Introduction to interfacial waves
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Lecture - 45
Rayleigh – Plateau capillary instability on thin film coating a rod

We were looking at waves and oscillations and Instability on a liquid cylinder. We had modeled it as being infinitely long, and in the base state, the fluid was quiescent. And there was a pressure jump due to surface tension and so, the pressure outside was taken to be 0, and the pressure inside was T by R naught, where T is the surface tension and R naught is the base state radius of the cylinder.

We had put a surface perturbation, and then we had calculated the dispersion relation for these perturbations.

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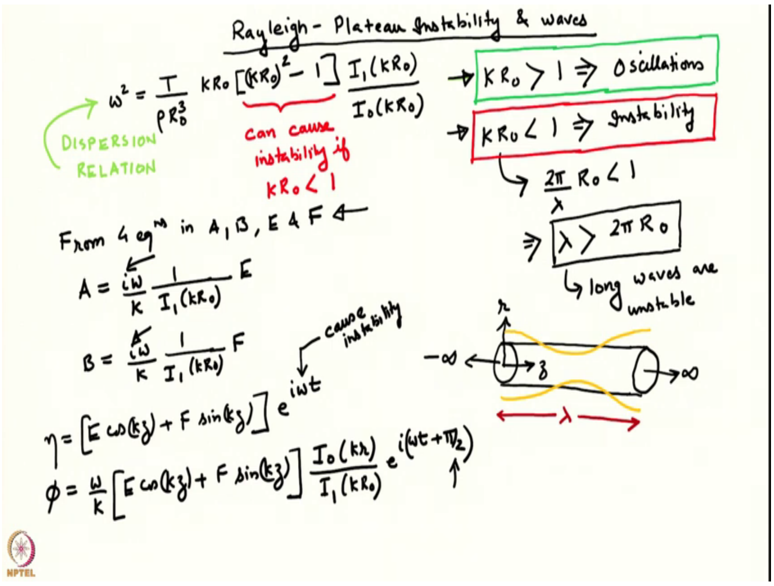

Note that we had found that the dispersion relation look like this. In particular, there was a qualitatively new aspect that we had found in this dispersion relation that this quantity which is contained within the square brackets and is highlighted in red here, can be actually negative in this case. This would imply that omega square would become negative, and thus, omega could be possibly purely imaginary.

I had argued that this would cause an instability causing exponential growth in time, as time became larger and larger. We are going to analyze this instability a little bit more in particular because this has implications for many practical engineering applications, especially in chemical engineering, where in certain dip coating applications, we take a thread and immersed it in a bath. And then it is pulled out slowly in order to coat the thread. And the thin film that the thread is getting coated with often becomes prone to the instability that we are studying here.

In this case, we do not have a thread as yet. But in the immediately next example, we are also going to discuss the case of a thread where there is a solid core inside and then there will be a film coating it. We will find that the same instability happens there as well. But let us finish what we had started. So, we had said, that this particular case depending on whether K naught depending on whether K R naught is greater than 1 or less than 1, one can have oscillations or instability.

Let us look at the instability case a little bit more in detail. Now, before we do that one can like before, one can obtain the dispersion relation was obtained through these four equations in these four quantities A, B, E, and F. And we had obtained four linear equations whose coefficients gave us a determinant and the dispersion relation was obtained from there.

Now, we can use two of these relations to express A, and B in terms of E and F. Once we have done that, we can write down our expressions for eta like the way we have written for the earlier cases. And phi will turn out to be omega by K E cos K z plus F sin K z to I 0 into e i omega T and plus pi by 2. This pi by 2 is coming like before from this extra factor of i, that is there in a as well as the expression for B.

So, e to the power i pi by 2 is just i, and I have absorbed that in the phase here. We have seen this in earlier examples also. So, now, you can see readily that if this omega is becomes purely imaginary, then it becomes i times omega i and then we recover e to the power minus omega i into t and if omega e is less than 0, then we obtain growth in time.

So, now, let us look at this. So, this can cause instability. So, let us look at this instability a little bit more.

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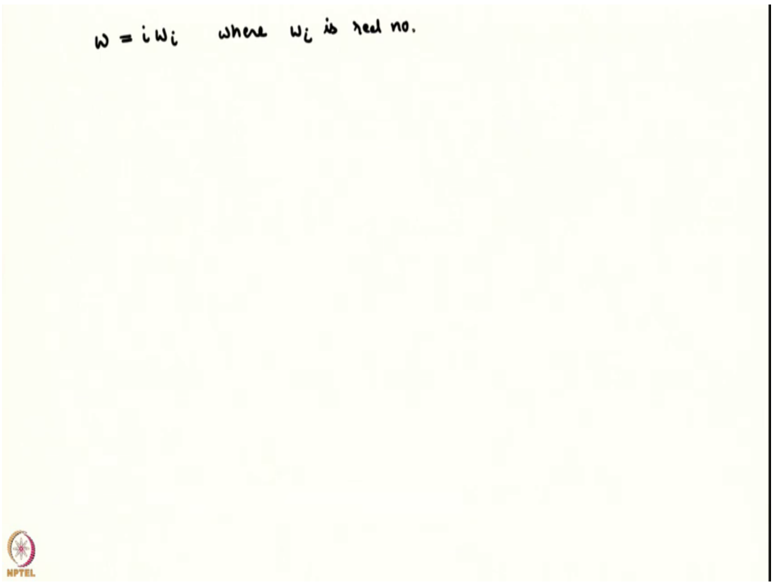

So, what I am going to do is I am going to just substitute omega is equal to i times omega i, where omega i is a real number, it can be positive or negative. But we can see that if it is negative then it will give us the growth rate. So, we will plug this back into the dispersion relation, ok. So, that I had already written the dispersion relation in the last slide. So, I am just going to plug omega on the left-hand side of the dispersion relation as i times omega i and then square it.

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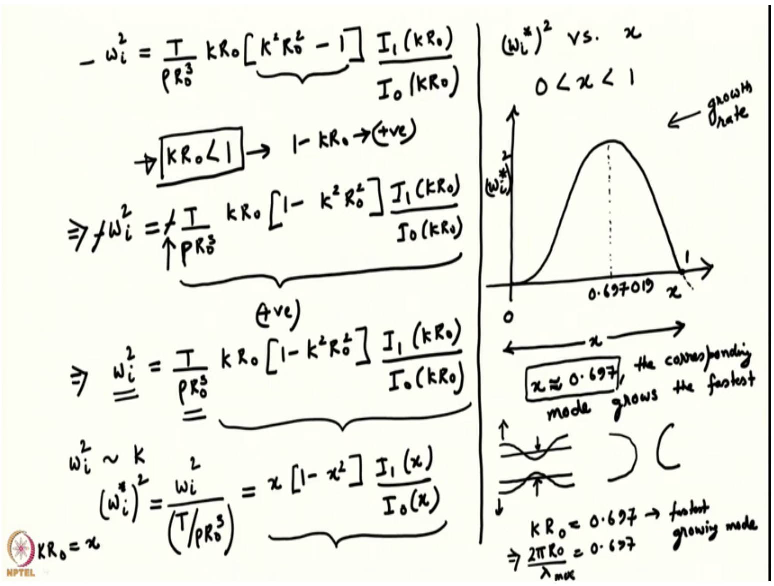

So, we are going to obtain. So, minus i square will give us a minus 1 or i square will give us a minus 1, and then we will have the rest of the right-hand side remains exactly the same. Now, we are going to get growth we said we are going to get growth when case K R 0 is less than 1, ok.

So, you can see that omega i square is going to be given by, so I am just going to reverse the right-hand side and write it in such. So, I am just going to reverse this part, ok. And make it positive, and pull out a minus sign. So, there is a minus sign here, and I will pull out a minus sign and reverse. So, this becomes 1 minus K square R 0 square. Why am I doing this? Because when K R 0 is less than 1, when K R 0 is less than 1 then 1 minus K R 0 is positive.

So, by reversing the thing I have make sure that this entire quantity is now positive. The negative part is contained here. And this negative and this negative gets cancelled, and so, I obtain omega i square is equal to T by rho R 0 cube K R 0 into 1 minus K square R 0 square into I 1 K R 0 divided by I 0 of K R 0.

So, now, you can see that omega i square is given by this quantity. And now if I take the square root, there will be a plus and a minus, and the minus part is going to give me instability. So, it is good to plot this omega i square. So, we are going to plot this omega i square as a function of K for different wave numbers. Recall that K is related to the wavelength of the perturbation that we imposed on the surface.

So, I have put this symbol in red here. So, K is related to 2 pi by lambda, and so, K R 0 less than 1 corresponds to lambda greater than 2 pi R 0. What is lambda? Lambda is the wavelength of the surface perturbation. So, we are saying that all those waves are unstable whose wavelength is greater than 2 pi R 0.

2 pi R 0 has a physical interpretation, it is the perimeter of the cylinder. So, any wave whose wavelength exceeds the perimeter of the undisturbed cylinder is unstable. So, this is also known as a long wave instability because waves above a certain wavelength all of them create instability.

So, now let us look at omega i square as a function of K. Now, I want to plot this in a non-dimensional way. So, this is related to my growth and I can non-dimensionalize this by T by rho R naught cube because both quantities because this part is non-dimensional. So, if I divide out omega i square divided by T by rho R naught cube, then I will call this some non-dimensional quantity omega i star square. And the right-hand side just becomes whatever way I have written.

I am going to use the notation K R 0 is equal to x, it is a non-dimensional quantity. So, this can be written as x, 1 minus x square and I 1 of x divided by I 0 of x. So, I am going to plot now omega i star square as a function of K R 0 or rather x. So, omega is star square versus x. This will tell us the growth rates of various modes which are present in the system.

Obviously, I am interested in this limit because I have said that we are looking at modes which grow. So, K R 0 has to be less than 1. So, I am interested in the possibility that x is less than 1, and x is bound below by this range. So, we are interested in this range, where x is between 0 and 1.

So, let us plot omega i star square as a function of x. This you can do in any of the software packages like MATLAB or Mathematica. And you will qualitatively see what I am going to draw by hand. So, you will see a graph which looks like this. So, you can find out what is the coordinates of this point, the graph intersects the x axis where omega i star is 0, so you can see that at x is equal to 1. This expression on the right-hand side is 0.

So, we start from here and this is the range we are interested in x between 0 and 1. Of course, the graph continues beyond this, but beyond this we will not get instability rather we will get oscillations because x would be more than 1, x is K R 0, so K R 0 would be greater than 1.

We have seen that K R 0 greater than 1. For a given R 0, if we choose a wave whose wave number satisfies K R 0 is greater than 1, then that particular mode would oscillate up and down it would not cause any growth. Here we are looking at growth, and so this is a plot of growth rate or rather square of growth rate, ok. So, omega i star indicates a non-dimensional growth rate. And if we square it, we will get the non-dimensional growth rate, ok.

So, now, the most interesting thing that you see here is that that all modes do not grow at the same rate. There is a mode which goes the fastest. There is a peak, well-defined peak, numerically it approximately happens at 0.697, it is actually 0.19, 019. So, that is where the maxima of the curve lies.

What does this mean? This means that at x is equal to 0.697, approximately. The mode whose x is 0.7; the corresponding mode grows the fastest. What are the implications of this? It implies that if you have a spectrum of modes present in your initial conditions and we are treating this by linear theory, and some of those modes are unstable that is they satisfy K R 0 is less than 1, then the mode whose K R 0 satisfies 0.697 or K R 0 is equal to 0.697 is the mode which is going to grow the fastest in time as time becomes larger and larger.

And so, it is going to increase exponentially in time beating out all the other modes some of which may also be unstable. So, sufficiently later in time we are going to see this mode dominating over all other modes. This has consequences because this mode can actually cause disintegration of the filament.

So, we have a mode which is like this, and it is going to grow in time, the crests are going to go outward, the troughs are going to go inward, and this will cause the disintegration of the filament into drops. This, this number can help us estimate the sizes of the drops that are going to be generated. So, this is the physical meaning of this number. So, K R 0 equal to 0.697 is the fastest growing mode.

Note that the criteria for the fastest growing mode does not depend on surface tension or any other physical parameter of the system. It purely comes out of, the it is purely a property of the geometry of the cylinder in the base state whatever is the radius it is related to that. So, this is telling us that if you put a wavelength, so K is 2 pi by lambda.

So, if you put a wavelength which satisfies 2 pi R naught divided by lambda is equal to 0.697, then that wavelength is, so I will put a lambda max here. So, that wavelength is going to cause the fastest growth. And that wavelength can help us determine the sizes of the droplets that are going to be generated when the cylinder breaks up into smaller and smaller drops because of the instability.

You may have seen this instability sometime, on a wintery morning when dew settles on spiders webs. You can sometimes see a string of droplets that happens because the spiders web is like a thin fiber on which liquid water is deposited through condensation, and then the film becomes unstable, and it breaks up into droplet us. This the sizes of the droplet us can be determined using the theoretical arguments that we have presented here.

Of course, they are also practical, more practical examples as I said before. There are dip coating examples where one immerses a fiber and pulls it out at a slow speed in order to coat the fiber with a certain liquid, and then the film that gets coated on the fiber also become susceptible to similar instabilities.

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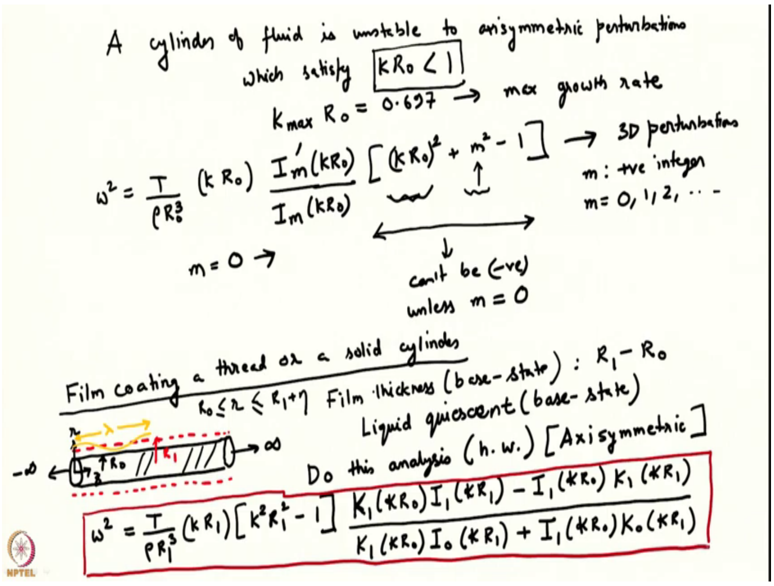

So, we have seen that a cylinder of fluid is unstable to axisymmetric perturbations. Recall that the all of this whatever we did was done under the axisymmetric approximation, ok. So, m, so if you put a cos m theta and do a 3D analysis, then this corresponds to m is equal to 0, which is by we got the modified Bessel function the subscript was I 0 and K 0.

So, now, we have seen that the cylinder of fluid is unstable to axisymmetric perturbations which satisfy K R 0 less than 1 and the maximum growth rate is given by, so K max does not indicate the maximum wavelength. This is the maximum; this is the wavelength whose growth rate is the maximum. So, K max into R naught is equal to 0.697 approximately will give you the maximum growth rate.

Now, one can ask what happens if I put in three-dimensional perturbations. So, it can be shown, so if one has to go back to the analysis and redo it without assuming axisymmetry. So, in your Laplace equation there will be a del square by del theta square term, and in general, you will have to take a linear combination of cos m theta and sin m theta.

So, once again we will use variable separation arguments. So, everything which is a function of theta will be separated out, and then you have to repeat the analysis. If you do that, you will find that the full three-dimensional dispersion relation for full three-dimensional perturbations, takes a similar form except that instead of getting I 0 and I 1, I 1 was d I 0 by d x. So, here I am putting a prime in order to indicate derivative with respect to the argument. And this would become K R 0 square plus m square minus 1.

You can see that if I substitute m is equal to, so this is for 3D perturbations. You can see that if I substitute m equal to 0 in the above, then it just reduces to what we already know the I 0 prime would just become I 1, we have seen that. So, this is this is the thing that we have seen.

Now, note that there is an m which appears here and this m is a positive integer. That is got to do with the fact that the theta direction is periodic. So, it can only be positive integers, ok. So, if this is a positive integer, then you can see that m is 0 in the axisymmetric case 1, 2, and so on. And so, in general the cylinder, you can see that this is no longer because this is a this is a positive quantity, this is a positive quantity and it is an integer. So, you can see that unless m is 0 you cannot make what is inside this square bracket negative, cannot be negative unless m is 0. Because if m is not 0, then m can be 1, 2 or whatever you know and you can see that it immediately becomes a positive quantity, it cannot be 0, ok.

So, this is telling us that this cylinder, if you put a 3D perturbation on it; it is stable, it will lead to oscillations. Whereas, if you put a axisymmetric perturbation, not all axisymmetric perturbations are unstable, only some of them are unstable. The once which satisfy K R 0 less than 1 are unstable, the once which satisfy K R 0 greater than one are stable, and they continue to oscillate like the 3D perturbations. So, this is the dispersion relation for the full 3D problem.

Now, let us come to the closely related problem of a film coating a thread, ok. So, there are many such examples which can be done. We are going to do a film coating a thread or a solid core whatever you want to call it. So, what we have in mind is a analogous problem. So, I have a solid rod, let us imagine of some radius I will call it R 0, and I am coating it with some film.

And this radius is R 1, the radius of the film, the free surface radius. And we can ask the same question here also. So, for simplicity, I am just going to assume that the rod is infinitely in in length, so it, so the coordinate system like before remains the same. This is z, this is R and it goes to plus infinity here and minus infinity there.

And we can ask the same question that what is the dispersion relation which governs perturbations on this film. You can note that this is analogous to what we did earlier in cartesian geometry. There we had found that waves on a dip pool had a certain dispersion relation. However, when I put the thickness of the pool as capital H, then the dispersion relation got modified. There was a tan hyperbolic of K H which came in when the depth of the pool was finite.

So, you can think that this is the cylindrical analog of a pool of finite depth because now here the film thickness is finite. Of course, in the previous problem also the film thickness was finite, but the film, but the liquid all the way went up to r equal to 0 small r equal to 0. Here the liquid film only goes from small r is equal to R 0 to small r is equal to R 1. So, the film thickness in the base state is just R 1 minus R 0.

Now, like before we can assume that the in the base state the liquid is quiescent. So, liquid is quiescent in the base state, and there is a pressure jump inside the film, the pressure inside the film is uniform, but different from 0. We will assume that the pressure outside in the air is just 0, and the pressure inside is a constant value which is governed by the radius of the film.

So, one can do this analysis. So, I give it to you as a homework problem. And I will just write down the dispersion relation, and I will just analyze the dispersion relation a little bit. We already have experience with doing this kind of things. If you look at the algebra that I have followed in the previous example, it should be very easy for you to do this case as well.

I am going to just give you the dispersion relation and leave it as an exercise for you to prove the dispersion relation that I have written. So, you can show that the dispersion relation is given by omega square is equal to T by rho R 1 cube. Recall that R 1, R 1 is the is the radius of the free surface, free surface of the thin film. So, T by rho R 1 cube into K R 1 into, so this is an axisymmetric dispersion relation. You can work out the corresponding full three-dimensional thing also, but let us first write down the axisymmetric dispersion relation.

Now, recall that this one in this dispersion relation we anticipate that there will be both I 0 and K 0, the modified Bessel function I 0 and K 0 will both appear. This is similar to what we did in the earlier example where we had done waves on a liquid pool of finite depth, along the vertical direction when the depth was infinite we had eliminated one of the exponentials e to the power minus k y was the coefficient of that was set to equal to 0 because otherwise it would diverge.

Similarly, here when we did a liquid cylinder in the earlier example, we set K the coefficient of K 0 to 0, because K 0 would diverge at small r equal to 0. But now you see that our domain for the liquid goes from R is equal to, so our domain goes from R is equal to R 0 because there is a solid cylinder at R 0, this is my solid cylinder and at the free surface it would be R 1 plus some eta, when I put a perturbation.

So, you can see that my small r never goes to 0. It goes from capital R 0 to capital R 1 plus eta. And so, we do not have to eliminate when we solve the Laplace equation using variable separation. Once again, we will find K 0 and I 0, but we now we do not have to eliminate K 0.

Analogously, in the finite depth problem we had e to the power k z and e to the power minus k z, and when the depth was finite, we did not have to set both of one of them to 0. Here also we will not set K 0 or I 0, the coefficient of them to 0, ok. And so, it is clear that both K 0 and I 0 will appear in the dispersion relation because their coefficients are not set equal to 0. So, we find that the next term becomes slightly complicated and it has this form.

So, because now there is a modified Bessel function. So, I am going to use a big K and a small k, the small k is the wave number. So, I am going to use a k like this, to indicate that this is different from. So, this k is related to the wavelength of the perturbations that we will impose, ok.

So, you can put a perturbation of wavelength lambda and k would be 2 pi by lambda. So, K 1 of a k R 0, I 1 of k R 0 minus I 1 of k R 0, K 1 of sorry this is k R 1, k 1 of sorry k R 1 divided by; so, the next term becomes slightly complicated because now in general both K 0 and I 0 will survive.

And then, when we look at the kinematic boundary condition, there will be derivatives of those, because there is a del phi by del R in the kinematic boundary condition and so, there will be derivatives of those with respect to the argument. And that will give us I 1 and a K 1. So, that is how in the final term, there is both K 1 and K 0 and I 1 and I 0.

So, this is the dispersion relation which governs perturbations on a solid rod of radius R naught, and which is coated with a film of radius R 1. We will analyze this dispersion relation slightly more in the next class.