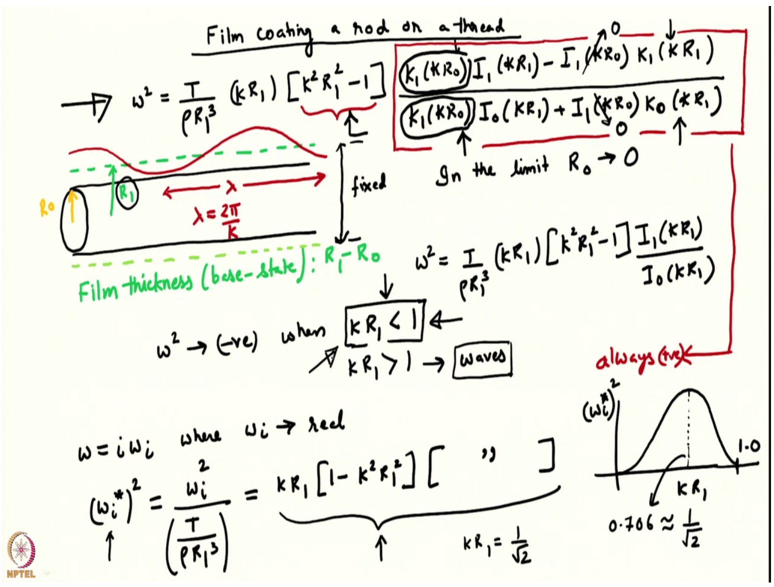
Introduction to interfacial waves
Prof. Ratul Dasgupta
Department of Chemical Engineering
Indian Institute of Technology, Bombay

Lecture - 46
Rayleigh – Plateau capillary instability of a cylindrical air column in a liquid

We were looking at perturbations on a thin film coating a rod of radius R naught. Without proving it, I had told you that one can show that if you put perturbations of wavelength lambda or wave number K, then the dispersion relation is given by the expression that I have written here.

(Refer Slide Time: 00:41)



Now, just to check for consistency, it is useful to take the limit R 0 going to 0. So, you can see that when R 0 goes to 0, R 0 is the radius of the rod, so I am shrinking the rod you can think of this process as one which holds this entire distance constant and shrinks the radius of the rod. So, the space that is occupied by the rod gets now occupied by the fluid.

Now, as the rod thickness shrinks to 0 in that limit this should reduce to a fluid thread or a fluid cylinder whose radius is R 1, and so we should the dispersion relation should reduce to the case that we have already derived earlier. This is just a check on the consistency of this dispersion relation. Now, this can be shown easily that when R naught goes to 0, then this part is finite, and this term is finite ok, this goes to 0 and this goes to 0. And so we are left with just the first term here and here. Note that the first term on in both numerator and denominator has a K 1.

And K 1, the argument if I set it to 0, K 1 diverges ok. However, there is a K 1 in the numerator and a K 1 in the denominator. So, this divergence will get cancelled out. And so we will just recover omega square is equal to T by rho R 1 cube kR 1 into k square R 1 square minus 1 into I am cancelling out the K 1 and the K 1. So, I am cancelling out this K 1 and this K 1, both of them will diverge, K 1 diverges at the origin.

So, as I said R naught to 0 for a fixed small K the argument will go to 0, and both the numerator and the denominator will diverge. But both of them will diverge in exactly the same way because they are the same functions. So, it can be cancelled out and so it remains what remains is I 1 of kR 1 divided by I naught of kR 1. And you can check that this is exactly the same dispersion relation where we have obtained earlier.

Now, we have replaced R 0 in the previous relation with R 1 now. And that is because as the keeping this length fixed, if I shrink the radius of the rod, then the space that was earlier occupied by the rod now gets occupied by the fluid. And the limit when the rod shrinks to 0 or just a line, then it just becomes a fluid cylinder. And so I should recover the same dispersion relation that we had got earlier except that now we are writing R 1 because the radius of the fluid cylinder is now R 1 ok.

So, it is this, so it is this number ok. So, this is consistent with what we have done so far and kind of generalizes what we have done for a fluid cylinder. So, now, let us analyze this dispersion relation. You can once again go to MATLAB or Mathematica and plot this quantity. And you will find that this quantity is always positive for all K, this quantity is always positive for all K, this can be shown.

So, once again this dispersion relation can have omega square can become negative, but the only place which can become negative is once again our old familiar term K square R 1 square minus 1. So, we get omega square negative, omega square is negative when kR 1 is less than 1. Analogous to what we had got earlier. Earlier it was kR naught less than 1, now it is kR 1, because R 1 is the radius of the free surface of the film that coats the rod.

Now, of course, for kR 1 greater than 1, you get waves or oscillations. So, there is no growth. Let us analyze this a little bit more. So, we will do exactly what we had done earlier. I am going to substitute omega is equal to i omega i where omega i is now real. That will introduce a minus omega i square on the left hand side like before. And I am going to reverse this part in red. So, that it is positive now because I am looking at this limit of kR or looking at this range of kR 1 less than 1.

So, if kR 1 is less than 1, then k square R 1 square minus 1 is negative. And so I am going to reverse it and bring out an overall negative sign like before the negative sign on the left and the negative sign on the right will cancel each other. And I can plot omega i square as a function of the right hand side. So, once again I will do the same thing. I will define a omega i star square which is a non-dimensional omega i square which is basically just omega i square divided by T by rho R 1 cube.

Note that I am working on the full dispersion relation ok. And so this is equal to it is just kR 1 into I have reversed this now, so I will write it as 1 minus k square R 1 square into this part it is a long one. So, I am not going to write it again. And so now, this is purely positive because I have reversed the part which would which is actually negative ok. So, I have reverse the part. And now I am going to plot the left hand side as a function of right hand side ok.

So, if I do that or rather if I plot it as a function of omega i square as a function of kR 1, then you will see once again the same thing. So, while doing this, you will have to choose a value of R naught ok. So, you will have to choose a value of the radius of the rod, and then plot it as a function of kR 1 ok. If you do that, once again you will get a very similar curve and the curve will look like this. Once again this will hit the x axis at 1, 1.0 ok.

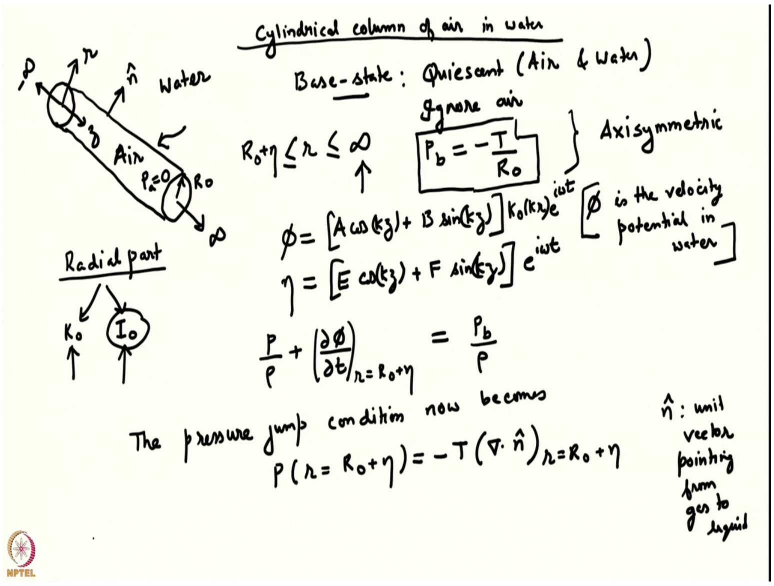
And that is because beyond kR 1 when it is greater than 1, omega i square is negative; or in other words, we are going to the regime where there would be oscillations and not instability. So, this is the place where we are getting growth kR 1 is less than 1. Does the growth rate get affected by the presence of the rod? That depends on what is the location of this maxima. Earlier we had seen that the location of the maxima was 0.697.

In this case, this maxima is approximately at 0.706 or approximately 1 by square root 2. So, it gets slightly shifted ok. So, now, this is the fastest growing mode. And so if this film breaks up into droplets or rather cylindrical droplets which will coat the rod, then this will give us the wavelength of those or the size of those droplets. This wavelength will be the fastest growing mode.

As you can see like before the criteria for whether it will grow or not grow is independent of surface tension; it is purely a geometric criteria which is given by this quantity kR 1 less than 1. So, any wave number which is less than any wave number which satisfies kR 1 less than 1 will exhibit instability and will cause exponential growth. Among all such wave numbers the wave number which grows the fastest satisfies kR 1 is equal to 1 by square root 2.

This is slightly different from what we had done earlier for a liquid cylinder. Now, this is a solid cylinder coated with a liquid film. Still prone to the Rayleigh-Plateau instability and the criteria remains the same. Let us now move over to yet another problem where we see the same instability and you will find that yet in this problem also the criteria for the instability remains exactly the same. It is a very analogous criteria that we have now seen two times.

(Refer Slide Time: 09:36)



So, let us the third problem is that of a cylindrical bubble ok or a cylindrical column of air you know in water let us say. Such columns of air often get trapped when waves break in the ocean, and it leads to air entrapment trapping a cylindrical column of air inside just below the surface of water ok. Now, we will simplify that problem. And we will ask the question that suppose, I have a cylinder, so it should of some radius R naught, and there is gas inside it ok.

So, let us say there is air inside it, and outside there is some liquid let us say it is just water. Now, due to the density difference between air and water, we are going to ignore the dynamics of the medium inside. We will model this column as being infinitely long. So, it goes from minus infinity. So, the coordinate system is again like before this is z and this is r. And so the interface between air and water, we are going to impose perturbations on it, and we are going to solve.

But now we are going to solve for the fluid outside not for the fluid inside. So, this is a filament with air inside it. We are going to impose axis symmetric perturbations on it. We are going to ask are those perturbations stable, do they oscillate or do they grow in time. We will find that some of those perturbations are still unstable to the Rayleigh-Plateau instability ok.

So, this is here the base state once again is quiescent. So, both air and water is quiescent. We are going to ignore air, the density of air is much much less than that of water. So, as a first approximation, we are going to ignore air. We have ignored air in all the examples that we have done until now except that in all the examples the gas was present either above or outside, now the gas is present inside. So, this is like a air bubble.

So, we are not solving for the air inside the bubble. We are going to solve for water outside the bubble that is still modelled by a Laplace equation. So, when we have perturbations the perturbation velocity potential in the water is going to be still modelled by the Laplace equation, let us do that. So, in the base state, the pressure once again is, so if I say that the pressure here air pressure here is 0. In the earlier example, when we had done it for a liquid cylinder, we had modelled the air pressure outside to be 0.

And so the pressure inside was more and so that was T by R naught. Now, we are saying the pressure inside is 0. The pressure inside because of the way it is curved is always more than the pressure outside. So, if I have set the pressure inside to be 0, then the pressure outside has to be less than the pressure inside. So, that is minus T by R naught. So, this is the pressure in the water outside. It is a uniform pressure, we are ignoring the presence of gravity, and we are going to solve for perturbations on the surface.

We have already done this geometry. So, I am just going to skip a few steps because they are identical to what we have done for a cylindrical column of liquid. So, like before using variable separation in cylindrical coordinates axisymmetric, we will find that the velocity potential the perturbation velocity potential is once again given by. And now for the radial dependence, we are going to have K naught and I naught.

So, for the radial part, so for the radial part of the velocity potential like earlier we will have two choices K naught and I naught. And it will be a linear combination of the two. Now, our domain, we are not solving for air inside. So, our domain extends from R is equal to R naught plus some eta that we will put on the surface to infinity. So, our domain goes from R naught plus eta to infinity.

So, small r does not go to 0. Recall that K naught diverges at small r, I naught diverges at large R. So, we our domain actually goes to infinity. So, I cannot include I naught in my calculation, because I naught will diverge as the radius becomes larger and larger. And we are solving for so we are solving for the water. And the water is unbounded. It is readily unbounded ok.

So, I will write this, so phi is the velocity potential in water. So, we are solving only for water, we are not solving for air. We are assuming the air to be quiescent, and the pressure in the air to be always 0. So, then we are going to get a K 0 of K r here, I hope you understand why. So, I am just going to put this bracket here. Earlier it was I 0 of K r because the domain included small r equal to 0.

Now, it is K 0 of K r because the domain includes small r is equal to infinity or small r going to infinity, and so I 0 diverges as small r becomes larger and larger. So, we have to set the coefficient of I 0 to 0 now and only K 0 will survive. Similarly, eta would be E cos K z plus f sin K z into e to the power i omega t. Like before eta is only a function of z and t. By definition eta does not depend on R, and these are my normal mode approximations.

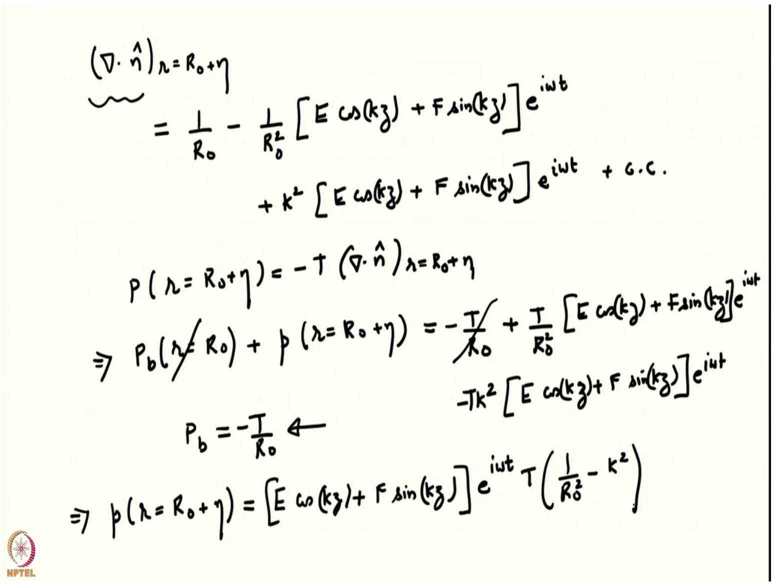
And so, like earlier we will write the Bernoulli equation the total pressure plus the total velocity potential in this case the total velocity potential is just the perturbation velocity potential, because this quiescent fluid in the base state. And this is equal to the Bernoulli constant. The Bernoulli constant is once again found by applying the same equation in the base state. And in the base state, it is just P b by rho, P b by rho in the base state is just it is just P b by rho.

And like before so the pressure jump condition now becomes P at R is equal to R naught plus eta is equal to minus T grad dot n. Earlier our n was pointing from the fluid where we were solving into the fluid where we were not solving. Now, also our n continues to point. But now we are not solving for air, but we are solving for water ok. So, the n is still radially outward and that brings in this minus sign ok. So, this is at r is equal to R 0 plus eta ok.

So, now so n points from, so n is a unit vector pointing from gas to liquid. You can check that this minus sign is necessary because this equation is also true in the base state.

And without this minus sign, you will not, so if you apply this equation in the base state and if you calculate the curvature in the base state that is just a constant, and you will find that it recovers this equation only if you take that minus sign. So, the minus sign is necessary. So, now, we will once again like before we are going to work out what is divergence of n. So, this is we have already done this before.

(Refer Slide Time: 18:08)



So, divergence of n at r is equal to R 0 plus eta is just going to be. So, I am just going to write down expressions that we have already written you know. So, we have done this before. So, I am just going to straight away give you the expressions, and you can check it for yourself. This is exactly the same procedure that we have followed.

You have to define a function capital F whose value is constant on the surface, and then you have to take the gradient of that the components of that. So, the denominator is not there in the linear approximation is a is an n approximately is just gradient of capital F. And then you have to calculate the various components of grad F in cylindrical coordinates, and then take the divergence of that quantity.

Once you take the divergence, then you will get this to e to the power i omega t plus K square. Please refer to the previous video where we have already done this. And of course, there is a complex conjugate which I am suppressing. And then the expression we have just written that the total pressure at R 0 plus eta is equal to minus T, this is not there. And then I can write my left hand side as P b.

And P b in the base state there is no eta, so it is just at r is equal to R 0 plus the perturbation pressure at R 0 plus eta is whatever we have calculated from this. So, I am just substituting. So, there is a minus sign overall. So, minus T by R naught plus T by R naught square into E cos K z plus F sin, and then minus K square T K square into E cos K z plus F sin K z e i omega t. And then I am I have to write a plus c c also, but I am not writing that.

And we know that P b in the base state is just minus T by R 0. In the base state, there is a uniform pressure in the water outside that pressure is just minus T by R 0. There has to be a pressure difference between inside and outside. If we say that the pressure inside in the air is 0 in the base state, then the pressure outside has to be lower and that is minus T by R 0 ok. It is only the difference which matters. So, then this quantity and this quantity get cancelled out using this.

And so like before we just recover an expression for perturbation pressure. So, P at r is equal to R 0 plus eta is just the same thing. So, T, so it is E cos K z plus F sin K z into e to the power i omega t T into 1 by R naught square minus K square. Very similar expressions, there will be a overall difference of minus sign because now in the pressure boundary condition there is a minus sign compared to what was there earlier, so that is the only difference.

(Refer Slide Time: 22:01)

B.C : $p(r=R_0) = -\rho \left(\frac{\partial \phi}{\partial t}\right)_{r=R_0}$

$$\Rightarrow \left[\left\{\rho i \omega K_0(kR_0) A + T\left(\frac{1}{R_0^2} - k^2\right) E\right\}\cos(kz) + \left\{\rho i\omega K_0(kR_0) B + T\left(\frac{1}{R_0^2}-k^2\right)F\right\}\sin(kz)\right] e^{i\omega t} + c.c. = 0 \quad \rightarrow (1)$$

K.B.C : $\frac{\partial \eta}{\partial t} = \left(\frac{\partial\phi}{\partial r}\right)_{r=R_0}$

$$\Rightarrow \left[\left\{i\omega E + k K_1(kR_0) A\right\}\cos(kz) + \left\{i\omega F + k K_1(kR_0)B\right\}\sin(kz)\right] e^{i\omega t} + c.c. = 0 \quad \longrightarrow (2)$$

So, once again you can go back and substitute it in into the linearized Bernoulli equation. So, the linearized Bernoulli equation is just once again like we have to find that the perturbation pressure has to be applied at r is equal to R naught, and not r is equal to R naught plus eta because that will bring in a non-linear term is equal to minus rho. Similarly, del phi by del t the perturbation the time derivative of the perturbation velocity potential also has to be evaluated at r equal to R naught.

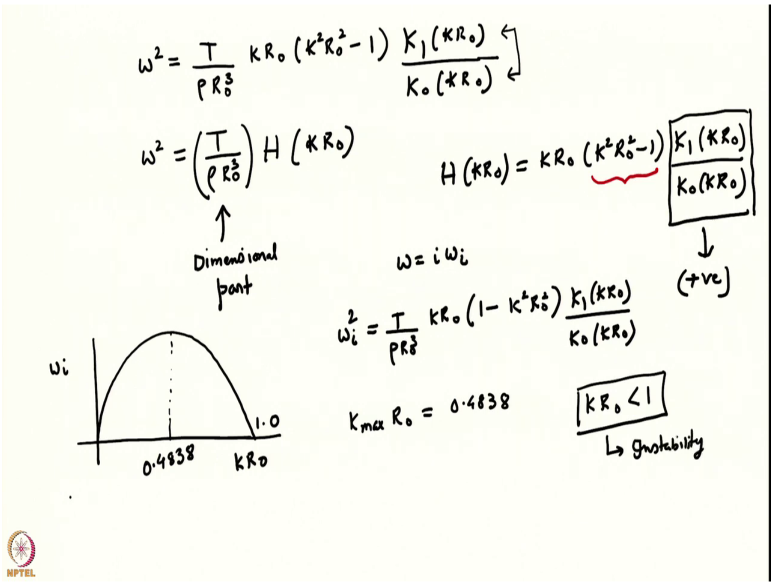
If you substitute whatever we have found in the last slide, we will recover the equation rho i omega, I am straight away writing the equation. So, compared to the last example where we had done this for a liquid cylinder, the only difference is there is a minus sign. And instead of I naught there, we have K naught here. I naught was the modified Bessel function, here this is the modified Bessel function the second kind ok.

So, instead of I naught here, we have K naught into A plus T into 1 by R naught square minus K square into E. And this whole thing gets multiplied by cos K z plus rho i omega K naught kR naught B plus T 1 by R naught square minus K square into F, this gets multiplied by sin K z, the whole thing is multiplied by e to the power i omega t plus c c is equal to 0. This will be my equation 1 coming from Bernoulli equation, linearized Bernoulli equation; second, so Bernoulli equation.

The second thing will come from kinematic boundary condition. And so we will have del eta by del t is equal to del phi by del r at r equal to R 0. This will involve derivatives of K 0 with respect to its argument. So, this will lead to I am straight away writing the final answer, this will give you i omega E plus small k into the modified Bessel function plus i omega F plus small k, this whole thing multiplied by e to the power i omega t plus a complex conjugate is equal to 0. And this is equation 2.

Like before we have to set the coefficients of cos and sin to 0, 4 equations and 4 unknowns A, B, E, and F, the determinant will determine the dispersion relation. I will write down the dispersion relation. It just requires a little bit of algebra. You can do it yourself and verify that this is correct.

(Refer Slide Time: 25:18)



So, the final dispersion relation looks like omega square is equal to T by rho R 0 cube into kR 0 k square R 0 square minus 1 into K 1 of kR 0 divided by K 0 small kR 0. You might be wondering where did all the minus the extra minus sign that we got, why did not that made it difference to the dispersion relation? You can see that this K 1 earlier we had a I 0 in the numerator and I 0 in the denominator. And we had seen that the derivative of I 0 with respect to its argument is I 1.

Here there will be an extra minus sign overall, but the derivative of K 0 with respect to its argument will be minus K 1. So, that minus and the overall minus will cancel and lead to exactly the same dispersion relation except that now we will have K 1 and K 0 here. So, this is the only difference, the rest of it remains the same.

You can see that this also has the same structure as before that this is something, so this is the part which has the dimensions of frequency squared. So, this is the dimensional part, and the rest of it is non-dimensional. So, I can once again write it as may be some capital H some function of small kR 0 and capital h of small kR 0 is defined as it is a non-dimensional function 0 by K naught of kR 0. Once again you can show this part is always positive.

So, if this dispersion relation has to admit instability that instability has to again come from this part. Once again we find the exactly this is the same criteria that all wave numbers which satisfy kR naught less than 1 are unstable. We have now seen three examples one of a liquid filament, one of a thin film coating a cylindrical rod, and one of an air bubble. In all of them the criteria for instability is the same the growth rates will be different.

In this case, the growth rate if you plot the growth rate, so here non dimensionalize omega square. So, you can substitute omega is equal to i times omega i, like before omega i is real this will give you a minus sign and then you can reverse. So, you can write it as omega i square is equal to T by rho R 0 cube into kR 0, and reverse this 1 minus k square R 0 square into K 1 of kR 0 and K naught of kR 0. And this whole thing will be positive in the range kR 0 less than 1. So, you can plot it as a function of kR 0 and it will look something like this.

Once again it will hit the x axis at 1. And the maximum growth rate will be about 0.4838. So, kR 0 or rather the mode which grows the fastest, so K max into R 0 is equal to 0.4838. So, you can see that whether we have a denser fluid inside or whether we have a denser fluid outside, whether we have a solid inside it makes no difference to the criteria of instability; it makes a difference only to the growth rates of the fastest growing mode.

So, we will find that we will try to understand the origin of this behavior that what is so special about kR 0 less than 1. And why does this cause instability in each and every case that we have seen so far. In all the cases, kR 0 greater than 1 gives you stability at least it is linearly stable. And so you expect small amplitude oscillations governed by the dispersion relation.

The dispersion relation is actually different in all the three cases. So, the frequencies are not the same. However, the criteria the boundary between stability and instability remains the same in all the three cases. We will examine the reason for this in the next lecture.