Introduction to interfacial waves
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Lecture - 48
Shape oscillations of a spherical interface

Up to now we have looked at oscillations and waves in different base state geometries in particular we have looked at oscillations in over which occur over base state which are described by rectangular Cartesian geometries. We have also looked at radial geometry, we have also looked at waves caused by surface tension on a cylindrical geometry in the case of the Rayleigh plateau instability as well as waves over the geometry.

Now, let us move on to another geometry which is this spherical geometry and now, we will look at oscillations occurring on a sphere. So, we are going to look at surface tension driven waves which are created on the surface of drops liquid drops and bubbles.

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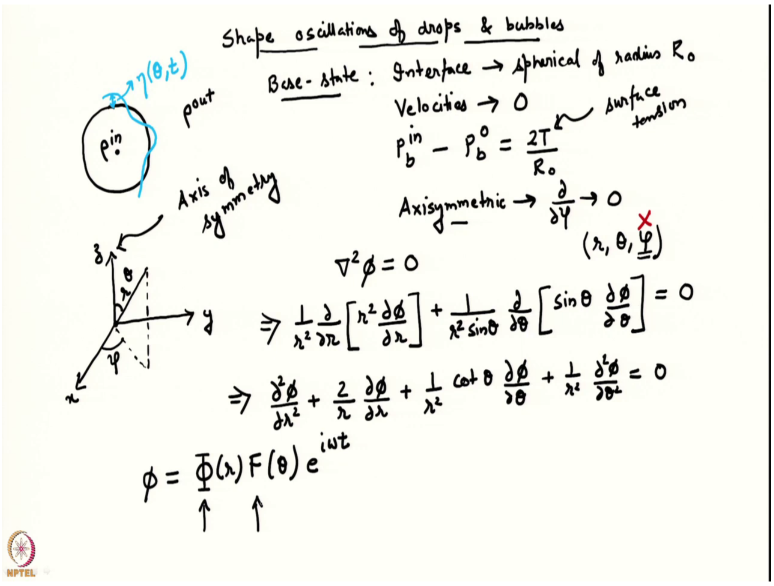

So, here my base state would be described by a sphere. We are going to use a spherical coordinate system. So, my perturb sphere would look something like this and like usual we will define a quantity eta which in this case will be a function of some angle and time.

So, let us write down first what is the base state. So, let us say we have some liquid inside. So, let us say we have some fluid inside which is given by rho in. We have some fluid outside whose density is given by rho out. In this case we are going to solve for both the fluids we will see that this will help us to reduce the expressions to the particular cases of drops and bubbles.

Now, in the base state the interface between the two fluids is spherical of let us say radius R naught the velocities R 0. So, quasi fluid both inside as well as outside and pressure will have a nontrivial base state because this interface is curved. So, the base state pressure inside minus the base state pressure outside will be given by 2T by R naught.

Notice the factor of 2; this comes because for a sphere there are two orthogonal directions in which there is curvature. For the case of a cylinder we had just one direction along which there was curvature, here T is the surface tension. So, we are going to use a spherical coordinate system now in order to look at oscillations on the surface of this spherical interface.

The interface could be an interface separating two liquids or it could be an interface which separates a gas from a liquid, the gas could be outside or inside. If the gas is outside and it is a liquid inside we will call it a drop; if it is the other way around where it is a liquid outside and a gas inside we will call it a bubble. But, right now let us say that we have two fluids of density rho in and rho out and let us do the analysis taking into account the density of both the fluids.

Now, our spherical coordinate system so, I will just drawing it outside the drop, but our spherical coordinate system. So, this is my Cartesian coordinate system which is centered at the center of the sphere. So, this is the center of the sphere and so, as is usual we define a radius vector whose projection.

So, this is let us say the x-axis, the y-axis and the z-axis. So, in spherical coordinate system any point has coordinates r an angle theta which the radius vector makes with the z-axis and an angle psi which is the projection of the radius vector on the x-y plane makes with the positive x-axis.

Now, in this particular case we are going to stick to the axisymmetric approximation like we have done before for the Rayleigh plateau case. So, this essentially implies we can see that axis the axis of symmetry would be the z-axis. So, axis of symmetry and so, we are going to deal with quantities which are just functions of theta and T they are not going to be functions of the angle psi. So, del by del psi of all quantity will be 0 or in other words quantities will not depend on the coordinate psi.

Now, with that approximation let us now go ahead. So, as is usual our governing equation for the perturbation velocity potential in both the fluids is given by the Laplace equation. In this particular case we will have to solve for two copies of the Laplace equation one representing the fluid inside and one outside. Before we do that, let us try to understand a little bit more about variable separable solutions to the Laplace equation in spherical coordinates.

So, the Laplacian operator in spherical coordinates and axisymmetric spherical coordinates so, my coordinate system is r, theta and psi and this is not there this variable is not there because of the axisymmetric approximation. So, this variable is not there. So, the Laplacian operator just becomes 1 by r square. This formula you can look up in any book on transport phenomena.

This is the form that the scalar Laplacian takes in a spherical coordinate system where we have set second derivatives with respect to psi to be 0 the axisymmetric approximation. Now, as is usual we will say that so, I can simplify this further and I can write this as del square phi by del r square plus 2 by r del phi by del r plus 1 by r square cot theta del phi y del theta plus 1 by r square del square phi by del theta square is equal to 0.

So, this is the form of the Laplace equation. Let us look for variable separable solutions. So, we are going to say that phi is some function capital phi of small r, some function F of theta and because we are going to do a normal mode approximation so, I will set it equal to e to the power i omega t. So, we are looking for perturbations about the base state. In the base state there is a pressure jump across the interface the interface is purely spherical and there is no velocity inside as well as outside.

So, we are going to look for perturbations about the base state and we are going to ask is the base state stable does it produce an instability or does it lead to oscillations. So, this is our formula for phi. We will have to write down a similar formula for eta, but let us first plug this formula into the Laplace equation and determine what is this function capital phi and capital F.

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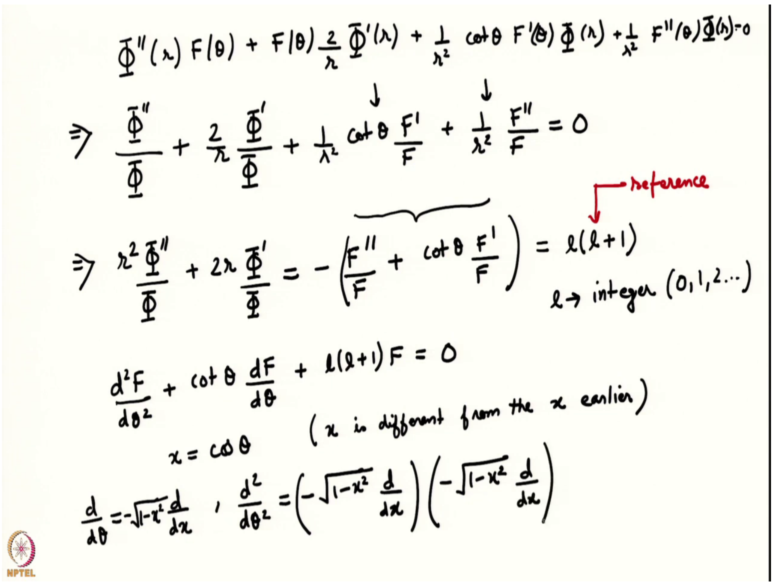

So, when we plug this form in into the Laplace equation we obtain capital phi double prime. So, prime represents derivative with respect to small r and F of theta plus F of theta 2 by r phi prime of r plus there was a cot theta F prime of theta into phi of r plus 1 by r square F double prime of theta into phi of r is equal to 0.

Now, I can divide throughout by capital phi and capital F. If I do that then I obtain plus is equal to 0. Once again as is usual in variable separable we try to separate everything which depends on small r and keep it on one side and separate everything which depends on theta and keep it to the other side. If we do that then we obtain, so, I have multiplied by small r square and after multiplication you can see that the last two terms of the equation above. So, this term and this term they become independent of small r.

So, I can separate it and take it to the other side and this just becomes F double prime by F I have written the fourth term first plus cot theta F prime by F. So, now, we have our variable separable form. So, on the left hand side we have a pure function of small r, on the right hand side we have pure function of theta they are independent and so, they can be varied independently. And, so, the usual argument is that that each of them must be equal to a constant.

We are going to set a particular form of the constant in order to understand why that particular form has been chosen I will give you a reference at the end of this video and you can go through that reference and understand better why we are choosing the separation constant to be of that form. So, I will choose the separation constant to be of a very specific form l into l plus 1. So, this is a constant and l is an integer. We will take it to be a positive integer 0 1 2 3 4 and so on.

I will give you a reference where you can understand why this particular form has been chosen, ok. So, now, notice that the separation constant is positive and has been chosen to be a particular form. What are the consequences? The consequences let us work out the consequences for capital F.

So, this implies that d square F. So, I am writing F double prime as d square F by d theta square plus cot theta into dF by d theta plus l into l plus 1 into F is equal to 0. So, that is the part that I get by equating the F part to the separation constant this gives me a differential equation for capital F.

Now, this equation you can see I will change it to a standard form because this equation is actually a well known equation. I will just change it to a standard form and then tell you what is the name of the equation. So, I am going to use the substitution x is equal to cos theta. Please note that this is not the same x as we had drawn in the coordinate system.

So, in the coordinate system we had x, y, z and then the angle that the radius vector made with the z-axis was theta. So, this is not the same x although I have used the same symbol. It is standard to use x as the transformation variable in this particular case. So, that is why we have chosen small x, but this x is different from the x earlier. Because we are going to do this analysis in a spherical coordinate system we need not worry about the x that we had encountered earlier in a Cartesian coordinate system.

So, x here just represents cos theta. So, with that we have to change all derivatives with respect to theta to derivatives with respect to x. So, you can see that d by d theta is basically d by dx and then dx by d theta which is minus sin theta which is minus square root 1 minus x square. Similarly, d square we need d square by d theta square for the first term is basically the operator operating on itself 1 minus x square d by dx operating on itself d by dx.

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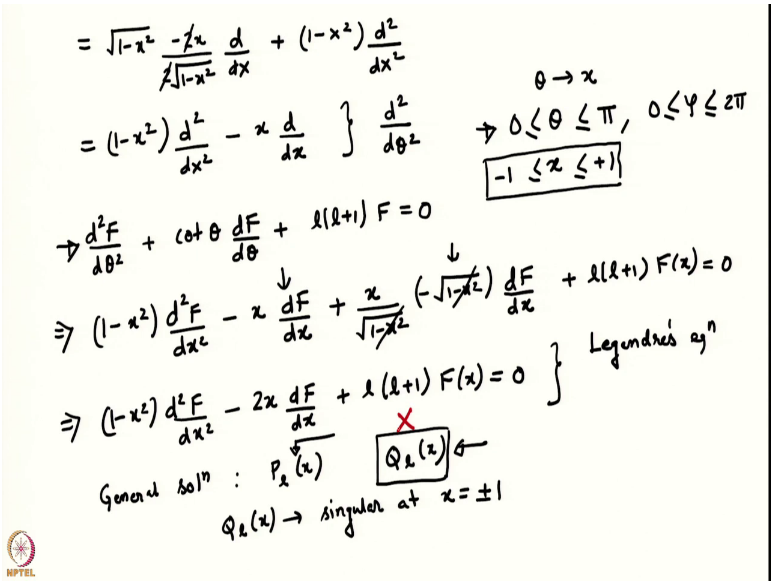

If you open that out if you let the first operator operate on the second one, then you will find the minus and the minus cancel each other out and then we find that we have square root 1 minus x square. So, I will take the derivative of so, 2 into square root 1 minus x square and then minus 2x and then d by dx plus 1 minus x square into d square by dx square.

So, this just becomes 1 minus x square I am writing the second term first d square by dx square this and this cancel each other and then this just becomes minus x d by dx. Therefore, my equation transforms my equation was d square F by d theta square plus cot theta dF by d theta plus l into l plus 1 into F is equal to 0, where l is a positive integer.

Now, we want to rewrite this equation in terms of derivatives with respect to x. So, substituting so, this is the expression for d square by d theta square. So, this just becomes 1 minus x square d square F by dx square minus x dF by dx that is just the first term. Then we have plus cot theta is cos theta by sin theta. So, cos theta is x, sin theta is root 1 minus x square and d by d theta we had seen is minus square root 1 minus x square into d by dx which is just dF by dx.

And, then the last term remains the same and now, F is a function of x. You can see that this term and this term cancel. These two terms have the same sign, they are exactly the same. So, it is just 2 times the first term. So, 1 minus x square d square F by dx square minus twice x dF by dx plus l into l plus 1 into F of x is equal to 0. Now, this is a very well known equation.

The reference that I will give you at the end you will find more details about this equation and how what are the solutions of this equation, how does one obtain solutions convergence solutions to this equation. This equation is known as the Legendre's equation named after a French mathematician who first studied this equation. Notice that this is a second order linear ordinary differential equation. So, we expect two linearly independent solutions.

In general, one can use series solutions to find out what are those solutions ok. So, I will tell you the what is the solution. So, the general solution to this equation so, the general solution is a linear combination of two functions which you can think of as being known analytically P l of x and Q l of x recall that l is an integer positive integer.

So, we will have to select particular values of l and for each such value there will be two functions P and Q. So, for example, if you select l equal to 0 it will be P 0 Q 0 if I select l equal to 1 P 1 Q 1 and so on. Now, because we have gone from theta to x recall that in a spherical coordinate system theta goes from 0 to pi, the variable psi goes from 0 to 2 phi; psi is the azimuthal angle which is not here in our analysis because we have assumed it to be axisymmetric.

So, we are interested in this angle theta because we have gone from theta to x cos theta is equals to x. So, when theta varies from 0 to phi x varies from minus 1 to plus 1. So, this is the range in which we will have to look for solutions to this equation and those are the solutions.

So, P l of x and Q l of x are to be thought off in the range minus 1 to plus 1 x between minus 1 to plus 1. So, it corresponds to going from the north pole of the sphere to the south pole theta is derived theta is defined as a angle with respect to the from the north pole from the top of the sphere. So, at the top cos theta is cos 0 is 1 at the bottom cos phi is minus 1.

So, now it turns out that Q l of x these are the Legendre functions and it turns out that the Legendre function Q l of x the Legendre's function Q l of x is singular at x is equal to it diverges at x is equal to plus minus 1. Now, obviously, we do not want quantity functions which diverge at the north pole and the south pole.

So, in order to keep things finite, whenever we write the solution to this equation as a linear combination of C 1 into P l of x plus C 2 into Q l of x we will set C 2 equal to 0, so that in further analysis this function is not going to appear. I hope it is clear why it is so. This function diverges at x is equal to plus 1 and minus 1, this function does not P l of x has a regular behaviour.

So, we are just going to retain one of these two P l of x and so, the solution to this equation will just have one function which is P l of x. So, in general some constant times P l of x.

Now, this is as far as the capital F dependence of phi is concerned. Let us now go back and find out the other dependence. So, in our last slide, we had written capital F by variable separation we had written capital F and we had solved this part. So, now, we are going to look at this part.

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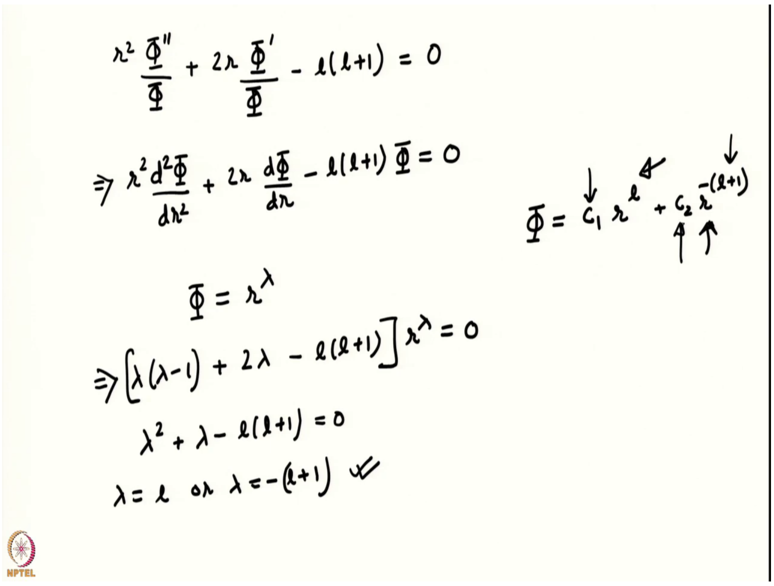

So, let us proceed. So, the small r dependence of phi is governed by this equation r square capital F double prime by capital, capital phi plus twice r phi prime by phi and we have chosen the separation constant to be l into l plus 1. So, it comes to the same side and becomes minus l into l plus 1. So, this just becomes r square d square phi by dr square plus twice r d phi by d r minus l into l plus 1 into phi is equal to 0.

So, the small r dependence of capital phi once again satisfies an ordinary differential equation a linear one and it is a second order ordinary differential equation. This is this does not have constant coefficients. So, we cannot just solve it in a straightforward manner by assuming some exponential dependence in r.

However, notice that this is the second derivative with respect to r and it is gets multiplied by r square, this is the first derivative with respect to r and it gets multiplied by r. So, there is a pattern. This suggests that if I look for solutions of the form r to the power some constant lambda then you can see that this is going to satisfy this equation.

Why? Because if you take the first derivative of phi it will give you r to the power lambda minus 1, but when you put it into the equation it will get multiplied by r. So, that will give you an r to the power lambda. Similarly, when you look at the first term two derivatives with respect to r will bring r to the power lambda minus 2, but it will get multiplied by r square which will again make it r to the power lambda.

So, each of these terms in this equation is going to give you r to the power lambda with some coefficient and then we are going to collect those coefficients and set them equal to 0 in order to determine lambda. So, if I substitute then you can see very easily from the first term that the coefficient is going to be the coefficient of r to the power lambda is just going to be lambda into lambda minus 1, and then for the second term is just one differentiation; so, 2 lambda and then l into l plus 1.

This whole thing multiplies r to the power lambda is equal to 0. We can solve this equation, this is a quadratic equation in lambda you can solve this equation. So, this can be written as minus l into l plus 1 is equal to 0. This has solutions lambda is equal to l or lambda is equal to minus l plus 1. You can use the formula for a quadratic and verify this.

So, now as expected we have found two linearly independent solutions. So, phi is going to be written as c 1 into r to the power l plus c 2 into r to the power minus l plus 1 remember that l is a positive integer. So, this is going to be a positive power of r and this is going to be a negative power of r; even if we substitute l equal to 1 the second term will give me c by r. So, r is going to appear in the denominator.

Now, here we have to make a choice. When we are going to use this solutions of the variable separable solutions to the Laplace equation for determining the form for the velocity potential inside as well as outside, you can clearly see that r with a positive power diverges when I go to very large distances.

So, I will have to set this constant to 0 when I choose the form for the velocity potential for the outer fluid. In contrast for the inner fluid, it is this term which is going to diverge because it contains terms like 1 by r 1 by r square and so on and a small r equal to 0 all of these terms will diverge.

So, for the inner fluid I will have to set c 2 equal to 0, I hope this is clear and this logic will decide what form we are setting for the inner as the velocity potential in the inner fluid as well as in the outer fluid. Now, we have now found using variable separation what are the forms that have to be chosen for small r and for theta. Now, coming back to the so, functions P l of x let us look at those functions a little bit and let us get a physical field for what they look like.

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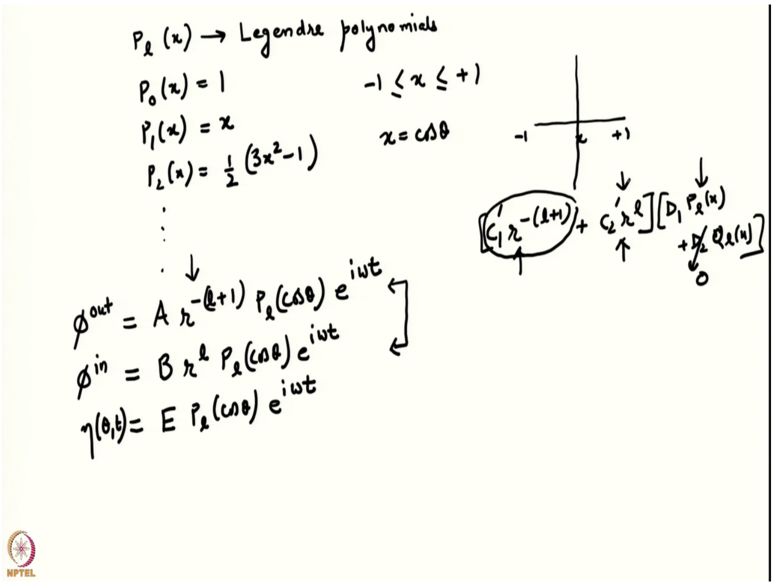

P we want to see what does P l of x look like, they are also known as the Legendre's polynomials they are actually polynomials in x Legendre polynomials. So, it is known that P 0 of x is 1, P 1 of x is x, P 2 of x is half 3 x square minus 1. All this can be analytically shown and so on and so forth P 3, P 4, P 5, P 6 and so on. You can plot all this recall that x has the limit minus 1 to plus 1. So, you can plot between x is equal to minus 1 to plus 1 you can plot these functions and get a physical feel for what these functions look like. Recall that x is equal to cos theta.

So, now with whatever we have learnt so far, I am going to set phi out the velocity perturbation velocity potential for the outer fluid to be equal to some constant which could be in general complex r to the power minus l plus 1 P l of cos theta, cos theta is basically x into e to the power i omega t. Phi in is some other constant B, again possibly complex r to the power l again P l of cos theta e to the power i omega t.

And, eta now we will have to set a formula for eta; eta by definition is not a function of r it is a function only of theta and t. So, eta will be some complex constant e into P l of cos theta into e to the power i omega t. You can try and understand in the light or whatever we have discussed you can try and understand how did we guess these forms. The small r dependence is a linear combination of r to the power minus l plus 1 and r to the power l.

The theta dependence is a linear combination. So, I will call some function D 1. So, I will put some primes here D 1 into P l of x plus D 2 Q l of x. D 2 has to be set to 0 because Q l has divergences at minus 1 and plus 1. So, it just leaves us with D 1 of P l of x, now when we set it for the inner fluid and the outer fluid we will have to we cannot keep both of them this as well as that. This part is fixed.

But, we cannot keep r to the power minus l plus 1 as well as r to the power l because one of them will diverge outside and the other will divert inside. One of them will diverge when small r goes to infinity the other will diverge when small r goes to 0. So, we will have to keep the one which does not diverge in its domain of definition; for the outer fluid the domain of definition goes up to infinity. So, we will have to ignore the one which diverges an infinity and keep the one which does not.

So, for the outer fluid we are keeping this; note that l is positive. So, this is going to give us decaying functions of r. For phi in we have to set the constant to 0 which the power of r which diverges. So, we will have to keep c 2 prime to 0 inside.

So, for inside we will have to set this to 0 and just keep this and so, this dictates the choice of phi out, phi in and eta. Using this we are going to conduct our normal mode analysis, we will write down our boundary conditions and you will see that this leads us to a dispersion relation.