Introduction to interfacial waves
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Lecture - 43
Rayleigh-Plateau capillary instability

Let us start by looking at waves on a cylindrical base state geometry. This particular problem is also known as the Rayleigh-Plateau capillary instability problem and is named after the two people Rayleigh and Plateau who were the first two study it.

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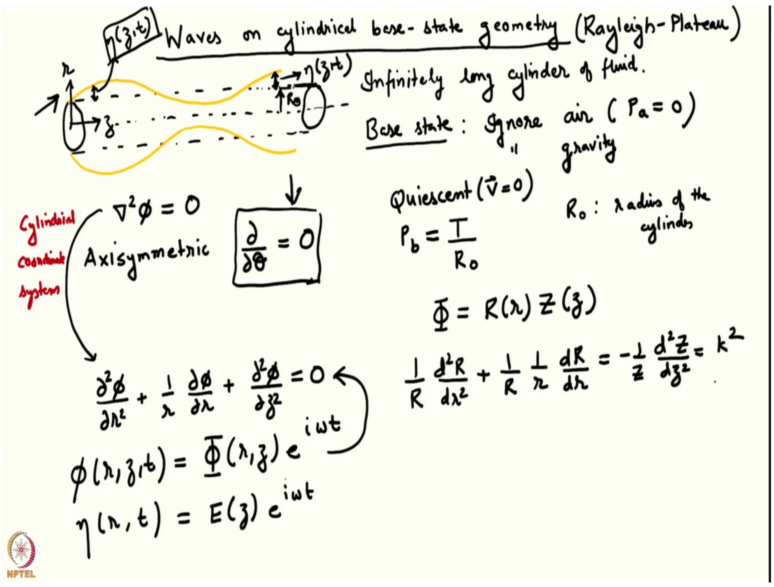

So, what we find here or what we are considering here is an infinitely long its modeled as an infinitely long cylindrical column made of fluid. So, you can see that in the base state, the column is infinitely long it is made of some fluid we will ignore the effect of the gaseous medium around it. So, we will set the pressure to be 0 there. We will ignore the effect of gravity ignore air ignore gravity. So, in this case if we find waves the restoring force will be due to surface tension alone.

So, you can see that in the base state the fluid is quiescent like before. So, no velocity anywhere, but pressure in the base state is not 0 and that is because the base state has a curvature ok. This is unlike what we have done until now where in the base state the interface was flat and did not have any curvature. If the surface tension coefficient is T then the base state pressure is given by T by R naught where R naught is the radius of the cylinder the unperturbed of radius of the cylinder. So, this is R naught this is the center line.

And now what we are going to do is, we are going to introduce perturbations on the surface of this cylinder and we are going to ask the question to these perturbations oscillate or do they growing time. In this particular case we will find that there are some perturbations which oscillate and pleat to waves we look for waves of the standing wave form and you will find that there are other perturbations which do not oscillate, but grow in time those are the unstable modes. So, let us begin our analysis.

So, once again like usual we will use the Laplace equation to represent the perturbation velocity potential, then the pressure in air is assumed to be 0. We are also going to make the approximation that my perturbations are axisymmetric. So, all derivatives with respect to theta are going to be 0.

Note that I am using a cylindrical coordinate system here, this is my radial coordinate and the horizontal direction is my axial coordinate along the length of the cylinder or along the axis of the unperturbed cylinder. In the azimuthal direction there the angle is measured as theta and we are saying the del by del theta is equal to 0. So, all my quantities will be independent of theta.

Now with that approximation we have to write down our Laplace equation. So, you can see the Laplace equation in a cylindrical coordinate system in a cylindrical coordinate system is given by del square phi by del r square plus 1 by r del phi by del r plus del square phi by del z square is equal to 0.

We are going to introduce surface perturbations and the surface perturbation is measured by this quantity eta. You can see like before that eta is measured from the base state. So, eta is the difference between the perturbation the local perturbation the height of it compared to the base state. So, this is eta and you can see that eta is a function of z and t. So, as you go along the axis of the cylinder eta varies. So, eta is a function of z and t, its not a function of theta because of the axisymmetric approximation. So, eta and phi are not functions of theta.

Now we will have to solve the Laplace equation, we have done this before. So, we will do this quickly. So, we are going to say that phi of r, z, t is some eigen mode r, z into e to the power i omega t that is my normal mode approximation, eta is just a function of r and t. And this is some e of z into e to the power i omega t that is the usual thing that we do.

If I plug this in into the Laplace equation, the e to the power i omega t does not matter and I get an equation for phi. Now again I am going to use variable separation. So, I am going to say that this is some function of small r and some function of small z. If we substitute into this equation and separate the small r dependence from the small z dependence the way we have done it until now then we will find in the following equations.

So, I am skipping one or two steps here because we have done this a few times before. So, I am separating all the small r dependence everything that depends on small r is on the left hand side everything which depends on small z is on the right hand side of this equality and I have chosen the separation constant is equal to plus k square you can think why I am doing this.

Now, if I like usual if I solve this equation these leave me to two equations to ordinary differential equations both of them are linear for capital R and capital Z.

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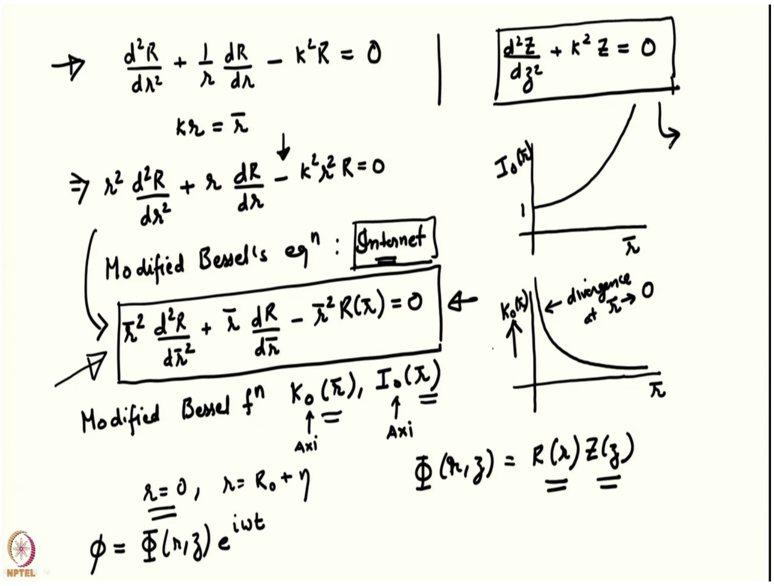

The equation for capital R turns out to be, the equation for capital Z similarly turns out to be let us work on the arch equation first. So, like before what we do is we introduce a non-dimensional quantity which is k r ok. So, I will call this r bar. So, I will write this as r square, I am multiplying both sides by small r square and then I will convert all the small r into small r bar.

Now note that this is not the Bessel's equation that we have seen earlier. This is related to the Bessel's equation, but this is not exactly the Bessel's equation because there earlier we had a positive sign here now we have a negative sign. So, now, this is known as what is known as the modified Bessel's equation.

You can show very easily you can convert this into in terms of r bar. So, this becomes in terms of r bar. And this has the form of a modified Bessel's equation you can look up modified Bessel's equation on the internet. In particular this is a modified so, the solutions to this equation will be the modified Bessel function which are given by K 0 of r bar and I 0 of r bar those are the two symbols that are used this is a linear second order ordinary differential equation its not exactly the Bessel's equation its the modified Bessel's equation.

The 0 here comes because in general there is a n square present in the modified Bessel's equation. Please look it up on the internet and you will find it that there is an n square in the modified Bessel equation and we have to put n equal to 0 in order to recover this equation this n is related to the axisymmetric approximation because we are not making the we are not looking for three dimensional perturbations.

So, we have put we are only putting axisymmetric perturbations if you had done the three dimensional exercise then we would have put cos m theta in the azimuthal direction and so, this corresponds to choosing m is equal to 0. So, that is why the 0 comes from the axisymmetric approximation.

So, what do the modified Bessel functions look like? So, I am just going to plot them qualitatively. So, I 0 of r bar as a function of r bar argument is non dimensional looks like this. Zeroth order modified Bessel function of the first kind I 0, it diverges as r bar goes to infinity starts from 1. K 0 of r bar has a singularity or it diverges r equal to 0 and it decays at large r a large r bar.

You can immediately see what are the consequences we are going to express the solution to this equation in terms of linear combination of K 0 and I 0. There will be two pre factor sitting which are the constants of integration. Because our domain extends from r equal to 0 to our domain extends from small r equal to 0 to small r is equal to the radius of the filament unperturbed plus some perturbation.

So, any function the diverges at r equal to 0 is not going to be acceptable to us. You can see that this K 0 of r has a divergence at small r and so, we have to set the pre factor of K 0 of r bar to be 0 that will just leave us with I 0 of r bar. This is a very similar exercise compared to what we did earlier when we had the Bessel function there we had j 0 and y 0. And we eliminated y 0 here we are eliminating K0 ok.

So, we with usual arguments we find that our velocity potential is capital phi of r comma z into e to the power i omega t and capital phi of r coma z is has something which depends on small r and something which depends on small z. We have already found out the r dependence we need to find the z dependence you can the z dependence is governed by this equation very easy to solve you can see that it is a linear combination of cos k z and sin case and we will retain both because none of them diverge as z goes from minus infinity to plus infinity.

So, I can write down the general form for phi and eta from whatever we have done so, far.

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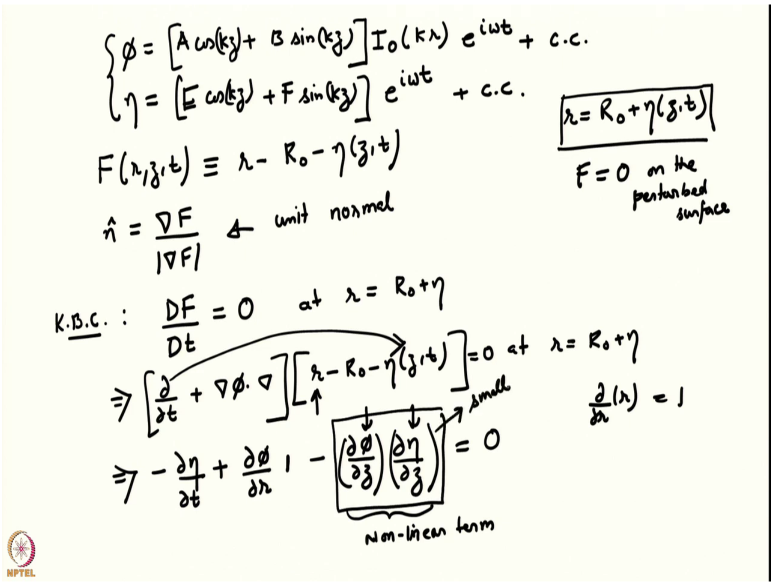

The general form of phi will be some A cos k z plus B sin k z into a linear combination of K 0 and I 0. I will eliminate K 0 by setting the pre factor to 0. So, we are left with only I 0 e to the power i mega t similarly eta is equal to I will put some different constants these are (Refer Time: 13:25) complex cos k z plus f sin k z into once again I 0 of sorry this is there is no r dependence here. So, this is e to the power i omega t.

And of course, we have to remember that we have to add the complex conjugates. So, now, let us proceed with these forms. Our boundary conditions remain the same we have a kinematic boundary condition and we have a Bernoulli equation applied at the free surface. Let us work on the Bernoulli equation first because that involves computing some unit normals and computing the divergence of the unit normal. So, here let us find out what is the unit normal first.

So, we define a function capital F like before whose value is constant on the perturbed surface. We have perturbed our surface in the form r is equal to R 0 plus eta which is a function of z and t. So, you can see that here, then my yellow line represents the perturbed surface that perturbed surface is given by small r is equal to capital R naught plus eta as a function of z comma t. So, we have to construct the function which will be constant on that yellow surface like before.

So, as we did earlier we will construct a function whose value is 0 on the surface and that function is defined as r minus R 0 minus eta of z comma t. So, because the equation of the surface is given by this. So, F is 0 on the perturbed surface at all time small n will point vertically outward if we define it has grad F by mod grad F. So, this is our definition for the unit normal. We will use this when we will have to calculate Bernoulli equation at the surface.

Let us come to the kinematic boundary condition. The kinematic boundary condition we have seen says that DF by Dt is equal to 0. The surface is a material surface and so, the value of f capital D by dt of that value is always 0 we have seen this also earlier all we have to do is apply this capital D by dt operator to the cylindrical coordinate system that we let us do that.

So, this is true at the surface which is given by R naught plus eta. So, I have del by del t plus grad phi dot grad, this is my d by dt operator and this is r minus R naught minus eta of z comma t at r is equal to R naught plus eta is equal to 0. If we work on that then we get the del y del t operator operates only on this quantity because other two are not functions of time. So, I get a del minus del eta by del t.

And then I have del phi by del r that is coming from the grad of phi into del by del del r of small r. So, I am doing del by del r of small r that is just 1. So, this is just 1 and then we have del by del z eta is the only quantity which is a function of z and so, we have minus del phi by del z into del eta by del z is equal to 0.

You can immediately see that this is a non-linear term phi is a perturbation velocity potential eta is also perturbation quantity if you have non dimensionalize this each of them would be epsilon time some order one quantity. So, the product of these two terms would be an order epsilon square quantity we want to do a linear analysis and so, this is neglected. So, this is small.

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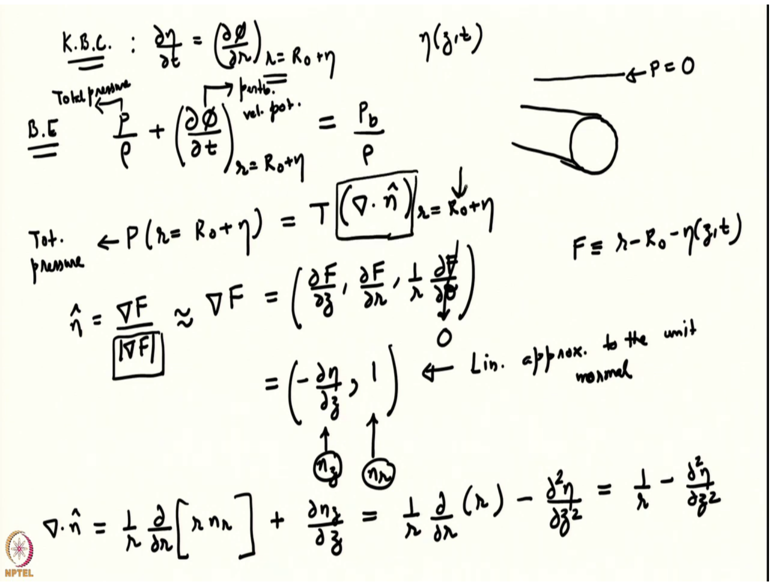

So, at leaning order our equation just becomes del eta by del t is equal to del phi by del r at r is equal to R naught plus eta. I do not have to worry about r is equal to R naught plus eta on the left hand side because eta by definition is just a function of z. So, eta does not depend on r. So, this r is equal to R naught plus z operates only on the applies only to the right hand side. So, this is my kinematic boundary condition linearized let us what come back to the Bernoulli equation.

So, the Bernoulli equation says P by rho plus the total pressure by density divided by the total velocity potential and I am doing Bernoulli equation at the surface. So, this is r is equal to R naught plus eta is equal to until now we have put the Bernoulli constant to be 0. Here we have to be more careful because until now we have looked at base state where the interface was flat.

So, whether we were looking at capillary waves or whether we were looking at surface gravity waves, the interface in the base state was flat and so, in the base state the pressure at the interface was 0. In this case our interface in the base state has a curvature, it is actually a part of a cylinder and there is a curvature on the cylinder.

So, we have to calculate the Bernoulli constant by applying the Bernoulli equation in the base state; in the base state there is no velocity because the fluid is quiescent and so, the only term which survives is P b by rho. So, that is the Bernoulli constant and so, in the perturb state we have the left hand side and that must be equal to the same equation applied to the base state. So, this determines the Bernoulli's constant for us.

So, pay attention that this is total pressure base state plus perturbation this is also total velocity potential, but in this case because there is no contribution to velocity potential in the base state as the fluid is quiescent. So, this can be thought of as perturbation velocity potential. So, now, let us proceed from here the pressure jump condition like before is given by P at r equal to R 0 plus eta and this P is again the total pressure is equal to surface tension times divergence of the unit normal which we have just described.

Now, the unit normal is grad F by mod grad F. F we had defined earlier as r minus R 0 minus eta of which is a functional z and t. In a linear as approximation we do not have to worry about the mod grad F because this will involve non-linear terms. So, this is approximately equal to grad F. So, we just have to compute gradient of F in a cylindrical coordinate system ok.

So, this basically has three components because we are under the axisymmetric approximation this is 0 and so, we are left with two components del F by del z which is just minus del eta by del z comma del F by del r which is just 1. So, this is the linearized approximation to the unit normal to the perturbed interface.

This is my r component of this is my z component of n and this is my r component of n. What is the formula for the divergence? We have to compute the divergence del dot n. So, del dot n is given by 1 by r del by del r of r n r; n r is the r component of n plus del n z by del z.

We already know what are the r and the z components of n we just have to substitute it into this formula. So, the first term becomes, it just becomes r because n r is just 1 and the second term becomes it just becomes a minus and it just becomes del square eta by del z square. So, the this reduces to 1 by r minus del square eta by del z square.

So, that is my expression for divergence of the unit normal. What we really need is the divergence of the unit normal calculated at the perturb interface r is equal to R 0 plus eta. So, let us do that.

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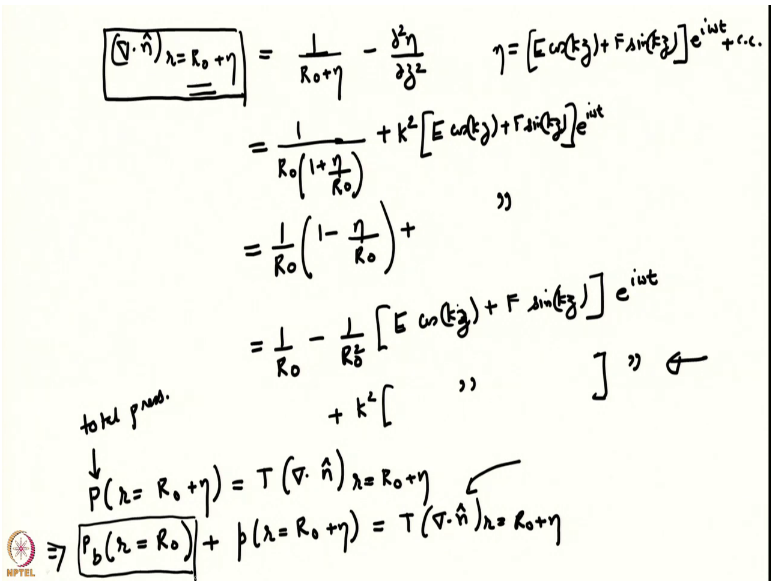

So, we need divergence of the unit normal calculated at r is equal to R 0 plus eta which in this case is 1 by small r and small r has to be evaluated at R 0 plus eta minus del square eta by del z square. Recall that we had written eta is equal to E cos k z plus F sin k z multiplied by e to the power i omega t and then we had a complex conjugate.

So, now, we are in a position to calculate this, I am going to work on this term. So, I am going to pull out r 0 and write this as one plane eta plus R 0 because there will be some linearization involved in this term and the second term can be calculated directly from this formula. So, this just becomes.

So, this will become plus k square into the same thing E cos k z plus F sin k z into e to the power i omega t and now i need to work on this term. So, I will write it as in a linearized approximation we only retain terms which are power unity in eta. So, I will just have this and the second term remains the same. So, this becomes 1 by R naught minus 1 by R naught square into E cos k z plus F sin k z this has a e to the power i omega t and then there is k square into the same thing.

So, now, I have an expression for divergence of the unit normal calculated at the perturbed interface. We have to go back and substitute this in the pressure boundary condition. The pressure boundary condition was P at r is equal to R 0 plus eta is surface tension times the divergence calculated at the perturbed interface. This is recall I told you was the total pressure. So, this is a sum of base plus perturbation.

In the base state filament or the fluid is in the shape of a cylinder. So, the radius is constant everywhere and it is just r is equal to R 0. In the perturb state this will apply at R 0 plus eta and so, this becomes T of divergence of n r is equal to R 0 plus eta. We are going to use this expression that we have derived for diversions of unit normal computed at R 0 plus eta. On the right hand side of this expression you will see that some terms cancel out.

Because there is a base state term here and this term and some term on the right hand side will cancel out because the base state satisfies P b is equal to T by R 0 and then we will recover an expression for the perturbation pressure. We will take this perturbation pressure and we will go back to the Bernoulli equation, linearized Bernoulli equation applied at the free surface and substitute it there and that will give us an equation for which will lead us to the dispersion relation.