Introduction to interfacial waves
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Lecture - 49
Shape oscillations of a spherical interface (contd..)

We were looking at perturbations of a interface which was spherical in the base state. In the base stage there was no velocity in the fluid inside as well as outside. The restoring force was purely due to surface tension we are ignoring gravity here and we have using variable separable solutions to the Laplace equation, we have guessed the form for the velocity potential the perturbation velocity potential in the fluid outside as well as inside, we also have guess the form for eta which is the perturbation at the surface.

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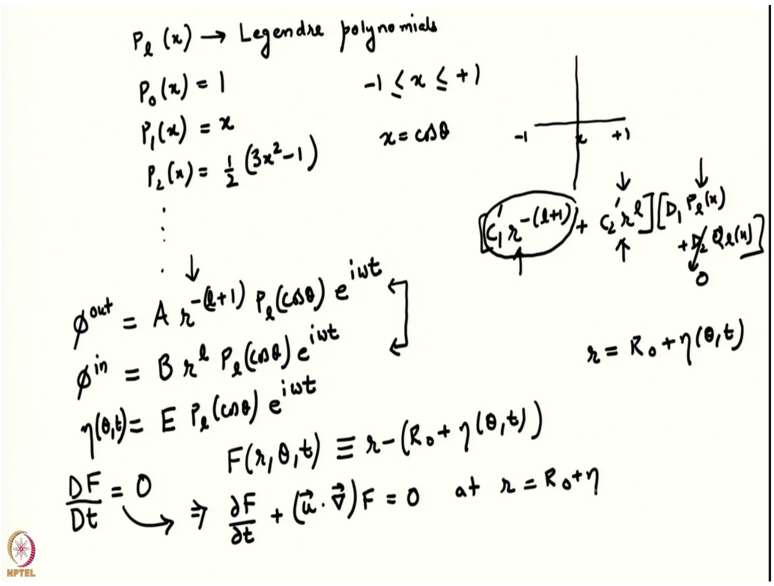

Now, using this we are going to conduct a normal mode analysis. Before we do that let us write down the boundary conditions. We have seen earlier that the kinematic boundary condition is expressed by DF by Dt the total derivative of a function F is equal to 0.

The function F is chosen in such a manner that its value is constant all over the surface. So, F in this case F in this case is going to be a function of r theta and t and this will be defined as r minus R 0 plus eta and eta itself is a function of theta comma t. You can see that the perturb interface is given by r is equal to R 0 plus eta.

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$$r^2\frac{\Phi''}{\Phi} + 2r\frac{\Phi'}{\Phi} - \ell(\ell+1) = 0$$

$$\Rightarrow r^2\frac{d^2\Phi}{dr^2} + 2r\frac{d\Phi}{dr} - \ell(\ell+1)\Phi = 0$$

$$\Phi = c_1 r^{\ell} + c_2 r^{-(\ell+1)}$$

$$\Phi = r^{\lambda}$$

$$\Rightarrow \left[\lambda(\lambda-1) + 2\lambda - \ell(\ell+1)\right] r^{\lambda} = 0$$

$$\lambda^2 + \lambda - \ell(\ell+1) = 0$$

$$\lambda = \ell \quad \text{or} \quad \lambda = -(\ell+1)$$

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$$= \sqrt{1-x^2}\,\frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}\,\frac{d}{dx} + (1-x^2)\frac{d^2}{dx^2}$$

$$= (1-x^2)\frac{d^2}{dx^2} - x\frac{d}{dx} \Bigg\} \frac{d^2}{d\theta^2}$$

$$\theta \to x$$

$$\Rightarrow 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi$$

$$\boxed{-1 \le x \le +1}$$

$$\Rightarrow \frac{d^2F}{d\theta^2} + \cot\theta\,\frac{dF}{d\theta} + l(l+1)F = 0$$

$$\Rightarrow (1-x^2)\frac{d^2F}{dx^2} - x\frac{dF}{dx} + \frac{x}{\cancel{\sqrt{1-x^2}}}\left(-\cancel{\sqrt{1-x^2}}\right)\frac{dF}{dx} + l(l+1)F(x) = 0$$

$$\Rightarrow (1-x^2)\frac{d^2F}{dx^2} - 2x\frac{dF}{dx} + l(l+1)F(x) = 0 \Bigg\}$$ Legendre's eq^n

General sol^n : $P_l(x)$ $\boxed{Q_l(x)}$ ✗

$Q_l(x) \to$ singular at $x = \pm 1$

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$$\Phi''(r)\,F(\theta) + F(\theta)\,\frac{2}{r}\,\Phi'(r) + \frac{1}{r^2}\cot\theta\,F'(\theta)\,\Phi(r) + \frac{1}{r^2}F''(\theta)\,\Phi(r) = 0$$

$$\Rightarrow \frac{\Phi''}{\Phi} + \frac{2}{r}\frac{\Phi'}{\Phi} + \frac{1}{r^2}\cot\theta\,\frac{F'}{F} + \frac{1}{r^2}\frac{F''}{F} = 0$$

reference

$$\Rightarrow r^2\frac{\Phi''}{\Phi} + 2r\frac{\Phi'}{\Phi} = -\overbrace{\left(\frac{F''}{F} + \cot\theta\,\frac{F'}{F}\right)} = l(l+1)$$

$l \rightarrow$ integer $(0, 1, 2 \ldots)$

$$\frac{d^2F}{d\theta^2} + \cot\theta\,\frac{dF}{d\theta} + l(l+1)F = 0$$

$x = \cos\theta$ (x is different from the x earlier)

$$\frac{d}{d\theta} = -\sqrt{1-x^2}\,\frac{d}{dx}, \quad \frac{d^2}{d\theta^2} = \left(-\sqrt{1-x^2}\,\frac{d}{dx}\right)\left(-\sqrt{1-x^2}\,\frac{d}{dx}\right)$$

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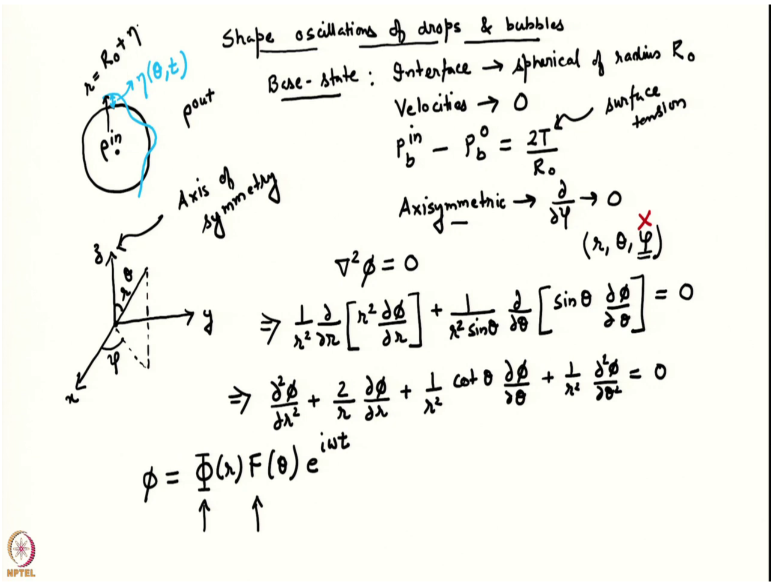

Recall the picture that we had drawn earlier. So, any point on the interface is given by r is equal to the unperturbed radius plus some eta. So, by definition capital F is 0 at the surface at the perturb surface. So, we have to. So, this is the kinematic boundary condition and we have to take the total derivative of F and equate it to 0. So, the total derivative is. So, we are continuing from here. So, this is del F by del t plus u dot grad; u I will write it as grad of phi in the perturbation velocity potential and this is to be applied at r is equal to R 0 plus eta.

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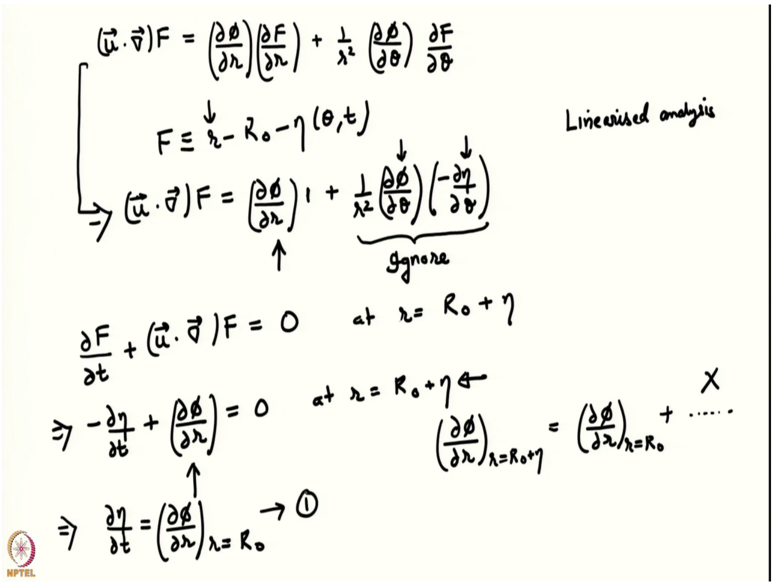

So, let us work out the form for u dot grad of F the second term in the total derivative operator. So, this in spherical axisymmetric coordinates would be del phi by del r into del F by del r plus 1 by r square del phi by del theta into del F by del theta. Remember that F is defined as r minus R naught minus eta which is a function of theta and t.

So, I can write the right hand side of this expression as u dot grad of F is equal to del phi by del r and del F by del r is just 1 is just the derivative of small r with respect to itself plus 1 by r square del phi by del theta will keep it that into del F by del theta which is minus del eta by del theta.

Now, recall that we are going to do a linearised analysis, that is our first approximation phi is the perturbation velocity potential eta is the perturbation at the free surface. So, you can see that this is a product of two perturb quantities. So, this is going to be an order epsilon square term if you had non dimensionalized carefully and we had found the size of every term. So, this would have been if we had done the perturbation expansion this would have been an order epsilon term. So, I am going to ignore this term. So, ignore.

And so, now, our kinematic boundary condition was del F by del t plus u dot grad of F is equal to 0. So, and this is at r is equal to R 0 plus eta. So, del F by del t is just minus del eta by del t and u dot grad of F is just this term because we have ignored the second term. So, plus del phi by del r equal to 0 at r equal to R naught plus eta. Now this condition applies only to this derivative because eta by definition does not depend on r. So, we do not have to worry about this condition on del eta by del t only the second term is what we will have to worry about that where does this derivative get evaluated.

Like before you can see that del phi by del r at r equal to R 0 plus eta may be written in a Taylor series as del phi by del r at r equal to R 0. It is not 0 here its R 0 because in the base state the drop or the bubble has a finite radius. So, this is del phi by del r evaluated at the base state and then we are doing an expansion in r.

So, you will see that there will be a second term in the Taylor series. I leave it to you the second time will also be evaluated at r equal to R 0, but I leave it to you to convince yourself that the second term will be an order epsilon square term. It will be a non-linear term because it will involve derivative of phi and then it will involve an eta.

So, the product of those two is an order epsilon square term we are going to ignore this because this is a non-linear contribution. So, like before our kinematic boundary condition just reduces to del eta by del t is equal to del phi by del r evaluated at the unperturbed surface r equal to R 0 let me call this some equation. So, I will call this may be equation 1 and then let us proceed from here.

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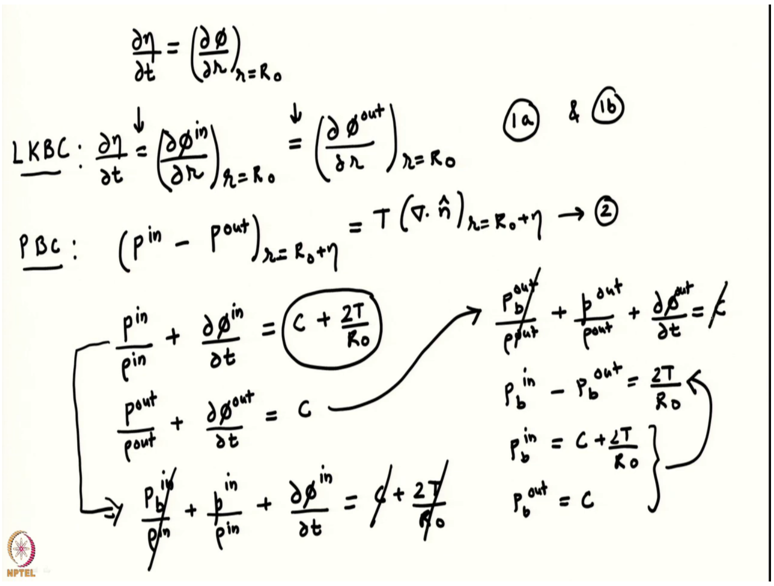

So, now we have found that our kinematic boundary condition, del eta by del t is del phi by del r evaluated at r equal to R 0, but you see now we have two fluids in the inner fluid the velocity perturbation potential is phi in and in the outer fluid the velocity perturbation potential is phi out.

So, this derivative can be evaluated using either of those two potentials that derivative has to be evaluated if we come from the inner side, then it becomes del phi in by del r and if we come from the outer side approaching the interface then it becomes phi out by del r. So, we expect that for this problem these two derivatives should be equal. It does not matter whether I use the inner fluid whether I approach from outwards or whether I approach from inwards, they should evaluate to the same value which is del eta by del t.

So, you can see that the kinematic boundary condition now has two equations there are two equalities this and this. I am going to call this equation 1a and 1b, this is our linearized kinematic boundary condition. So, linearized kinematic boundary. So, that takes care of one boundary condition what about the other?

So, the other boundary condition is the pressure boundary condition. So, the pressure boundary condition which basically says that the difference between p in minus p out at r is equal to R 0 plus eta in the perturb state is T times the divergence of the unit normal and this divergence has to be evaluated at the perturb free surface we have seen this boundary condition before. Recall that p in and p out are the total pressure they are they can be written as a some of base perturbation.

So, let me call this maybe equation 2 and we also have the linearized Bernoulli equation. So, the linearized Bernoulli equation in this case is p in by rho in plus del phi in this is the unsteady Bernoulli equation linearize. So, I do not have the half grad phi square term is equal to once again the Bernoulli constant is not zero here because in the base state there is a pressure jump across the interface.

So, if you evaluate this equation in the base state, it will just give you a pressure jump across the interface. So, I am just going to write it as some arbitrary constant plus twice T by R 0 twice T by R 0 is the magnitude of the pressure jump in the base state between p inside and p outside the difference between the two pressures.

I can write a similar equation for p out by rho out plus del phi out by del t is equal to C. C is just an arbitrary constant, it reflects the fact that only differences of pressure are known the absolute value is not known. So, C is an arbitrary constant and you will see that C will get eliminated in further in our analysis. So, the difference between these two is just twice T by R 0 as we have seen when we wrote down the pressure difference in the base state. So, now, let us take this equation and work further on it.

So, I can write this equation as base plus perturbation the perturbation pressure is written in small small letters rho in plus for the velocity there is nothing more contribution from the wave base. So, the base is 0 and its just pure perturbation velocity potential. So, the same thing is equal to C plus twice T by R 0. Similarly, this equation gives us p b out by rho out plus small p out this is the perturbation pressure in the fluid outside by rho out plus del phi out by del t is equal to C.

You can see that I can cancel out the base state contribution we know that p b in minus p b out is equal to twice T by R 0. If I said p b in is equal to some constant plus twice T by R 0 as I have done here and p b out to be just the same constant C, then you can see that the difference between them satisfies this. So, I will cancel out the base state contribution. So, p b in and this and this are equal to each other.

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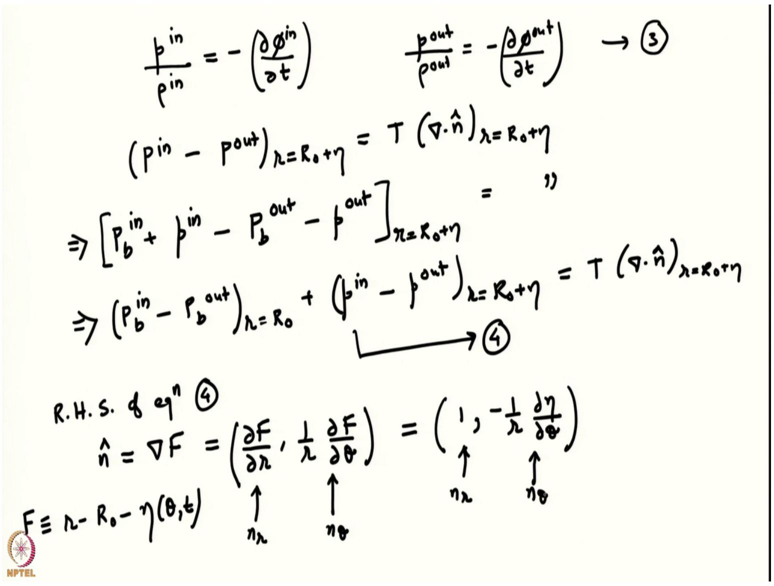

Similarly, you can cancel out the base state contribution these two and so, we are left with two equations for perturbation pressure p in by rho in is equal to minus del phi in by del t and p out by rho out very similar minus del phi out by del t let us call this equation 3. So, now, we return to our boundary condition our boundary condition recall was p in minus p out the pressure boundary condition at r equal to R 0 plus eta is T times divergence of the unit normal evaluated at the perturbed interface.

I can split the pressures as a sum of base plus perturbation similarly base plus perturbation it will become this whole thing evaluated at r equal to R 0 plus eta is equal to the right hand side. I can write this further as p b in minus p b out in the base state r is just equal to capital R naught. So, this is just capital R naught plus p in minus p out evaluated at R naught plus eta is equal to t of divergence of n at R naught plus eta there is no hat I will call this equation, equation 4.

Let us work on the right hand side of equation 4. So, right hand side of equation 4 for the right hand side we will need an expression for n, we have seen that n is evaluated in the linearized approximation as just grad F it is actually grad F divided by mod of the same thing the mod actually is a non-linear contribution. So, the denominator is just 1 at linear order. So, this in spherical coordinates is just del F by del r into 1 by r del F by del theta we are not writing the psi component because this is axisymmetry.

So, that is the r component of n radial component and that is the theta component of n. F is defined as r minus R naught minus eta and eta itself is a function of theta and t. So, I can work on this and I can write it as 1 it is just del r by del r and then this is minus 1 by r del eta by del theta that is n. Once again recall that this is the r component or the radial component and this is the theta component of n. We need the divergence of the unit normal in equation 4, right hand side.

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$$\nabla \cdot \hat{n} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 n_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(n_\theta \sin\theta\right)$$

$$n_r = 1, \quad n_\theta = -\frac{1}{r}\frac{\partial \eta}{\partial \theta}$$

$$= \frac{1}{r^2}2r - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\eta}{\partial\theta}\right] \qquad \frac{2}{R_0\left(1+\frac{\eta}{R_0}\right)}$$

$$= \frac{2}{r} - \frac{1}{r^2}\cot\theta\frac{\partial\eta}{\partial\theta} - \frac{1}{r^2}\frac{\partial^2\eta}{\partial\theta^2}$$

$$\left(\nabla\cdot\hat{n}\right)_{r=R_0+\eta} = \frac{2}{R_0+\eta} - \frac{1}{(R_0+\eta)^2}\cot\theta\frac{\partial\eta}{\partial\theta} - \frac{1}{(R_0+\eta)^2}\frac{\partial^2\eta}{\partial\theta^2}$$

$$\approx \frac{2}{R_0}\left(1-\frac{\eta}{R_0}\right) - \frac{1}{R_0^2}\cot\theta\frac{\partial\eta}{\partial\theta} - \frac{1}{R_0^2}\frac{\partial^2\eta}{\partial\theta^2}$$

Let us compute the divergence. So, the divergence of n is given in spherical coordinates by the following expression 1 by r square del by del r of r square the r the radial component of n plus 1 by r sin theta del by del theta the theta component of n into sin theta. Once again this is a standard formula you can look it up in any book on transport phenomena.

Now, we have seen that the radial component of n is 1, the theta component of n is minus 1 by r del eta by del theta. If I substitute it in the expression above then I obtain 1 by r square del by del r of r square because n r is 1. So, that will just give me a 2 r and then I will have minus because there is a minus in n theta the expression for n theta contains a minus sign.

So, I pull that out outside and I write this as minus 1 by r sin theta the expression for n theta contains a 1 by r I can pull that r outside and make this r square and then I am left with del by del theta of sin theta del eta by del theta. So, this can be simplified to 2 by r minus 1 by r square cot theta, I am carrying out the derivative inside and so, this just becomes cot theta into del eta by del theta minus 1 by r square del square eta by del theta square.

The expression for divergence of n is slightly more complicated as the geometry becomes curvilinear we have seen such expressions even in cylindrical coordinates. It was the simplest in Cartesian coordinates where we just had one term at linear order. Now, recall that this expression has to be evaluated at r equal to R naught plus eta. So, at r equal to R naught plus eta and we have to linearized, we have to retain only terms which are linear in eta.

So, this just becomes R naught plus eta minus 1 by R naught plus eta whole square cot theta del eta by del theta minus 1 by R naught plus eta whole square into del square eta by del theta square. Let us linearize and retain only term which are linear in eta you can see that the first term. So, I can write this as. So, I can write the first term here as 2 by R naught into 1 plus eta by R naught. Eta is a small quantity because I am going to set eta equal to some surface deformation and that is a order epsilon quantity.

I can use binomial theorem to take it to the top and then this just becomes 2 by R naught into 1 minus eta by R naught. You can see that in the second term if I do the same and if I want to retain only up to order eta not beyond that I do not want anything which is order eta square eta cube and so, on then this term will be evaluated just at in the denominator we just have to keep R naught square.

There will be no further contribution and it will just become cot theta into del eta by del theta. Similarly, this one will become R naught square del square eta by del theta square. I encourage you to try doing an expansion and convincing yourself that this is what we would obtain at if we retain terms only up to order eta. Now you can see that all the terms here have an eta in them except in the first term.

So, this is the term which basically comes from the base state and it will get cancelled out let us see how. So, we go back to our equation that we had written equation 4. So, now, we have worked on the right hand side of equation 4, we already have the expression for the right hand side. So, we will use the previous equations to obtain an expression for the base state pressure and the perturbation pressure and we will write down a final equation.

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$$\left(\cancel{p_b^{in} - p_b^{out}}\right)_{r=R_0} + \left(p^{in}-p^{out}\right)_{r=R_0+\eta} = T\left[\frac{2}{R_0}\left(\cancel{1} - \frac{\eta}{R_0}\right) - \frac{1}{R_0^2}\cot\theta\frac{\partial\eta}{\partial\theta} - \frac{1}{R_0^2}\frac{\partial^2\eta}{\partial\theta^2}\right]$$

$$p_b^{in}-p_b^{out} = \frac{2T}{R_0}$$

$$\left(p^{in}-p^{out}\right)_{r=R_0+\eta} = -T\left[\frac{2\eta}{R_0^2} + \frac{1}{R_0^2}\cot\theta \frac{\partial\eta}{\partial\theta} + \frac{1}{R_0^2}\frac{\partial^2\eta}{\partial\theta^2}\right]$$

$$\Rightarrow \rho^{out}\left(\frac{\partial\phi^{out}}{\partial t}\right)_{r=R_0} - \rho^{in}\left(\frac{\partial\phi^{in}}{\partial t}\right)_{r=R_0} = -\frac{T}{R_0^2}\left[2\eta + \cot\theta\frac{\partial\eta}{\partial\theta} + \frac{\partial^2\eta}{\partial\theta^2}\right] \rightarrow 5$$

So, we go back to equation 4 now. So, equation 4 is basically p b in minus p b out and this is not really a function of r, but I will still write it at r equal to R naught because in the base state the pressure is uniform inside as well as outside. p in minus p out at r is equal to R naught plus eta is equal to t times divergence of n evaluated at r equal to small r equal to capital R naught plus eta and we have worked that out up to linear order in eta.

So, up to linear order in eta it is just t into 2 by R naught into 1 minus eta by R naught that is the first term then we will get 1 minus 1 by R naught square cot theta del eta by del theta minus 1 by R naught square del square eta by del theta square. We have already seen that the base state pressure satisfies the relation p b in minus p b out these are uniform pressures is equal to twice T by R 0. You can see that I can cancel out I can use that equation to cancel out this term with the first term. So, the first term if you open the bracket will be twice T by R 0.

So, I can cancel that out that is a base state contribution. So, we are left with just the perturbation pressures which in general has to be evaluated at R naught plus eta, but we will soon argue that using Taylor series that these perturbation pressures have to be evaluated at capital R naught and not R naught plus eta ok.

But let us come to that later. So, you can see that in all the three terms. So, this is one term, this is another term and this is another term all of them have a minus sign. So, I can pull the minus sign outside and then write this as twice eta by R naught square plus 1 by R naught square cot theta del eta by del theta plus 1 by R naught square del square eta by del theta square.

Now, we have to use our expressions for perturbation pressure. In our earlier equation we have already obtained in equation 3. Equation 3 there are two equations at the top of this slide there is two expressions for the perturbation pressure inside and outside. We need the difference of them if we take the difference then we can using these expressions we can substitute and find that the left hand side of the above.

So, I am just coming from here the expression for p in and p out is already available before. So, if I use those expressions and take the difference then it just turns out to be rho out into del phi out by del t. And now you can see where this derivative should be evaluated phi itself is a perturbation velocity potential if you expand it in Taylor series the first term will be evaluated at R naught, but the next term will have a derivative of del phi by del t with respect to r and then an eta, eta itself is a order epsilon quantity phi itself is an order epsilon quantity.

So, the product will be an order epsilon square quantity. So, we will not written it and so, just the first term is enough. And similarly for the other term it will be rho in note that for pressure it was p in minus p out for velocity potentials it will become out minus in that is because there is a minus sign in the expressions for pressure. So, in equation 3 you can see that p in and p out both have a minus sign on the right hand side.

So, that causes a flip. So, minus rho in into del phi in by del t at r is equal to R 0 is equal to I can take the R 0 square out and just write it as twice eta plus cot theta del eta by del theta plus del square eta by del theta square let us call this equation 5. So, I have now two equations one comes from the kinematic boundary condition which we have already written earlier.

So, we have already written equation 1 a and 1 b and there are two equations here from the kinematic boundary condition this is because we are now dealing with two fluids both inside as well as outside. In all the previous examples we have taken only one fluid this is the first example where we are considering both the fluids. So, the kinematic boundary condition gives us two equations this is two equations and then there is a pressure equation which comes from the pressure boundary condition equation 5.

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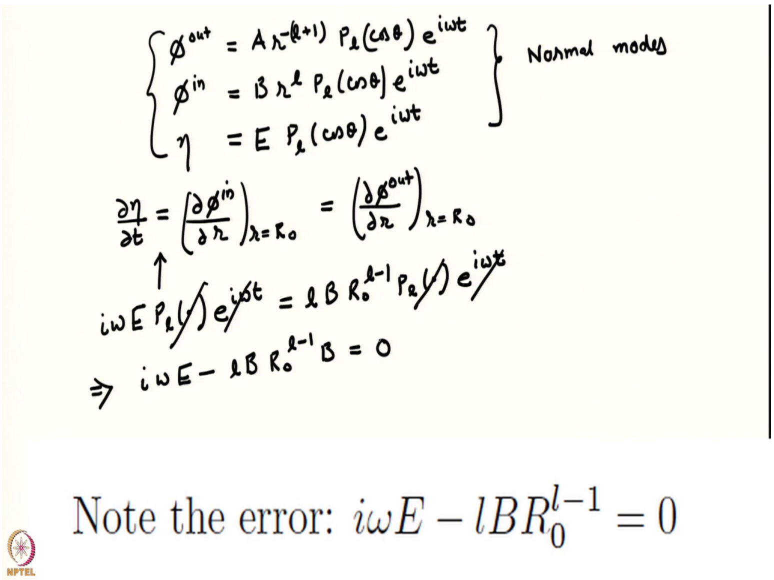

Let us use these two equations and so, recall. So, we had set phi out to be A r to the power minus l plus 1 P l of cos theta e to the power i omega t. Note that I have again gone back to theta we had called x is equal to cos theta, but now I have gone back to theta using theta is the independent variable.

So, then phi in is equal to some complex constant B into r to the power l p l of cos theta e to the power i omega t and eta was some complex constant into P l of cos theta e to the power i omega t this was our normal modes. We have to plug in these forms into our three equations 2 of these equations will come from the kinematic boundary condition and the third one will be the pressure condition which we have just derived. So, let us do that.

So, recall that our kinematic boundary condition which we have already written linearized kinematic boundary condition is just this evaluated at R naught. If I use this equality and equate the first two and use these expressions, then you can see that I get the expression for eta I get i omega E P l of cos theta will indicate that as a dot into e to the power i omega t is equal to del phi in by del r. So, that will bring out l and then this will be B and then it will be r to the power l minus 1, but small r is r naught.

So, I will make it R naught to the power l minus 1. And then again P l of cos theta represent it as a dot e to the power i omega t. These two are not 0 in general and so, I can equate this and I can get one equation which is this. Note that we need three equations because there are now three constants A B and E. A B and E. So, we expect to get three equations three homogeneous equations in these three unknowns and the determinant of the coefficient matrix will give us the dispersion relation.