Introduction to interfacial waves
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Lecture - 47
Mechanism of the Rayleigh-Plateau instability

We have looked at three cases so far; we have looked at perturbations imposed on a liquid cylinder, we have also looked at perturbations imposed on a thin film coating a solid rod and we have looked at perturbations imposed on the surface of a cylindrical gaseous air bubble.

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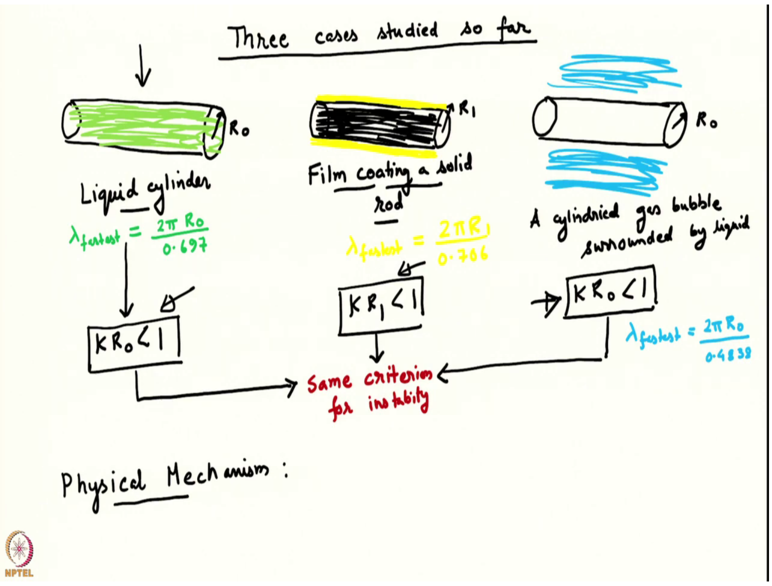

In all the three cases, we have found that the criteria for instability remains the same; in all cases there is an instability, some perturbations produce oscillations, others lead to exponential growth in time. In all cases the boundary between stability and unstability remains the same, namely k into some radius less than 1.

So, in the left most case for a liquid cylinder of radius R 0 in the base state, the modes which satisfy k R 0 less than 1 were seen to be unstable. In the case of a solid rod on which there is a thin film being coated, the criteria for instability was k R 1 less than 1, where R 1 is the radius of the free surface of the thin film. In the case of the cylindrical bubble, it was once again analogous to the liquid cylinder it was k R 0 less than 1, where R 0 is the radius of the bubble in the base state.

Now, let us try to understand physically, why do we get the same criteria for instability and what is the reason that this criteria is independent of any fluid property, like surface tension or density and so on. So, for that, we will do a simple calculation. So, we are looking at the physical mechanism.

Now, before we begin this calculation it is useful to recall that, we are looking at when we have stability, then we have oscillations and when we have instability, we have growth. Now, let us try to understand why do we have oscillations. We have discussed many times in this course that, the basic ingredient of an oscillation is an interplay between a restoring force and inertia.

In this case the restoring force comes from surface tension; because we are ignoring gravity and there is only surface tension which provides the restoring force. Now, surface tension has the dimensions of force per unit length or I can think of it as energy per unit area. If I multiply numerator and denominator by length; then numerator becomes force into length, which has the same dimensions as energy and the denominator becomes length square, which is basically an area. So, that is energy per unit area.

So, if I think of surface tension as energy per unit area, then it is clear; then that in order to minimize the surface energy of a system, a system would try to minimize the area that it exposes. So, an attempt to minimize the surface energy is equivalent to an attempt to minimize the surface area.

So, let us first calculate what is the surface area and what is the volume when we impose a perturbation on a liquid cylinder. For simplicity I am just going to do this calculation on this example; but it will be relevant to the other two. And this will give us some idea as to why are we getting the same criteria in all the three cases. So, let us begin this calculation.

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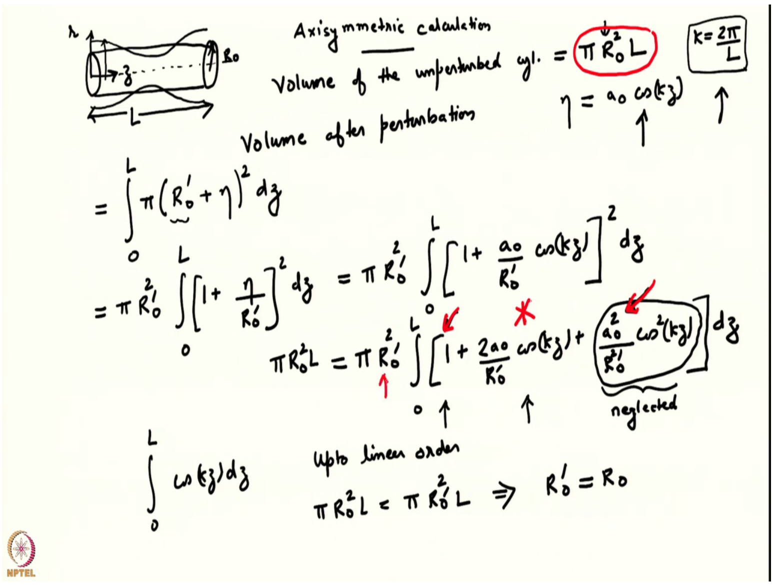

So, let us say that we have a liquid cylinder and I will say that it is base state length is some length L and this is the cylindrical coordinate system r and z centered on the axis of the cylinder. And let us put some perturbation on the surface. So, we will be putting perturbation.

And we will do an axisymmetric calculation; because we have seen that, the modes which are unstable are the axisymmetric modes. So, our calculation is going to be axisymmetric. So, we are going to put axisymmetric perturbations. Now, the volume of the unperturbed cylinder is. So, this radius is let us say R 0 in the base state, so pi R 0 square into L.

Let us calculate the volume after perturbation. Clearly due to incompressibility; the volume of the liquid before and after perturbation should be the same, if I assume that there is no liquid coming in through the ends or leaving through the ends. Let us calculate the volume after perturbation. If I say that the free surface is perturbed as some a 0 into cos k z, where I am the way I have written it k is 2 pi by L.

So, I am going to put a perturbation, whose wavelength is the same as the length of the cylinder. So, k is equal to 2 pi by L; it is 2 pi by lambda, but I am assuming that lambda is equal to L. So, the volume after perturbation; you can calculate this easily as the volume of a solid, which is generated by rotating it about the, by revolving it around the z axis. So, the volume is just given by 0 to L pi into some radius of the surface and the radius, so this this distance.

And so, I am going to write it as sum R 0 prime; I am not writing it as the same R 0 here, you will see shortly why plus an eta, some constant will be basically plus and eta in square. So, this is like pi into radius square into d z. Now, I am going to work on this integral. So, I will pull it out and I am going to pull out R 0 prime square. So, this the integral becomes 0 to L 1 plus eta by R 0 prime square d z.

Now, eta is given as this. So, we will take eta to be actually equal to this. So, if I do this, then we have pi R 0 prime square and if I do this integral 0 to L; this is 1 plus a 0 by R 0 prime cos k z square d z. Now, I can write it like this. And what is inside the integral, there will be three terms; 1 plus twice a 0 by R 0 prime cos k z plus a 0 square by R 0 prime square into cos square k z d z.

Now, this is the volume after perturbation; this has to be equal to the volume before perturbation from incompressibility. If there is no fluid coming in or exiting the system through the two ends. So, this should be equal to pi R 0 square into L. Now, if we do a linear calculation, then we are supposed to retain only terms up to order a 0. So, I can clearly see that, if I retain terms up to order a 0; then this, so this is then neglected, because there is an a 0 square sitting here.

And so, pi R 0 square L is pi R 0 prime square into this plus that; you can convince yourself that the second integral is 0, ok. So, the second integral is just 0 to L cos k z d z and k is 2 pi by L. So, if you do that integral, you will find that it is just 0. So, up to linear order, we just find that pi R 0 square L is equal to pi R 0 prime square into the first term, which is just 1. So, that integral just gives you an L.

So, this just tells you that up to linear order, R 0 prime is the same as R 0. And so, if you are just doing a linear calculation, then this is fine; however note that, this term the integral of this term is not zero. So, in linear calculation, up to linear order we have volume conservation; however there is a small amount of volume, which is proportional to a naught square, which is left behind and which is not taken into account in a linear calculation.

So, let us do the volume balance exactly; let us say that the volume before perturbation and the volume after perturbation is exactly the same, without doing any linearization. And let us calculate what is the relation between R 0 prime and R 0, ok. We will find that at quadratic order, there is a difference between R 0 prime and R 0 and that is because the that is a non-linear effect.

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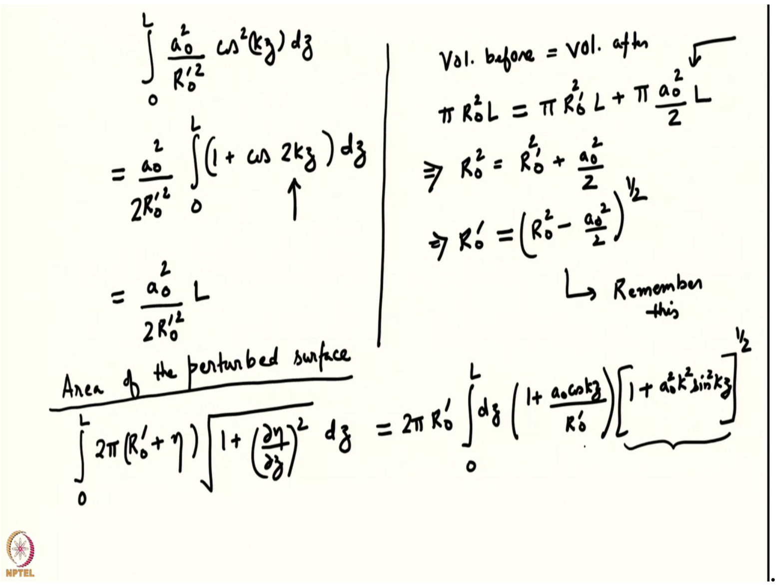

So, if we just do the calculation. So, for this we just need to do the second integral, which we had ignored in the linear approximation. So, the second integral was just a 0 square by R 0 prime square cos square k z. Again this integral is very easy to do; these are just constants, they come out and then I will put a 2 here and write it as 1 plus cos 2 k z. The cos 2 k z term will; so I will write this as 0 to L 1 plus cos 2 k z d z. You can cross check that this integral will again go to 0; but the first integral is not zero. So, the first integral just is square into L.

So, if you do a volume before is equal to volume after. So, volume before perturbation is equal to volume after perturbation; then the volume before was pi R 0 square L and this is equal to. So, this is pi R 0 prime square into L plus pi, this is the second term; a 0 square and there is a overall R 0s times square, so it multiplies and then it gets cancelled out and so this is just this. So, all I am doing is, I am doing the volume balance exactly. So, I am equating this term to the first term from here and a contribution from the third term; there is no contribution from the second term, because the second term integrates out to 0.

So, the first term just gives me pi R 0 prime square into L, the third term gives me two terms one of which is 0 and another of which actually has a 0 square by R 0 prime square; but there is an overall R 0 prime square and that cancels out and so in the third term I do not get a. So, in this term I do not get a R 0 prime square.

So, now, this is the exact statement of volume conservation. And so, this tells me that R 0 square is equal to R 0 prime square plus a 0 square by 2 or in other words R 0 prime. The new mean, because I am writing the location of the interface as some mean radius plus a perturbation and this is telling us that at non-linear order, the mean is slightly shifted from the base state value, ok.

So, R 0 prime is equal to R 0 square minus a 0 square by 2 to the power half; we are taking only the positive square root, because R 0 prime is a radius, ok. So, now, we have to remember this, we are going to use this. So, now, let us come to; we have done the volume calculation and we have found that at non-linear at order a 0 square, there is a small shift in the mean. So, now, let us write the statement of area. So, let us calculate the area of the perturbed surface.

Until now we have calculated the volume of the perturbed surface, let us now calculate the area of the perturbed surface. So, we are going to calculate the area of the perturbed surface. So, it is an integration, integration of something which is a solid of revolution and it can be written as 2 pi into some radius; I am going to use the correct mean plus some eta into 1 plus del eta by del z whole square into d z. This you can look it up in any text book on calculus, this formula.

So, this is the area. So, now, let us evaluate this. So, this is I can pull out R 0 prime; so this will become 0 to L d z, let me write that first. Then this is 1 plus eta by R 0 prime; eta is itself a 0 into cos k z divided by R 0 prime. And then I have to put this inside a square root and then what is happening is del eta by del z; eta is a 0 cos k z, it will bring out a minus a 0 into k into sin k z, the minus sign is not important, because it will be squared.

And so, I will be left with a 0 square k square sin square k z. Now, this is the correct expression for area of the perturbed surface. This integral is slightly complicated, it can be done exactly; however we will do it approximately and we will in the process, we will understand that why does k R 0 less than 1 always lead to instability.

Let us take this integral and let us use. So, let us the complicated part of this integral is this term; because this this term, the first term is easy to integrate, it is this term. So, let us use the fact that a 0 square is a small quantity and so, we are going to expand this in a infinite series and retain up to some terms. So, let us do that.

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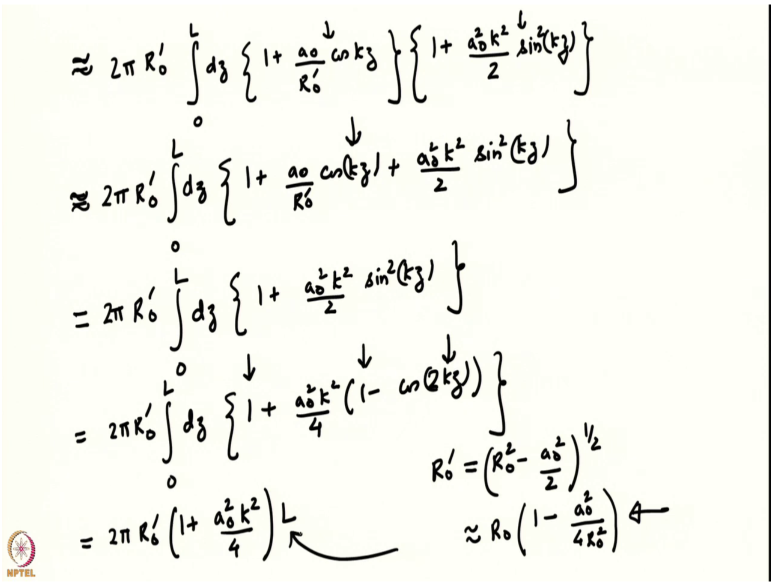

So, my integral becomes this just becomes or this is rather an approximation 2 pi R naught prime; this is the prefactor we had already pulled out 0 to L d z 1 plus a naught by R naught prime cos k z, this is the first part, I am not doing anything to it yet.

And the second part we have decided that, we will expand it in a series. So, we are just going to retain up to square and a factor of half sin square k z. So, now, there is no square root. Now, let us work on this; this is 2 pi r naught prime 0 to L d z. And I am going to retain only up to quadratic terms. See if I multiply both these brackets, I will get cubic terms, I am going to ignore those cubic terms. So, this is again an approximation. So, 1 plus a naught by R naught prime cos k z plus a naught square k naught square by 2 sin square k z.

So, I am going to stop here and not continue and not write the product of this term and that term; because that will give me something which is cubic in a naught. Now, let us see, you can also understand this intuitively as to why we are going up to quadratic terms. Recall that a naught is a displacement.

So, in a linear theory, we retain things up to a naught; energy by definition is quadratic in the displacement, think of a spring mass system, force is linear in the displacement in a linear theory, energy is quadratic in the displacement. This area as I said before is proportional to the surface energy of the system.

So, in a linear calculation, energy calculation has to be done up to quadratic order ok, to remain consistent with linear theory, ok. So, we are going up to a naught square. So, this would tell twice by R naught prime 0 to L; you can readily see that this term is going to integrate out to 0.

So, I will just write the first term and the third term, sin square k z. And once again we can do, we can solve this integrals in a easy manner. So, 0 to L d z 1 plus a naught square k naught square by 4 and then this becomes 1 minus cos twice k z, twice k z. Once again this integral cancels out, this integral will just go to 0. And so, we will just be left with twice by R naught prime into the contribution from here and the contribution from the first term inside this bracket, ok. So, that is just.

So, this these are just constants. So, I can pull that those two terms out, 1 plus k 0 a 0 square k square by 4; this comes out of the integral and then 0 to L integral d z, which is just L. So, that is my expression for area, which is proportional to surface energy up to quadratic order in a naught.

Now, we had seen a relation between R naught prime and R naught, which was given earlier as this; we have derived this earlier when we did a volume balance. So, that is the relation that I have just written down. And this is also the expression is quadratic, but there is a square root.

So, in order to be consistent, we are going to use the same expansion now. And so, this is approximately equal to R 0 square. So, R 0 square can be pulled out; because there is a square root, it will come out as R 0 and then this will be 1 minus a 0 square by 4 R 0 square and that is it ok, this is correct two order a naught square.

So, if I substitute this expression for, this expression for R 0 naught into this expression; then I get an expression for the area of the perturbed surface.

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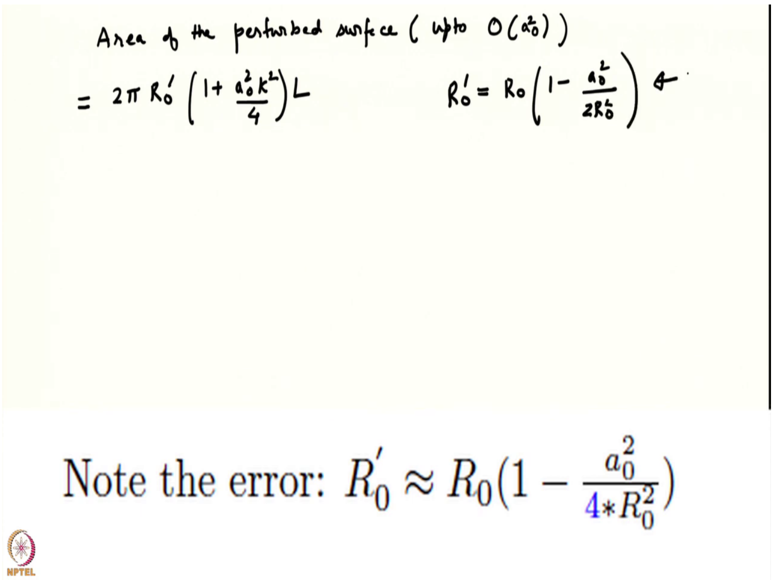

So, area of the perturbed surface up to order a naught square is. So, we had written it as twice pi R naught prime 1 plus a naught square k square by 4 into L and R naught prime we had written it as R naught into 1 minus a naught square by twice R naught square.

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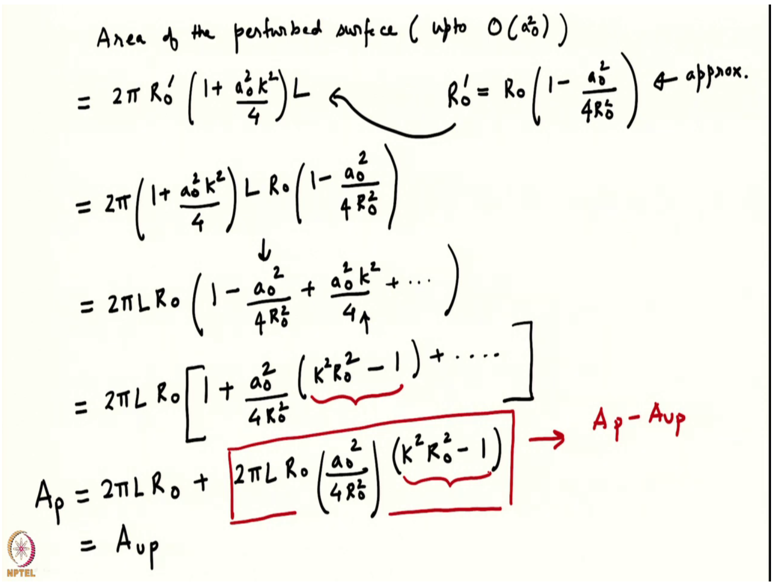

This is an approximation; this is a approximation, not the exact expression. If I now substitute this expression; so use this here into L into R naught 1 minus a naught square by twice R 0 square.

So, this is twice pi L R naught into the product of these two brackets. Once again we have to ensure that while taking the product, we do not retain anything beyond a 0 square; you can see that there will be a nau0ght to the power 4 terms, but we are not going to retain that, because we have only kept up to order a naught square.

If you do that, then you will get 1 minus a naught square by twice R naught square plus a naught square k square by 4 plus dot dot dot dot which we are not going to write; this is twice pi L R naught into, now I can write this as 1 plus a naught square.

Let me pull out a 4 R naught square common. If I do this, then this term becomes k square R naught square and this term becomes minus this should be 4, because we had written this as 4. So, I have to correct it here. So, this has to be 4. If I do the multiplication, then this is 4. And so, this is just 1 plus dot dot dot.

Now, you can see what is the connection between the two, ok. So, this is the area of the perturbed surface; this I can write it as twice pi L into R naught plus; plus some quantity which is given by twice pi L R naught into a naught square divided by 4 R naught square. You can simplify that, but the important part is this part that, you can see is coming out already; this is the important part, k naught k R naught square minus 1.

Now, you can see that this is area of the perturbed surface 2; 2 pi L R naught is nothing but area of the unperturbed surface. And so, this expression that we are getting on this this term that we are getting is the difference between A p and A u p or the delta A; the change in area between the perturbed surface and the unperturbed surface.

You can see that everything here, the change in area all terms are positive, except this part. So, the change in area can actually be negative or positive depending on whether k square R naught square minus 1 is negative or positive.

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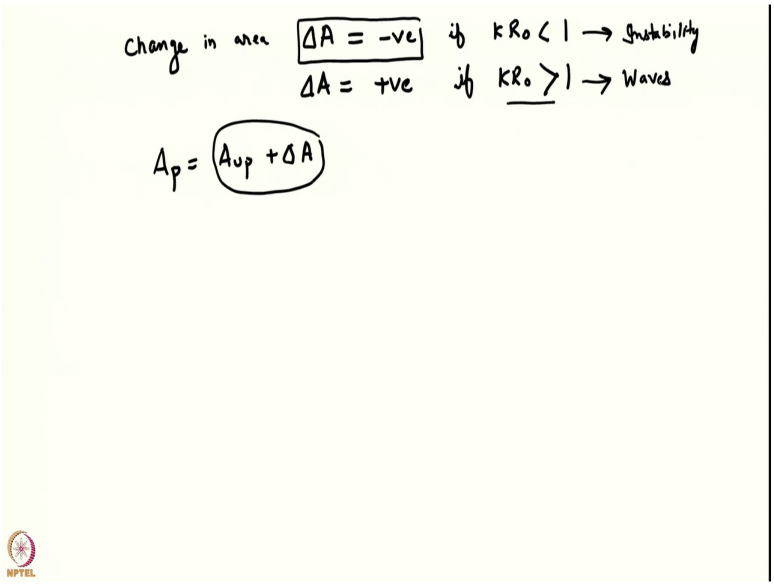

So, we find that change in area or delta A is negative if k R naught is less than 1; delta A is positive if k R naught is greater than 1. Note that this comes from a pure geometric calculation; we have nowhere said that this this filament which we are deforming is made up of fluid, what are its properties.

We have said nothing of that sort, you know we have just taken a surface of revolution and we have just calculated that if we impose a perturbation on it, what is the change in area; if you impose a Fourier mode perturbation, what is the change in area.

And it is already telling us that k R 0 being greater than 1 or less than 1 is the deciding factor, which decides whether the area in the perturbed state is actually more or less than in the area in the base state. You can see this criteria; if k R naught is less than 1, then the area in perturbed state is actually less than the area in the base state. Recall that it is A perturbed is equal to A unperturbed plus delta A.

So, if delta A is negative, then this quantity is less than A unperturbed, ok. So, the A in the perturbed state is less than A unperturbed. So, the system is going towards a state where it is; because the area is a direct measure of the surface energy of the system, the system if k R, if you impose k R 0 less than 11 kind of perturbations on the surface, then it takes the system to a state, where the system has a lower surface energy than it had compared to the base state.

The system is always trying to lower its surface energy, when it moves to equilibrium. So, it likes to stay in that state more than it likes to stay in the base state or in other words, there is no restoring force which will bring it back to the base state.

In contrast if k R 0 is less than 1, then the perturbation increases the surface energy of the system; the system does not want to be in that kind of a state, it rather wants to come back to its base state, where it had a lower surface energy. So, the interplay between whether the surface energy increases or decreases in the perturbed state, decides whether the system wants to come back to the base state or whether it wants to go further and further away from the base state.

If the system goes further and further away from the base state, we have instability, which is reflected by the exponential growth in time; there is no attempt to come back to the base state. If the system does not like its perturbed state, because that is the state of higher surface energy; it wants to come back to the base state and in the process sets up an oscillation, because of inertia, this produces waves, in this case standing waves.

Now, we understand the physical reason for why we were getting the same criteria in all the three cases. In all the three cases, the criteria k R 0 or k R 1 less than 1 is dictated by the surface energy of the system. It is slightly intuitive, it is slightly non intuitive that, there are access metric, whose area is actually less than that of the unperturbed state; this is non-intuitive, because we not intuitively think that a curved surface has more area than a smooth surface.

The base state here is the smooth surface; we are finding that there are axisymmetric perturbations, where the surface has some curvature and that is still in that state, the surface has still lower area than it would if it was in the base state. This has got to do with the fact that there is curvature in this base state.

This is not true in a flat surface, in the flat surface there was no instability; we have seen capillary waves on a flat surface and there was no instability there. That is because whenever you put a perturbation, the perturbation always increases the area; the system does not like to be in that state, because there it has a higher state of surface energy and so it what wants to come back to the base state and in the process sets up oscillations.

In the cylindrical base state, there are certain perturbations which actually have a lower surface area than the base state itself. If you choose such a perturbation, then the perturbation will not want to come back to the base state and it will cause instability. For perturbations which increase the area, it gives oscillations; for perturbations which decrease the area, it leads to instability. That is the physical reason for why all of these things have the same criteria for instability.