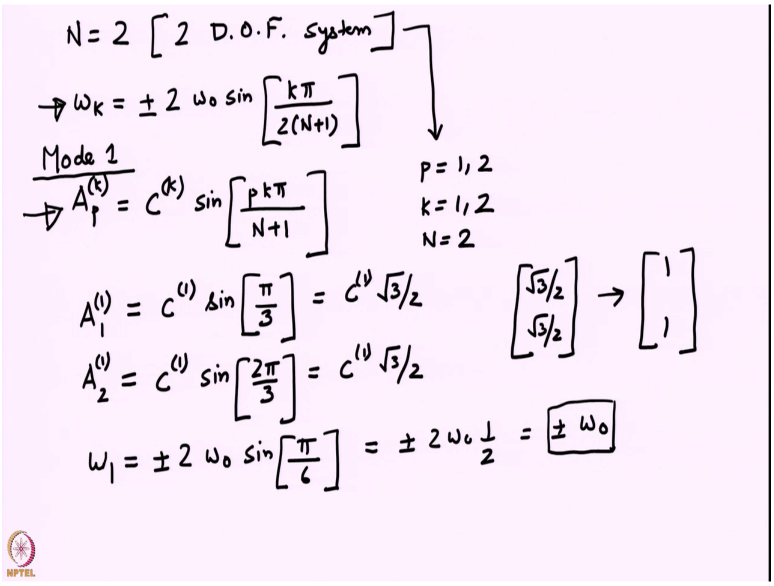

Now, we had applied this to the case of a two degree of freedom system which by definition can only have at most two normal modes. So, mode 1 we had found was exactly the same as what we had found earlier which was eigen the eigen mode was 1 1 and the eigen frequency was plus minus omega naught mode 2 was 1 minus 1 and the frequency was plus minus root 3 omega 1 which was also identical to what was obtained earlier.

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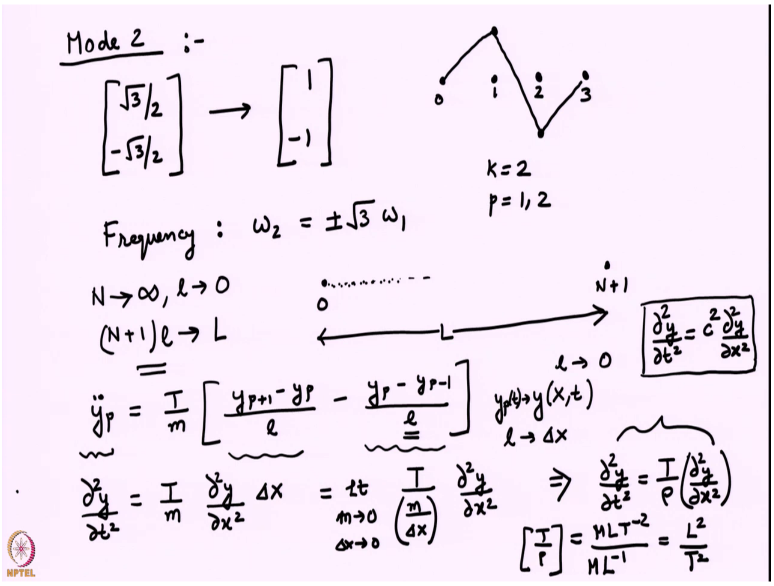

Now, we also said that it would be instructive to take the limit of the number of masses going to infinity. We are going to continue further with this in particular we had applied this limit of capital N the number of masses going to infinity. If I put if I put more and more masses in between two walls separated by a fixed distance capital L then the gap between the masses goes to 0.

So, N goes to infinity capital N goes to infinity small l goes to 0 in such a way that capital N plus 1 into l the gap between the two walls is remains fixed. So, that limit goes to capital L. Now in this limit we had shown that our if you have to write down the ordinary differential equations of governing the pth mass and then if we apply this limiting process, we had argued that it goes over to the linear wave equation.

The linear wave equation was of the form del square y by del t square is equal to T by rho into del square y by del x square. So, this is a recap of what we have done until now let us continue from there.

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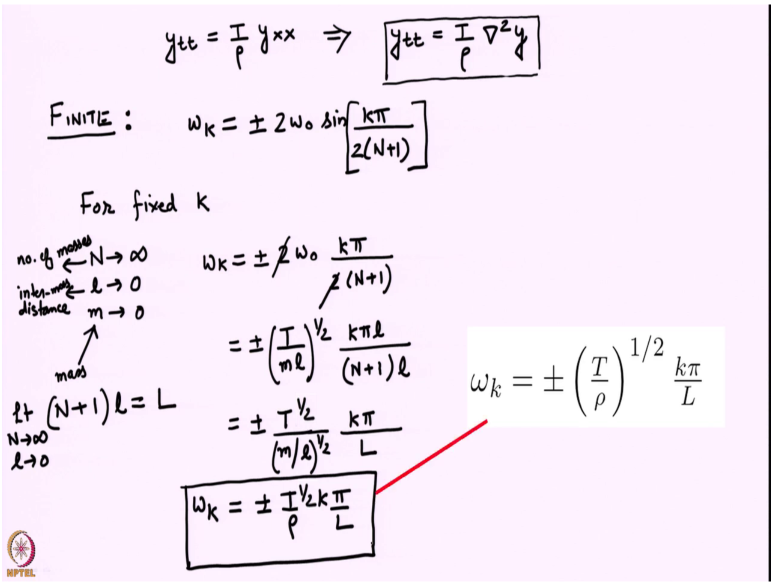

So, we have found the linear wave equation to be y tt is equal to T by rho y x x. In the more general case where there is more than one spatial dimension this would generalize to yt t is equal to T by rho grad square y where grad square is our usual scalar Laplacian and it will have del square by del x square del square by del y square y in that case would be another spatial dimension we will come to one such example shortly.

Now, you can see that we have gone over from a finite number of degrees of freedom system to an infinite number of degree of freedom system or namely a continuous system. So, our equation governing oscillations has become a partial differential equation. Let us explore this limit of going from a finite number of degrees of freedom system to an infinite degree of freedom system a little bit more.

For the finite degree of freedom system, we had found earlier that the eigen frequency of the kth mod is given by this formula. Now let us apply this limiting process and find out what happens to the eigen frequency as I put more and more masses into the system or in other words as capital N goes to infinity.

So, in this process we will have to do this limit for fixed k. Recall that k is an index for the which mode of oscillation is contained in k. So, if k is the second mode then we fix k. So, suppose we are taking the limit N going to infinity I am holding k fixed which means that if for a finite N.

So, suppose my system contains 10 degree degrees of freedom or in other words 10 masses and suppose I am looking at the second mode of oscillation the second normal mode. So, k is equal to 2. So, I hold k constant and I take and I put more and more masses into the system ok.

So, N becomes more and more 10, 100, 1000, 100, 1000, but I am always looking at the second mode of the system ok. And I am interested in what happens to this frequency as the number of masses capital N goes to infinity. So, you can immediately see that this if I put capital N going to infinity then this becomes this is like what I have in square brackets is like sin of theta.

So, for because capital N is in the denominator. So, capital N going to infinity will make theta going to 0. So, for sufficiently small theta sin of theta is just theta the first term in the Taylor series expansion.

Now, we have to do something more than this because you see if I just leave it here then k is fixed; if N capital is going to infinity. So, this frequency will just go to 0. Now that is not correct because recall that omega 0 was defined as T by ml to the power half. Now what I am going to do is I am going to multiply numerator and denominator by the quantity l small l.

Recall that as N capital N goes to infinity small l goes to 0 and small m also goes to 0. Capital N is the number of masses, capital L is the inter mass distance and small m is the mass each mass ok. So, in the limit when we are looking at a continuum if I take a smaller and smaller distance then the amount of mass that is contained in that distance become goes to 0.

So, now I so, now, I have to work out the limit here. So, now, you can immediately see that this becomes T; this l in the denominator is l to the power half and then there is a l in the numerator. So, it cancels out and m goes to the numerator whereas, l goes to the numerator as l to the power half I am bringing it down and writing it as m by l to the power half.

And then I am left with the k pi here and we have already seen that under this limit. We are taken this limit in such a way that N plus 1 into l is the distance between the two walls and I had maintained that distance fixed. So, limit N goes to infinity, l goes to 0 this was the inter wall spacing.

So, this is why I have multiplied numerator and denominator by small l and consequently we get this and what is this small m by small l in the denominator? It is nothing but the linear density of the string. So, this gives me T by rho k pi by L omega k let me put this in a box. So, we have the frequency in the continuum limit.

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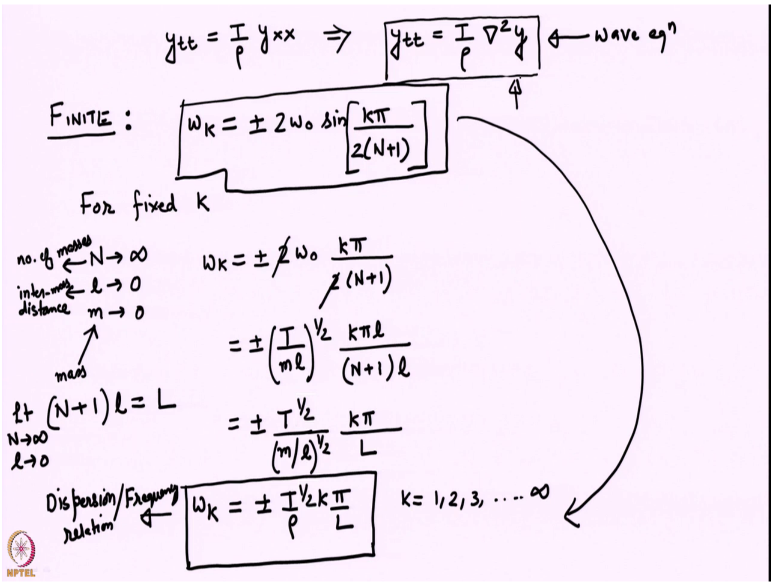

And now k goes from 1, 2, 3 up to infinity. It goes to infinity because now instead of having capital N number of degrees of freedom my capital N has gone to infinity. So, I would expect a countable infinite sequence of frequencies at which the system can vibrate and the frequency relation or the dispersion relation. We will encounter this word many times in this course dispersion or frequency relation is given by this; this formula.

So, now we have started with an ordinary set of coupled linear ordinary set of differential equations. We have taken the continuum limit, we have record recovered a linear partial differential equation namely the wave equation. We have also taken the corresponding frequency relation for finite number of masses and then we took again the continuum limit and we obtained the continuum version of the frequency relation or the dispersion relation.

So, this is what we have done so, far. Now we all also know that for the finite number of masses we know how to write down the solution to our set of equations. It is a linear combination of the eigen modes which we have done earlier let us write it for this case of capital N number of masses.

And you will see that process will actually teach us how to write down the most general solution to this equation for the particular set of boundary conditions that we have been following until. Now namely fixed it is a wall on the left and it is a wall on the right.

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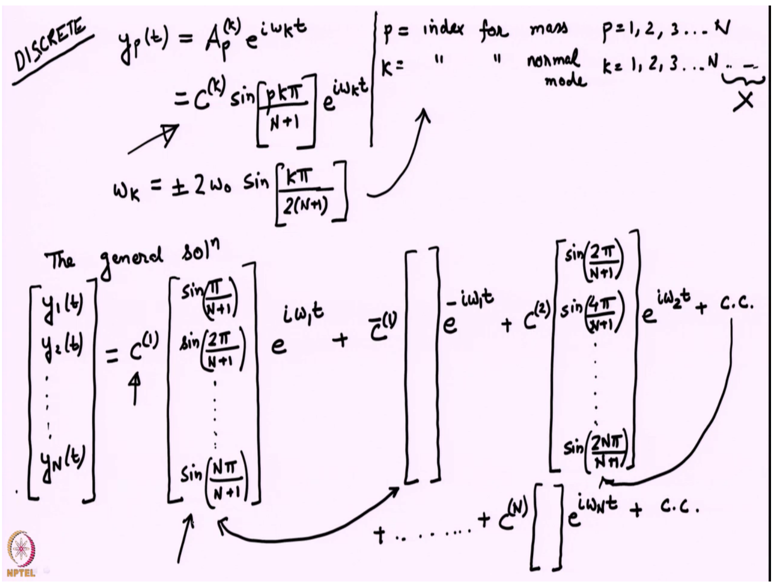

So, let us proceed with that. So we have found in the discrete case we have found that. So, this is discrete. By discrete I mean a finite number of degrees of freedom capital N is finite. So, in the discrete case we had found that Y P of t is basically A P k into e to the power i omega k t. I am just writing what we had already written earlier. Again recall P is an index for mass. So, the Pth mass. So, P goes from 1, 2, 3 up to N and k is an index for normal mode.

And k also goes from 1, 2, 3 up to N and we have convinced our self that anything beyond this N plus 1 N plus 2 all the way up to infinity is irrelevant. So, k is also going from 1 to N. We also found a formula for A P k and that is C k sin and the corresponding eigen frequencies these are all the discrete case.

I am just summarizing the discrete case. Now let us use this to write down the most general solution for the discrete case and we will use that to come up with a solution for the continuous case. So, in all of these things P and k have those things. So, the general solution which if I write it in matrix form these are all the vertical positions of the masses there are capital N of them and the general solution to the equation that we had seen earlier. Let me just recap that equation once.

So, it is this equation that we are talking about. So, the general solution to this is can be written following an analogous procedure that we have done until now in the case where the number of masses is capital N, it will be we expect capital N number of linearly independent eigen modes and capital N number of eigen frequencies and it will be a linear combination of those eigen modes.

So, we can write it as. So, I am going to put a 1 on top of C. So, that this identifies the first eigen mode or the first the constant associated with the first normal mode. So, I will write it like this. Now, this is my formula for the eigen vector. So, I just have to.

So, this is the first eigen vector or the first mode of oscillation. So, k is 1 and I will write N such elements in this column vector for each element starting from first k is always 1 and P will go from 1 to N. So, you can see that the elements in the column vector would be sin pi by N plus 1, then next k again remains 1, P becomes 2.

This is the second mass and like this until we last reach the last mass for which t is capital N. So, that is our eigen vector 1 for the first mode of oscillation e to the power i omega 1 t. Now in order to make this real I have to add the complex conjugate of this. This in general is a complex constant we have seen this earlier. So, I will add the complex conjugate of this the eigenvector remains the same. So, I am not going to write it again.

So, this and this are the same things and then I will add the complex conjugate of this. So, this is there is a plus minus in omega 1. So, they takes into account that and makes it real, but this is only for k equal to 1 we also have more such modes in the system all the way until k is equal to capital L.

So, we will write more and now C will become 2; because we are now dealing with the second normal mode. You can guess what the structure would be now for all the elements in this column vector k will always be 2 and P will go from 1 to N. So, the first term will be 2 pi by N plus 1, 4 pi by N plus 1 k is 2 P is also 2 now for this element and then you have twice N pi by N plus 1 k is 2 and for the last term P is N.

So, 2 N and then the similar procedure e to the power i omega 2 t plus the complex conjugate C C is a term which is very frequently used complex conjugate of this part ok. So, you will add C 2 bar the eigenvector will remain the same and you will multiply it by e to the power minus i omega 2 t. Even that is not enough the expansion does not stop here the expansion continues and you can guess what is the going to be the structure of the last normal mode.

So, it is going to be C N, then you will write an eigen vector I leave it to you to write what its elements would be. For the last eigen vector k will be equal to capital N and P as you go from top to bottom in the vector P will vary from 1 to N. So, the first term would be sin N pi divided by N plus 1 and the last term in the column vector would be sin of N square pi divided by N plus 1 ok.

So, you can do that yourself and then this would get multiplied by omega N into t plus the complex conjugate of this. So, you have to add a C bar of N eigen vector remains the same and then e to the power minus i omega N t. So, each of these terms will have two parts for each mode we have a part plus its complex conjugate.

So, this is the general structure and we have a total of N such pairs ok. So, each term when it is added to its complex conjugate gives you a real answer. So, the whole thing is real. So, this is the structure of our capital N number of degrees of freedom this is the way we would solve it.

And of course, you can now go back to real notation and convert from e to the power i omega 1 t to cos omega 1 t plus sin omega 1 t. So, the coefficient of that would be C 1 plus C 1 bar you can C 1 plus C 1 bar is a real quantity. So, it would be some real number into cos omega 1 t plus i times C 1 minus C 1 bar i times C 1 minus C 1 bar is again a real quantity.

So, it would be some another constant times sin omega 1 t ok. So, you can again write this fully in real notation and it is also clear how those constants are going to be determined they are going to be determined from initial conditions. Now having written down the most general solution to this coupled linear set of equations when the number of degrees is finite, but arbitrary capital N.

Let us now again take the limit capital N goes to infinity and see what can we learn about the solution to our wave equation which we had found in the continuum case. So, let us now explore that a little bit. So, we have seen. So, you can the first thing that you can immediately see is that that as you take capital N goes to infinity. The number of elements in each of these column vectors that I am writing that I have written here.

So, this column vector for example. So, this column vector you can see that the number of elements is related to the number of the number of elements in the vector is related to the number of masses is the number of masses goes to infinity and then this vector will also have more and more elements going over to infinite elements ok.

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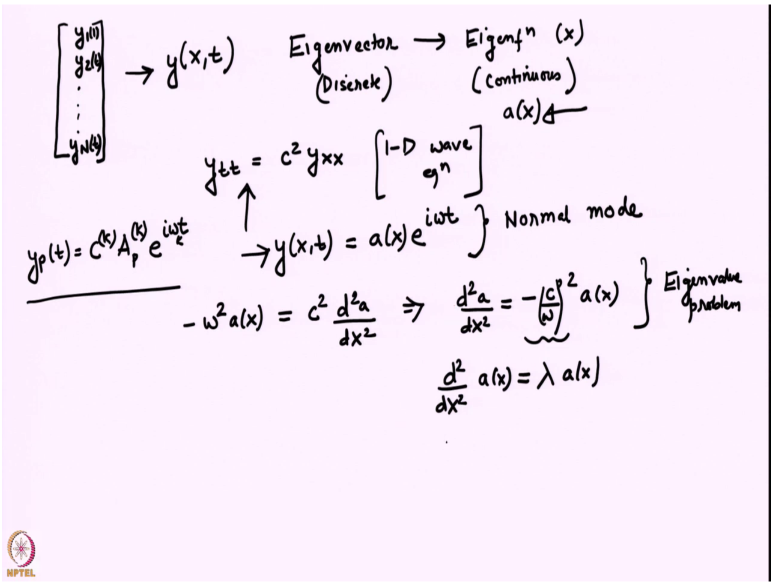

So, intuitively you can imagine that the solution y 1, y 2 in the discrete case where each of these was were functions of time. In the continuous case I would no longer write y 1, y 2 because there are an uncountable number of them and so, I would write y. So, instead of having a discrete index I would have a continuous index and so, y would be a function of xt.

Similarly, each of the eigen vectors that I have written here would also become continuous and so, the eigen vectors would become functions of. So, eigen each of the eigen vector in the discrete case would go over to an eigen function and the function would be of x in the continuous case.

So, you can imagine this intuitively let us call this function a of x and so, I am going to now use this basic idea to now do a normal mode analysis of the wave equation which was obtained by taking the continuous limit of the discrete number of ODEs that we had got ok. So, the wave equation that we had got was y tt is equal to c square y x x.

This is the 1 D wave equation the 1 D refers to 1 spatial dimension. So, now, I am going to do a normal mode analysis earlier in the discrete case I would have done y pt is equal to C k A p k into e to the power i omega k into t in the discrete case. So, in the continuous case this A k p which is basically my eigen vector the column vector goes over to an eigen function.

So, I should write y of x comma t is equal to my eigen function a of x which is what I had written here into e to the power i omega t. Normal mode. So, this would be the our guess for the normal mode form for this equation. Now remember that whenever we make a normal mode of assumption and we substitute it back into our equations of motion it leads always to an eigen value problem. Let us see how does this substitution into that equation lead us to an eigen value problem.

So, if you substitute this you can immediately see that you would get an minus omega square a of x. I would cancel out e to the power i omega t on both sides and then you would get c square d square a by d x square a is just a function of x. So, it is a ordinary derivative. Now you can see that if I write this as d square a by d x square is equal to minus c by omega whole square into a of x this has the structure of an eigenvalue problem.

How? You can see that I can write this remember that a of x is my eigen function it is the equivalent of the eigen vector in the discrete case. So, if I write this as d square by d x square operating on a of x is equal to some quantity which I will call as lambda times a of x.

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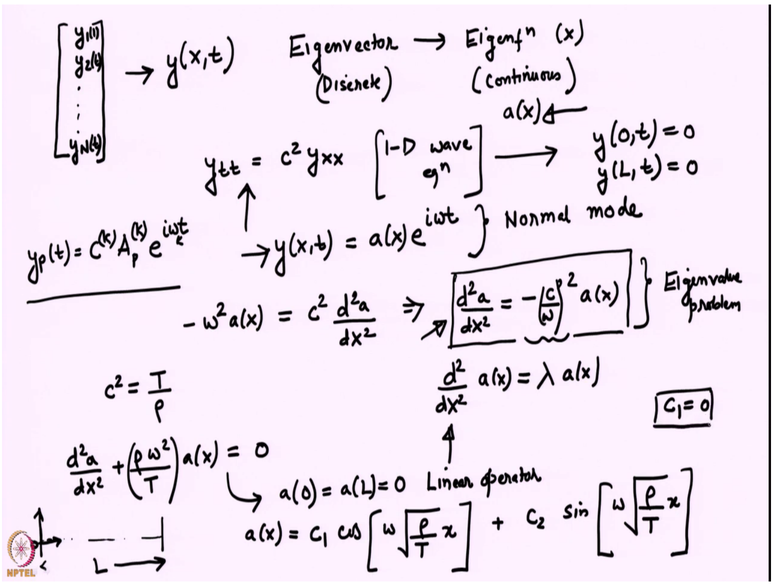

Notice that this is a differential operator, but it is a linear operator. So, one can express it as a matrix. So, a matrix operating on the eigen function is equal to lambda times. The same eigen function this is exactly the prescription of a regular eigenvalue problem ok.

So, you can see that even here this represents our eigenvalue problem and as expected the eigen value is related to the frequency of the allowable frequencies at which the system can oscillate the eigen functions will contain information about the shapes of those oscillations.

Except now the things are slightly more complicated here. Because earlier we had to deal with matrices now we will have to deal with solving differential equations for the eigen functions and boundary conditions will play a very important role here ok. So, now, that we have written it as an eigenvalue problem. So, let us convert it into. So, remembering that C square in the continuous case C square was the tension in the string divided by its linear density rho.

So, if I just take that equation and substitute C square is equal to T by rho I basically get d square a by d x square plus rho omega square by T into a of x is equal to 0. So, this is my equation which governs a of x more importantly we will find that the eigenvalue lambda which is basically minus C by omega square can only be certain values only for certain values will this equation have a non trivial solution ok.

Why this is so? Because there are boundary conditions to be respected. Recall that we are solving this problem for fixed boundary condition. So, we are dealing with a string which in base state is in tension and is flat is held horizontal and there is its base state length is L and we if the string is attached at both sides to a wall.

So, a which is the eigen mode and which is the only quantity which depends on x as far as y x comma t is concerned has to satisfy 0 and 0 at both ends at all times. So, we will have to solve this boundary condition with the restriction that a of 0 is equal to a of L is equal to 0.

This is just coming from the fact that for the corresponding y the boundary conditions were y of 0 at all times is 0 and y of L at all times is also 0. Notice that I have chosen my origin at the left hand corner. So, now, you can see that this is going to impose some restrictions on the possible values that C by omega square or minus C by omega square can take and those values will be the eigen allowed eigenvalues of the system.

So, if I write a of 0. So, the general solution to this equation is easy this was the constant coefficient ordinary linear ordinary differential equation. We all know that a of x is given by C 1 cos omega square root rho by T into x plus C 2 sin the same thing. If you substitute a of 0 is 0; then you can see that the sin term vanishes on its own we are only left with the cos term.

The cos term becomes unity at x equal to 0 and this just tells you that if you have to satisfy the left boundary condition you have to choose C 1 is equal to 0. So, this implies that a of x does not contain a cosine term it only contains the sin term. So, let us work on the sin term.

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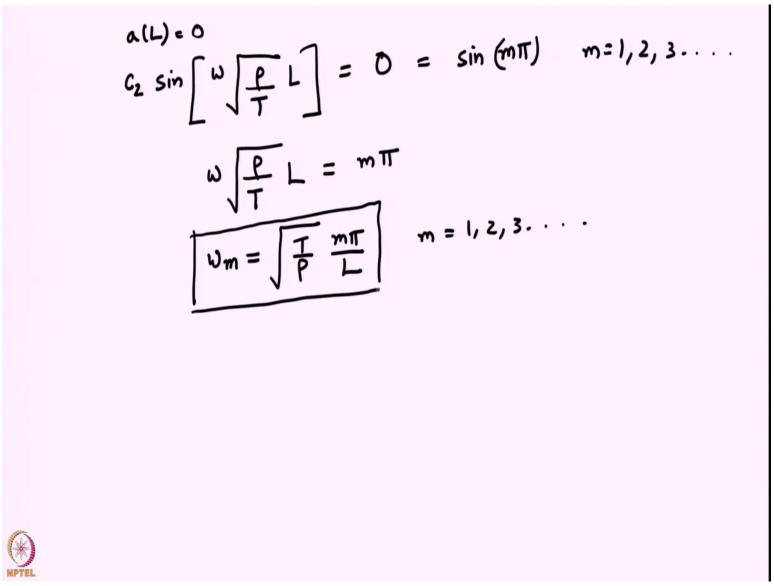

So, a of L is equal to 0 will determine our second constant and that is C 2 sin of omega root rho by T into x is equal to L is 0 and this is equal to sin m pi where m goes from 1, 2, 3 all the way to in positive integers up to infinity. And so, this tells you that omega cannot be any arbitrary quantity; omega it has to satisfy is equal to m pi because m is an integer. So, omega gets discretized.

So, I am going to use an index omega m and omega m is just going to be square root T by rho m pi by L m again going from 1. So, you can see that we have unlike the finite degree of freedom case we have an infinite number of frequencies here. Pay attention that this frequency is exactly what we had recovered earlier this frequency now has been recovered from normal mode analysis.

But we had recovered the same frequency earlier when we had taken the limit of our systems the number of masses in our system going to infinity. You only need to replace k, k is also an integer here. So, you can only need to replace k by m and you will see that this is indeed the same dispersion relation that we are getting in the case.

So, we are going to study this relation, we are going to work out the eigen modes and we are going to plot them in the immediately next video.