Introduction to interfacial waves
Prof. Ratul Dasgupta
Department of Chemical Engineering
Indian Institute of Technology, Bombay

Lecture - 44
Rayleigh–Plateau capillary instability (contd.)

We were looking at waves on a cylindrical base state geometry.

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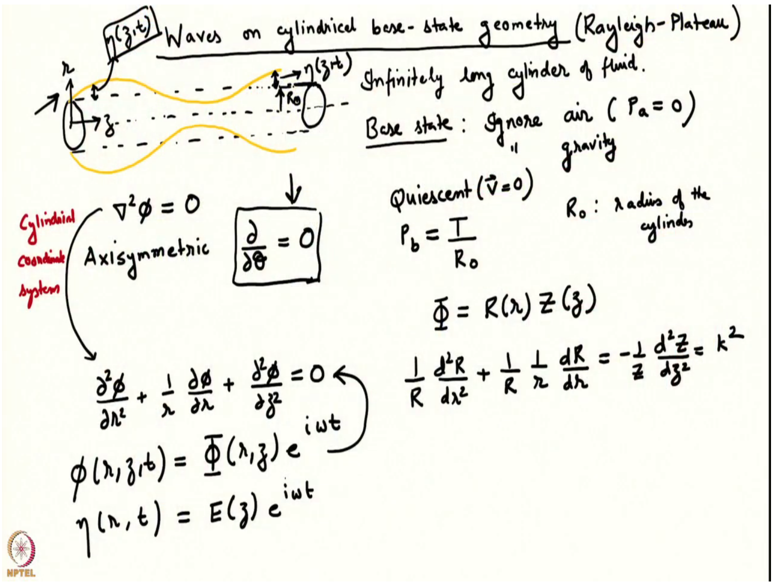

Where this was a fluid cylinder in the base state, quiescent fluid, with a pressure jump inside compared to outside due to surface tension. We had ignored gravity and we were looking at perturbations surface perturbations on the cylinder. So, we had solved the Laplace equation and this case we had found a modified Bessel function.

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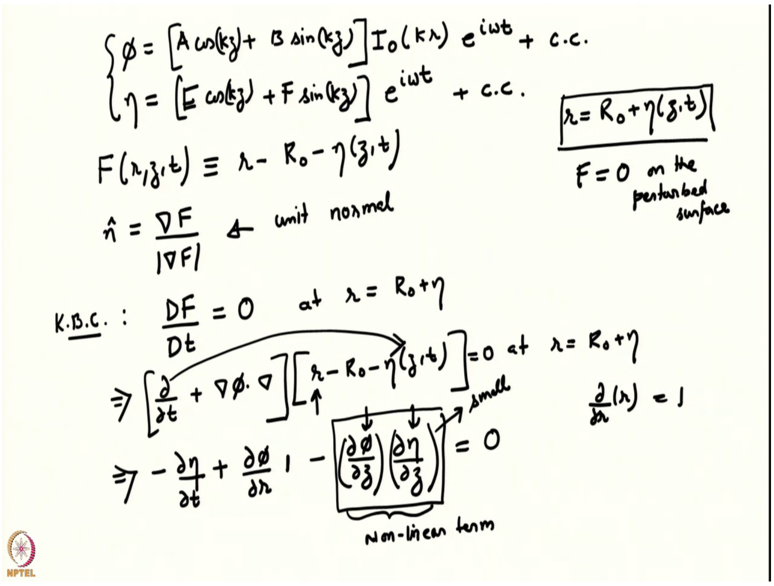

And we had expressed the R and the Z dependence of the velocity potential, the perturbation velocity potentia, and the surface perturbation in terms of the modified Bessel function. So, the I 0 had appeared in phi and then there were these trigonometric functions which are appear in both phi and eta and then we did a normal mode analysis. Now, we had worked out the Linearized Kinematic Boundary condition.

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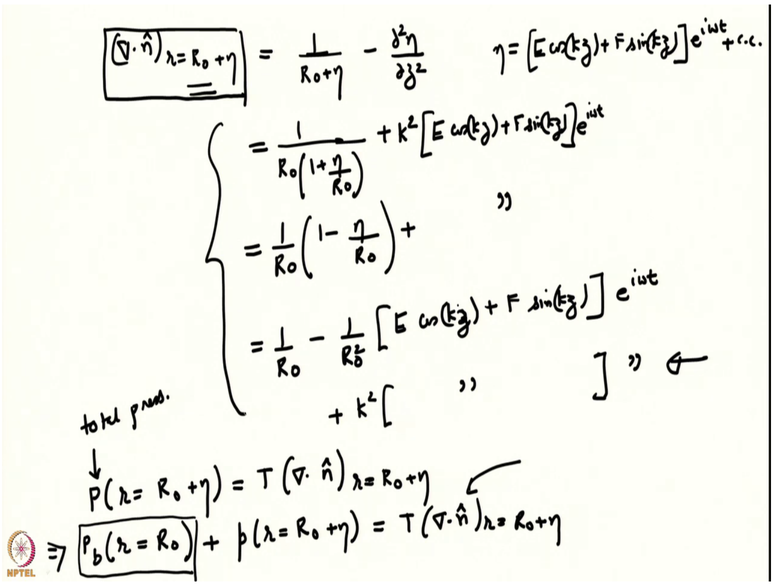

And we had also looked at the expression for the diversions of the unit normal, evaluated at the perturbed interface. Let us continue from there.

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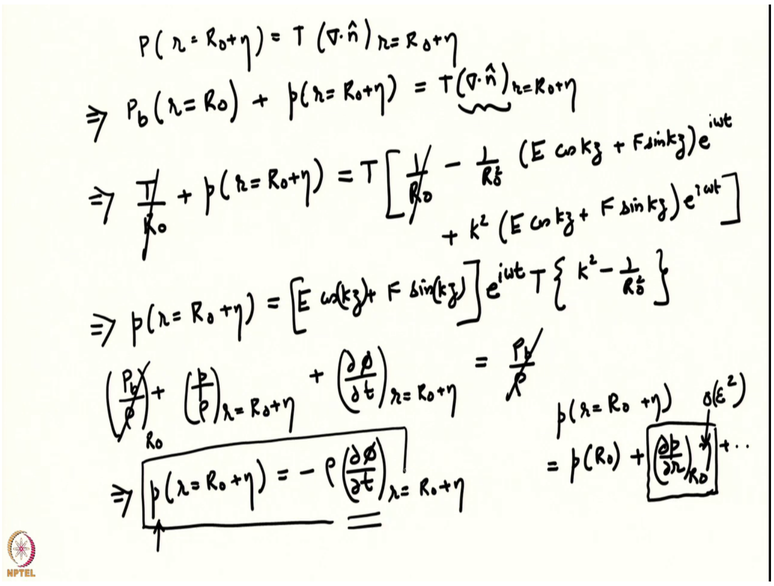

So, we had seen that the pressure boundary condition is just R 0 plus eta is equal to T divergence of n also evaluated at R 0 plus eta and this pressure on the left hand side is a sum of base. In the base state the radius of the cylinder is constant and a perturbation pressure which has to be applied in general that R 0 plus eta is equal to the right hand side.

We have already evaluated the expression for this earlier. So, you can see that we have already evaluated the expression here. So, we are going to use this expression in the equation that we have just written. So, we write p b is T by R 0, we have seen this earlier plus the perturbation pressure is equal to T times divergence of n evaluated at R 0 plus eta, that we have seen earlier is just this 1 by R 0 minus 1 by R 0 square plus k square the same thing.

And of course, there is a complex conjugate which has to be added which I am not writing, explicitly. Now, you can see that the first term here gets cancelled by the first term here that is just subtracting out the base state from our equation. So, that we are left with an equation for the perturbation.

The perturbation pressure is just this expression. So, it is E cos kz plus F sin kz, that part is common and then we will have E to the power i omega t and then the T can be pulled out and what we have here is k square minus 1 by R 0 square. So, this is my expression for the perturbation pressure. We have seen the linearized Bernoulli equation earlier we have already written it down in the previous class.

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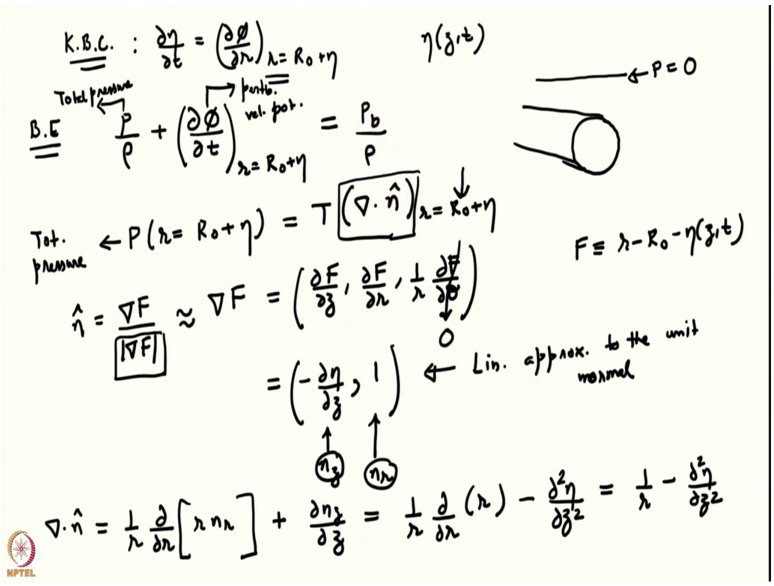

So, the linearized Bernoulli equation is just this, you can see that the quadratic term is missing and that is because we have linearized it ok and in particular notice that the Bernoulli constant here is not 0 and this is because the this is the linearized Bernoulli equation applied at the free surface and. So, the Bernoulli constant has been determined by applying the Bernoulli expression in the base state.

In the base state the velocities are 0 and so, we are just left with P b by rho this quantity is not 0 in the base state. So, using that equation we obtain. So, I can split the pressure into two parts, this term is also applied at R 0, but this term is anyway a constant the first term is anyway a constant. So, I can skip writing the R is equal to R 0, is equal to P b by rho the Bernoulli constant.

Now this and this will cancel each other, P b is just a constant and so, I obtain p at r is equal to R 0 plus eta is minus rho del phi by del t at r is equal to R 0 plus eta. Now while proceeding further, we will have to remember that we have to do a Taylor series expansion in order to decide whether these expressions are to be evaluated at R 0 plus eta or at R 0.

You can see that p, small p is a perturbation pressure, phi is also a perturbation velocity potential. So, like before we will have to write these as so, p at R 0 plus eta can be written in a Taylor series as p at R 0plus del p by del r also evaluated at R 0into eta plus so on. You can see that this quantity is an order epsilon square quantity because p itself is order a epsilon because this is a perturbation pressure and eta is also a surface perturbation.

So, if I had not done it properly with non dimensionalisation, expansion would start at as epsilon eta 1 plus epsilon square eta 2 epsilon p 1 plus epsilon square p 2 and so on. So, this would give me an order epsilon term. So, which is neglected in a linear theory. Similarly you can see that this quantity also has to be expanded in a Taylor series and so, it is clear that this equation in the linear approximation will be just applied at the undisturbed interface, which in the base state is just small r is equal to capital R naught, this is the same as what we had done earlier.

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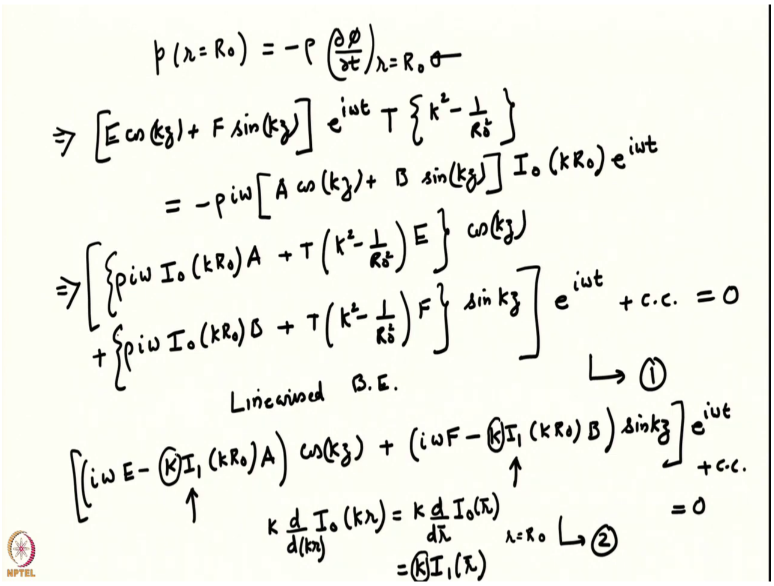

So, we obtain p at r is equal to R naught. This is the linearized version of the Bernoulli equation is minus rho del phi by del t at r is equal to R naught. We already have the expressions for p and phi. So, we can plug them in and if you do that then you will get the expression, E cos kz plus F sin kz, e to the power i omega t into surface tension into k square minus 1 by R 0 square is equal to we have to take a del phi by del t.

We already have the expression for phi, earlier we have already written it earlier. So, at the top of this slide you can see that we have already written the expression and so, we have to just to take the time derivative of this expression. It will lead to an i omega and then the rest of the part remains the same and we have to remember that we are applying this at small r is equal to capital R naught.

So, with that we obtain minus rho i omega into A cos kz plus B sin kz into I 0, now the small r has to be applied at capital R naught because of this into e to the power i omega t and of course, we are suppressing the complex conjugate part, now I will like before we can combine all the terms which have a cos kz and all the terms which have a sin kz and later we will equate the coefficient to 0.

If I do that then I obtain rho i omega I 0 of k R 0 into A plus Tinto k square minus 1 by R 0 square into E, this whole thing multiplies cos kz, plus a similar thing multiplying sin k z into B plus T into k square minus 1 by R 0square to F multiplying sin kz and this whole thing of course, gets multiplied by E to the power i omega t plus complex conjugate is equal to 0.

So, this is my equation 1, which has come from the linearized Bernoulli equation. It was slightly more complicated than last time because in this problem my base state has a curvature. In the last problem that we have seen so far, in the base state the interface was flat there was no curvature in the base state and so, the calculations were a little bit easier.

Here the base state has a curvature. The natural coordinate system here is a cylindrical coordinate system. So, expressions get a bit longer and we have to deal with modified Bessel function in the radial direction. So, this is my equation 1. We now we will go back to the second equation, which is our kinematic boundary condition.

The kinematic boundary condition I have already written at the top of this page we have already written the kinematic boundary condition in this linearize form. Once again this del phi by del r can be expanded in a Taylor series and you can convince yourself that this has to be applied at r is equal to R 0and not r equal to R 0plus eta, because that would contain a non linear contribution.

So, the equation for the kinematic boundary condition leads us to another equation. I am going to leave it for you to work out that equation and I am going to write down the final answer. The final answer is just this. I will tell you where I got the I 1 from in a moment, let me write it down.

This whole thing gets multiplied by e to the power i omega t plus complex conjugate is equal to 0 and this is my equation 2. Now where did I get these I 1's from? Note that the kinematic boundary condition contains a derivative of phi with respect to R phi the radial part of phi contains an I naught.

So, we have to do this derivative d by dr of I. naught of kr. It is easy if you this derivative becomes easier if we express numerator and denominator in terms of kr. So, I am multiplying and dividing by kr. So, that this actually becomes k d by dr bar of I naught which is a function of r bar. This actually turns out to be I 1, the modified Bessel function the first order modified Bessel function of the first kind the 0th order was I 0 the first order is I 1 ok.

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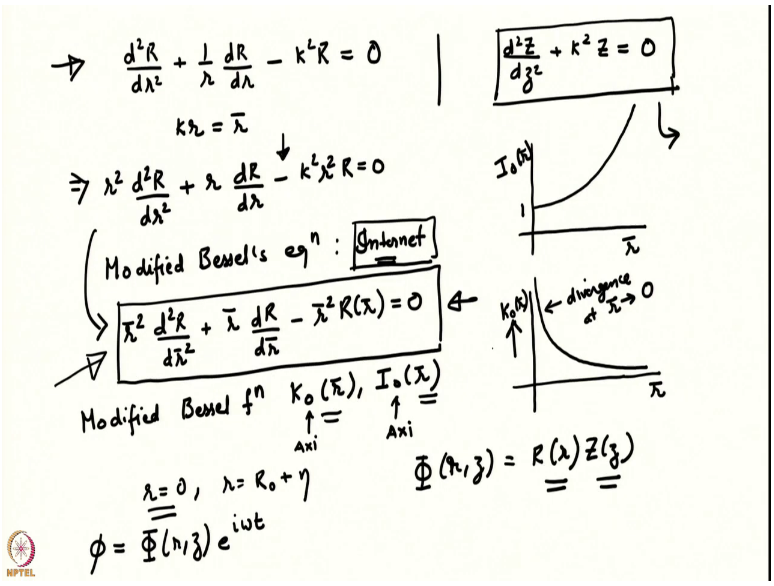

You can see from the graph that I had drawn earlier that I 1 is just related to the slope of I 0 because I 0 is shape like that I 1 in that range is always going to be positive. So, that is how I get my I 1 and then we have to substitute r is equal to small r is equal to R 0 ok which is why. So, this additional k that I am getting is why we have this additional factor of k here and here. So, those are my two equations 1 and 2.

Like before, we will have to set the coefficient of sin and cos in both the equations to 0. We will be left with 4 equations in 4 unknown A B E and F. I am going to write down those 4 equations and then straight away write the dispersion relation which will come by setting the determinant. Those will be linear inhomogeneous a linear homogeneous equations in A B E and F. We have done this a few times now. So, it should be easy for you to follow this.

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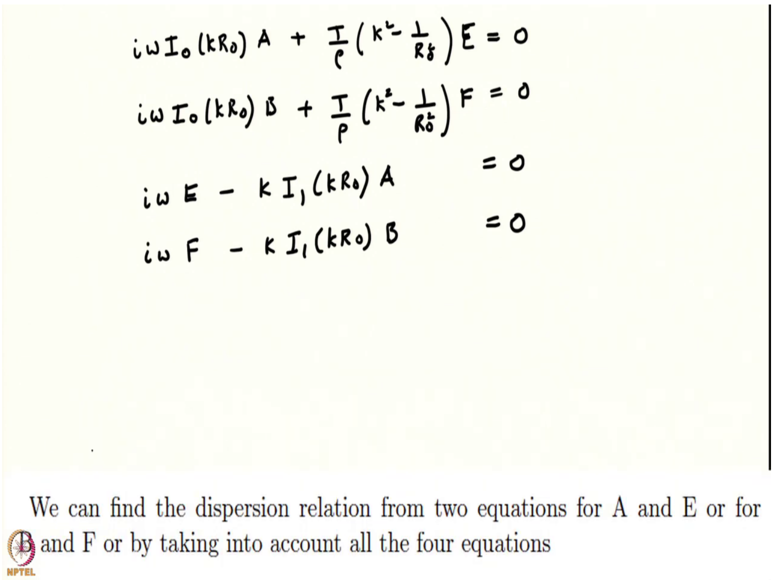

So, we will have the four equations are I have divided out by rho. So, that becomes T by rho k square minus 1 by R naught square t is equal to 0, that is equation number 1. Then we have from the second part from equation 2, it has an I 1 these are just the coefficients of sin k z and cos k z, equal to 0. Once again we have to set the determinant equal to 0 the determinant you can write it down the determinant you can.

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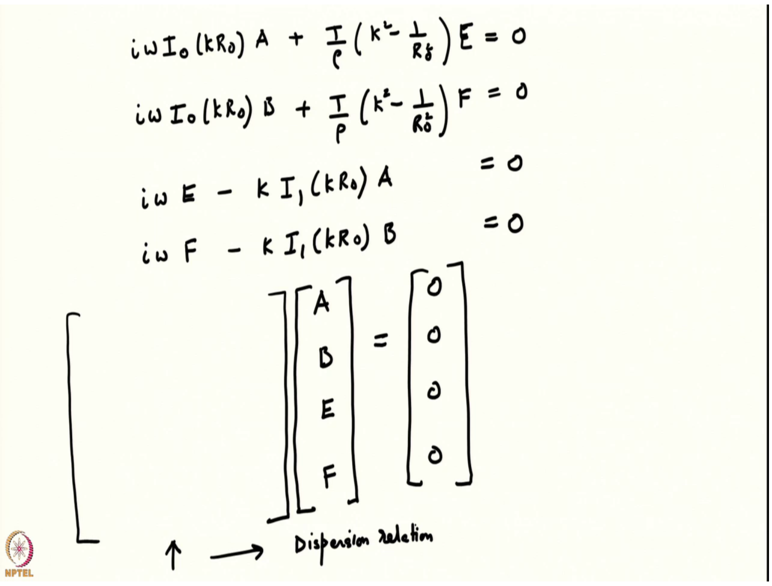

So, you can write it down as a matrix times the unknowns A B E F and this is a homogeneous equation. So, you can write this as 0 0 0 0 and once again like before we do not want trivial solutions. So, we do not want A B E F all 0 for non trivial solutions. The determinant of this matrix has to be equal to 0 and that will lead us to the dispersion relation. You can see that it is going to be a quadric in omega. So, it will involve products of omegas, but it can be factorized into a quadratic.

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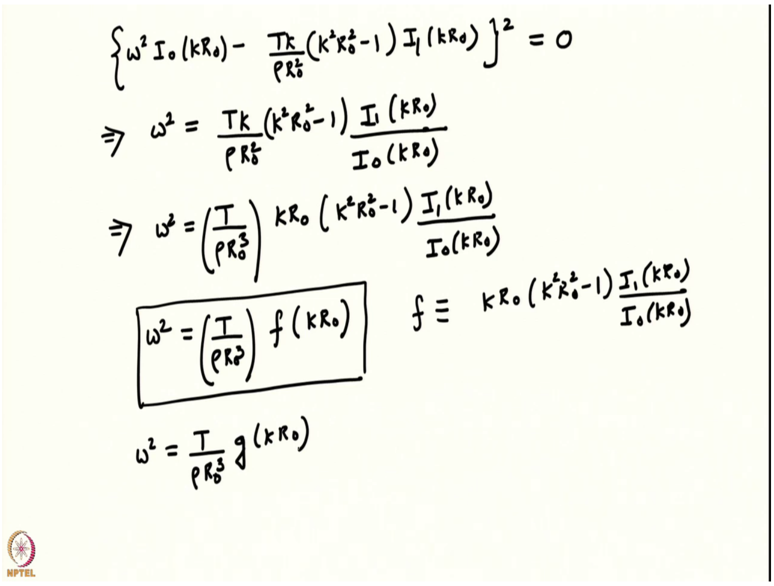

So, I am going to write down the dispersion relation and I leave the algebra to you. Algebra is not difficult you just have to work it out. So, the dispersion relation can be written in this form, omega square I 0 of kR 0 minus Tk by rho 1 0 square into k square R 0 square minus 1 into I 1 of kR 0, this whole square.

So, it factorizes into a quadratic and so, this just tells me my dispersion relation which is Tk by rho R 0 square, k square R 0 square minus 1 into I 1 of kR 0 divided by I 0 of kR 0. So, that is our dispersion relation it can be written in a slightly more compact manner, omega square is equal to.

Note that kR 0 is a non dimensional quantity and so, this I will write it as T by rho 0 R cube into so, I am multiplying and dividing by R 0. So, that makes the denominator R 0 cube and the numerator as kR 0 and the rest of the expression remains the same. Notice the analogy of this.

So, I can write this as T by rho R 0 cube. This is the part which has the dimensions of frequency square the rest of it is non dimensional. So, I can call it a some function of kR 0, kR 0 is a non dimensional argument, f itself is a non dimensional function. So, you can see that this analogy.

So, this is our dispersion relation and here in this case f is defined as kR 0, k square R 0 square minus 1, I 1 of kR 0 divided by I 0 of kR 0. Notice the analogy of this with what we had done earlier. For pure capillary waves on a pool described by in a rectangular Cartesian geometry there we had found rho R 0 cube into some function g of kR 0, if the pool was a finite depth.

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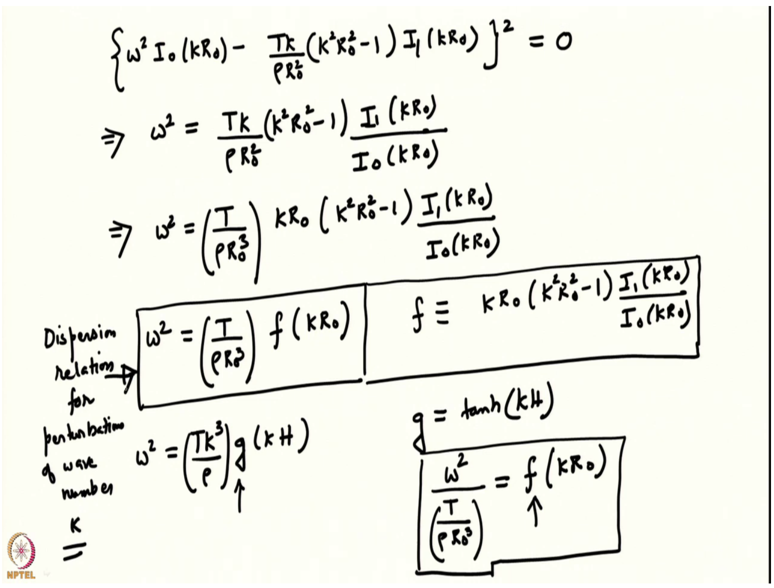

Or rather in that case it would be g of kH, in this Tk cube by rho, we had found and this is some non dimensional function and in this case g had turned out to be tan hyperbolic kH. Notice the analogy purely from dimensional arguments, you can see that whether we are solving for capillary waves on deep water or capillary waves on a cylindrical filament, we can always argue that omega square divided by something which has the dimensions of frequency squared.

In this case it is T by rho R 0 cube, T by rho into some quantity which has the dimensions of 1 by length cube ok. So, it could be k cube in the Cartesian case or 1 by R 0 cube in the cylindrical case. So, this is a non dimensional quantity. So, this must be a function of another non dimensional quantity in the rectangular pool it was kH.

In the cylindrical case it is kR naught. So, this much you can anticipate from dimensional reasoning. What precisely is the functional form of f or functional form of g cannot be inferred from dimensional reasoning and one has to do a detail calculation to figure out what is the functional for dependency of f on kR 0 or g the function g on the non dimensional combination k into H.

So, now let us analyze this dispersion relation. So, this is our dispersion relation, dispersion relation for perturbations of wave number k. So, this is the dispersion relation. Let us analyze this dispersion relation and we will find that there is something interesting about this dispersion relation which was not there in the earlier example that we have seen so far.

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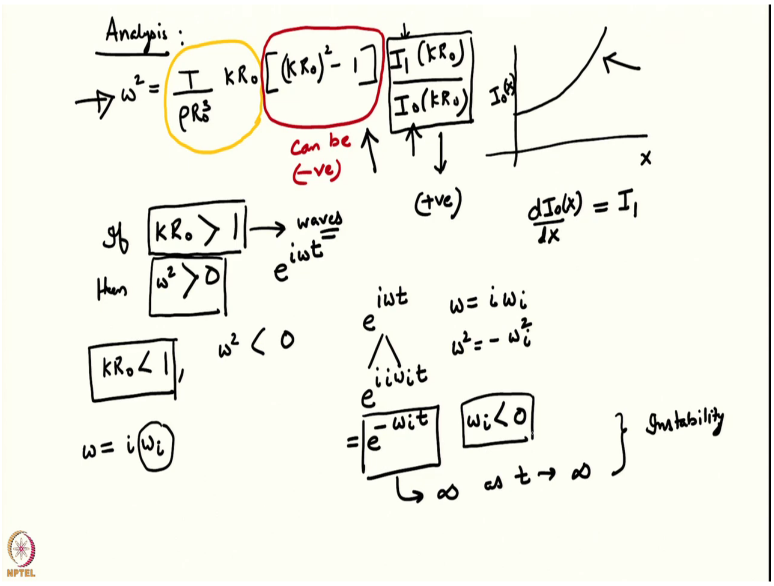

Analysis of the dispersion relation: So, we have seen that omega square is equal to T by rho R 0 cube into kR 0 into case kR 0 square 1 into I 1 of kR 0 divided by I 0 of kR 0. Recall that I had told earlier that the plot of I 0 as a function of some non dimensional argument x, if we plot then it goes like this we have also seen that dI 0 of by dx if I 0 is a function of some non dimensional variable x is just I 1, the way this is curved you can see that I 0 of x is positive and its slope is also positive.

So, I 0 of kR 0 is always going to be positive in the interval kR 0 between 0 and infinity I 1 of kR 0 which is just the derivative of I 0 with respect to small r and then evaluated at kR is also going to be positive, for any value of k and so, this part of the expression is always positive, these quantities are anyway positive.

However, you can see that the quantity which in between can be negative this can be negative. Why are we interested in negative values? We are interested in negative values because on the left hand side we have omega square. If we have a the square of a quantity being negative then the quantity can become complex in this case it will become purely imaginary when the right hand side becomes negative.

So, what is the criteria for the right hand side becoming negative? The right hand side becomes negative when so, if kR 0 is greater than 1, then omega square is greater than 0, recall that we have done a normal mode analysis where we substituted e to the power I omega t. So, I expect omega square to be real and omega square to be greater than 0.

So, I am going to get oscillations or waves when omega square is greater than 0. So, when kR 0 is greater than 1, then it leads to waves or oscillations, these are capillary waves. We have looked at the standing waveform. You can also put travelling wave form and you will recover exactly the same dispersion relation.

Now, let us ask the question what happens when kR 0 is less than 1. If kR 0 is less than 1 then omega square is clear is less than 0 because all the other parts are positive and this part in red here is the only part which becomes negative. So, omega square is negative which implies that omega becomes a purely imaginary quantity.

Now, this has consequences. You can recall that we had done e to the power i omega t if I said omega to be a purely imagining quantity, I would write it as i times some real quantity omega i. This implies omega square is minus omega i. So, you can see that e to the power i omega t, there is a square here.

So, this, would give me e to the power i into i omega i into t and so, this is giving me e to the power minus omega i t and. So, if I have an omega which is less than 0, this term is going to go to infinity as time goes to infinity. We will see that this is indeed what happens when this relation is satisfied.

So, this the fact that we have this quantity diverging to infinity implies that instead of oscillations about the base state, we get an exponentially growing we put a perturbation and the amplitude of the perturbation grows exponentially in time. So, this is what is known as an instability.

Until now we have not looked at any instability. We have only looked at situations where the base state was stable. So, if I introduced a perturbation about the base state it would oscillate about the base state. This is because the perturbation has a restoring force which wants it to bring it back to the base state, but when it arrives at the base state it arrives with the non zero inertia. So, there is an overshoot and the moment there is an overshoot there is again a non zero restoring force which again tries to bring it back to the base state.

So, this interplay between inertia and restoring force leads to oscillations and we have seen a number of these example so far, both discrete mechanical systems as well as fluid interfaces, where we found dispersion relation governing the frequencies of those oscillations. This is the first example where we are finding that the dispersion relation contains a particular case where which if satisfied can cause instead of oscillations can cause exponential growth in time.

We will try to understand the meaning of this and including the physical reason why this exponential growth happens, but first lets analyze the growth in a little bit more detail. We are going to do write omega is equal to i times omega i in this dispersion relation I am going to substitute it here and work out.

So, this omega i clearly will tell me something about the rate at which things are growing. We will see that it will give us a quadratic equation for omega i. There will be a positive value of omega i and a negative value omega i. The negative value is of interest because as I said here the negative value of omega i will is what will give me a divergence in time the positive value will decay to 0 exponentially first. We will continue this in the next class.