Introduction to interfacial waves
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Lecture - 42
Axisymmetric Cauchy-Poisson solution visualisation: the pebble in the deep pond problem

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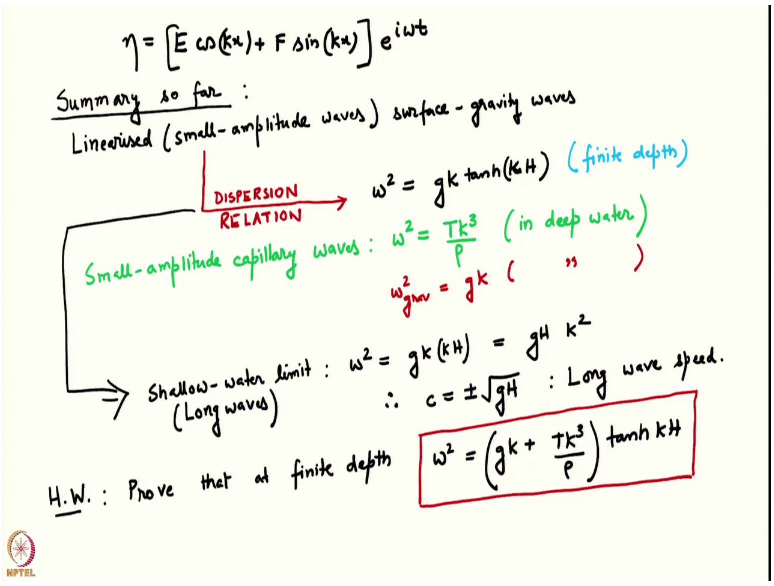

We were looking at Capillary Gravity Waves on a Deep Pool. Let us summarize what we have seen so far. We have seen that the dispersion relation for small amplitude surface gravity waves on a pool of finite depth H, is given by omega square is equal to g K tan hyperbolic KH. Here capital H is the depth of the undisturbed pool.

We have looked at a limit, namely the deep water limit of this relation. In that limit, we have also included the effect of surface tension. We have seen that pure capillary waves are governed by the dispersion relation TK cube by rho in deep water. Similarly, pure gravity waves are governed by the dispersion relation, omega square is equal to g K. The combination is governed by the dispersion relation, g K plus TK cube by rho.

We have also seen the shallow water limit, the reverse limit, where the wavelength is much longer than the depth in particular for such waves as the wave gets longer and longer, the effect of gravity dominates over surface tension. So, in that limit, the relation omega square is equal to g K time hyperbolic KH reduces to just omega square is equal to K square into g H. This leads to a phase speed which is independent of the wave number K. So, the long wave speed is a constant speed. All long waves travel at the same speed which is given by square root g H.

Now, one important difference that I have pointed out earlier was that, that the long wave limit and the short wave limit or rather the shallow water limit and the deep water limit, are qualitatively different. In deep water, waves are dispersive. Namely, the phase speed is a function of the wave number K. Whereas in the shallow water limit, the waves are non-dispersive; c p is a constant, it is not a function of K.

I leave it to you to prove as a simple homework exercise, that if you have both capillarity and gravity present, in a pool of finite depth, then one would recover a dispersion relation which is given by this. This is a very simple exercise using whatever we have learnt so far. You can prove this very easily.

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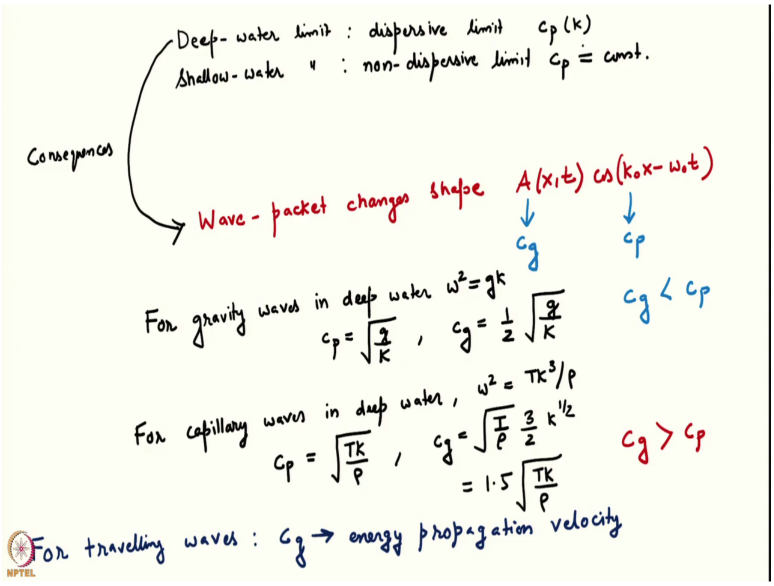

Now, let us go further. So, we have seen the consequences of the fact that the deep water limit is dispersive. In particular, we have seen that because the limit is dispersive or in other words every wave travel at its own phase speed, as a consequence, when we excite at time t equal to 0, a whole range of wavelengths, a spectrum of wavelengths, then the resultant wave packet keeps changing shape as it moves.

Using the method of stationary phase, we have also seen that at long times, and if you move with a constant speed x by t, then the wave packet may be described by this, this form. Some envelope, A of x comma t into cos K 0 x minus omega 0 t, where K 0 and omega 0 are also functions of x and time, some local wave number. In particular, we have also seen that this a of x comma t which represents the envelope of the wave packet moves with the local group velocity of the wave whereas, the this part cos K 0 x minus omega 0 t, moves with the phase speed.

Now, let us look at the dispersion relation once again in deep water. We have seen that in for pure gravity waves in deep water, the group velocity is less than the phase velocity, ok. It is exactly half of the phase velocity. This we have seen earlier. Let us do the same exercise for pure capillary waves in deep water.

You can see with a little bit of algebra, if you take this relation omega square is equal to TK cube by rho, then you can immediately see with a little bit of algebra that the group velocity in this case is greater than the phase velocity. So, this has consequences for how the what we will see when we travel along with the envelope. So, let us see that in a movie.

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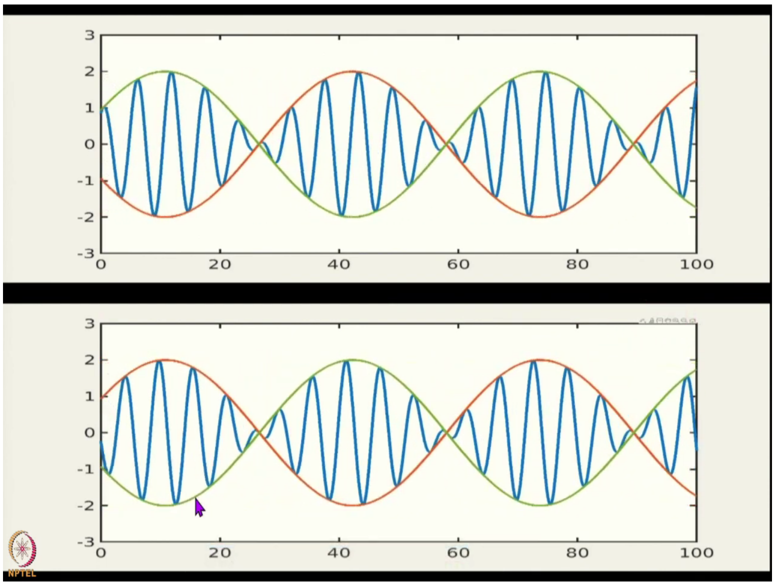

So, you can see two images here, the top and the bottom. Both have been plotted as a summation of two wave numbers. I have shown you a similar movie earlier. Now, at the top we have a pure gravity wave, the two waves of pure gravity wave, so I have used the dispersion relation omega is equal to square root g K, whereas, the one in the bottom panel is a pure capillary wave, where I have used the dispersion relation omega is equal to square root TK cube by rho. You can see a qualitative difference between the two.

So, here the envelope does not change shape, but if you follow the envelope, so for example, if you follow the peak of the envelope, the green curve, you follow the green curve, you will see that the blue curve which are the local phases they are overtaking. This is because this is a combination of two surface gravity waves.

For surface gravity waves the phase speed is more than the group speed. So, the group, the envelope moves with a group velocity, and so it moves slower than the local phase. So, you will see that if you follow the envelope, let us say I follow the peak, then you can see that the blue curves are overtaking me, ok.

Now, the reverse is happening here. Here if we follow the peak of the envelope you will see that the envelope is actually going ahead, and the with respect to the envelope, the blue curves are going backward. You can see that very easily. So, let us follow this. And you can see that with respect to that point the blue curves hat seem to be travelling backward, and that is because in the case of pure capillary waves the phase speed is less than the group speed or the phase velocity is less than the group velocity.

Now, I should also mention here; now an important point to notice and which can be proved, and so, I will say this without a proof here that for travelling waves it can be shown that the group velocity represents the energy propagation velocity of a Fourier mode. This can be shown easily. So, one must remember that the group velocity has a physical interpretation of energy propagation velocity.

Now, with that, we let us go over to the one last thing which we have not covered so far which is the axisymmetric Cauchy-Poisson problem. We have looked at the Cauchy-Poisson problem in two-dimensions, we have looked at the approximation, we have solved it for a delta function initial condition, and then we approximated the solution using the method of stationary phase, which gave us the concept of group velocity.

Now, let us go and ask the question what happens if our pool is cylindrical and if wave start spreading out readily. We have already done this before, where I showed that the final answer is expressible as a Hankel transform. So, let us go back to that solution.

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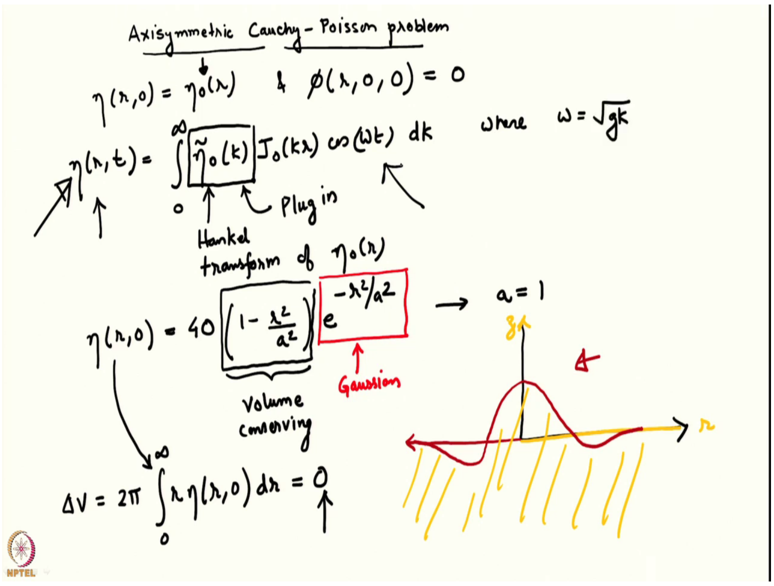

So, we come back to Axisymmetric Cauchy-Poisson Problem. We have already seen that subject to initial conditions eta of r comma 0, is some eta 0 of r, so some surface perturbation and no impulse. So, no impulse at the free surface at time t equal to 0. Our solution for the interface is given by this integral.

We did this in deep water and for pure gravity waves. One can put also surface tension into this, and this will modify the dispersion relation, but the expression will remain the same. Let us understand the physical meaning of this integral, and let us visualize it as to how it looks if we put a particular kind of initial condition.

We have done this exercise in Cartesian geometry for a delta function initial condition. Here we will choose a initial condition which is slightly different, and you will see the physical content of this integral. So, recall that this is just the Hankel transform of eta 0 of r, eta 0 of r is the initial perturbation. So, let us solve this integral for a particular initial condition.

So, the initial condition that I have chosen is taken to be this. This number is just for visualization, 40. We need to choose a large number, so that the waves are visible, but otherwise this number is not so important. So, why did we choose this kind of initial condition? You can immediately see that this is like a Gaussian hump, this is like a Gaussian and we are going to use a is equal to 1. So, I am going to replace a is equal to 1 everywhere in this formula.

This pre-factor just ensures that this perturbation is volume conserving. What does that imply? This implies that if at time t equal to 0, this is my base state where the interface is flat, and this is r, and this is z. And so, if I introduce a perturbation in the form of eta 0 of r. Sorry, it could be of something of this form. So, this is an axisymmetric problem. So, I will just reflect it about the other axis.

So, it is enough to consider just the right half because the left half is just a mirror image because of axisymmetric. There is a pool of liquid here, and if we ask the question how much is the volume of the pool before perturbation and after perturbation, then the volume should be the same or in other word the change of volume should be 0. So, this is manifested itself as the condition that delta v is equal to twice pi 0 to infinity r eta of r comma 0, dr is equal to 0.

This just expresses the volume of the perturbation. This says that the perturbation does not introduce any more liquid then was present in the base state. So, the perturbation preserves the volume. There is no change in the volume. So, we have taken this pre factor in such a way that this the integral if you plug this form into this integral, this integral will give you 0, ok. So, that is the rational for choosing this form. So, let us look at this form.

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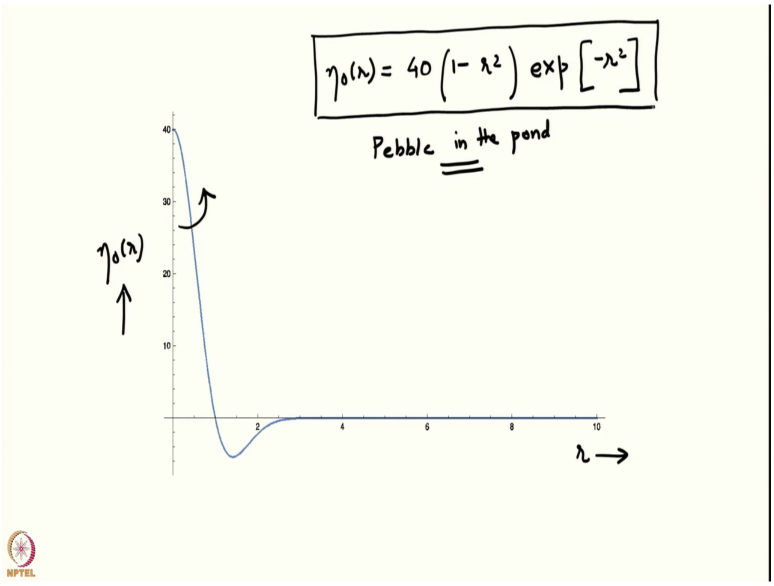

So, I have plotted it here. So, this is the form. I have taken a is equal to 1, and the pre factor to be 40. So, this is eta 0 of r. So, this is the shape of the interface as at time t equal to 0. Note that this is different from what we had done in the Cartesian geometry. There we had put a delta function initial condition. This had cost all wave numbers to get excited because the Fourier transform of the delta function is a constant. So, all wave numbers would get excited there.

Here we will see that we are not exciting all wave numbers. Here is an upper cut off, there is an upper cut off in K, and beyond that K there is no more wave numbers present in the system initially. So, eta 0 of r here is given by the initial condition 40, 1 minus r square, I have chosen a to be 1 exponential of minus r square. So, this is what? So, this is eta 0 of r and this is r.

So, I am perturbing the pool like this and I am asking what happens as a result. Remember that this is a cylindrical geometry. So, I have to rotate my curve about my axis of symmetry which is a vertical axis and the resultant will tell me how does the surface evolve in time.

Now, this is formally what I had mentioned earlier as a pebble in the pond problem. If you throw a stone of a certain size, the stone disturbs the surface of water. The disturbance that is created at the surface is equivalent to a perturbation which is related to the size of the stone, and you can think of this being equivalent to throwing a stone into water, and asking how is the surface, how are the waves going to get created, and how are they going to propagate outwards. So, let us look at them.

So, for that we need to as I showed you earlier, we need to solve this integral, we need to solve this integral. So, for that I have to plug in the Hankel transform of the initial condition into this integral. So, I have to work out the Hankel transform of this initial condition. So, I need the Hankel transform of this function.

Now, the Hankel transform can be worked out by looking up a hand book, you can look up a hand book which contains integrals of Bessel functions and get the Hankel transform from there. Alternatively, you can use a software package like Mathematica. So, I have plotted the Hankel transform. I will tell you the formula for the Hankel transform.

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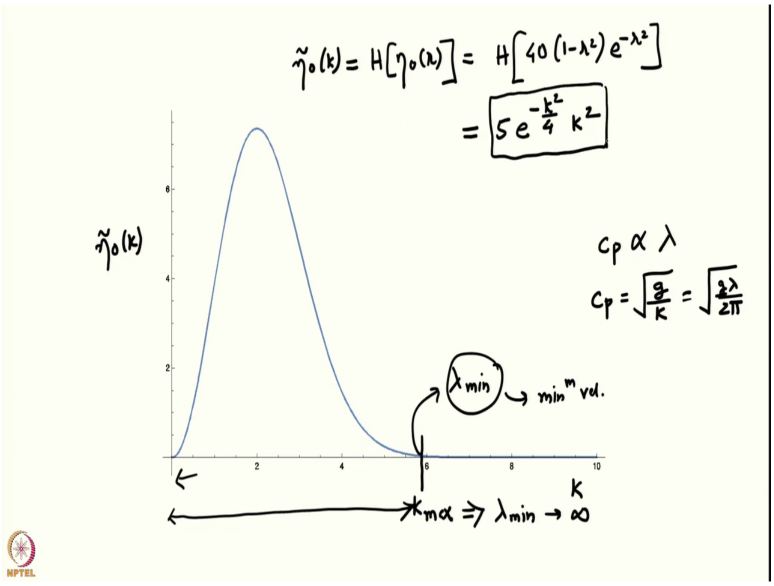

So, the Hankel transform of the eta 0 of r, which was written in the last slide turns out to be, so in this case it is 40 into 1 minus r square e to the power minus r square. And the Hankel transform of this is 5 e to the power minus K square by 4 into K square. I have plotted this Hankel transform. This Hankel transform I will call it eta 0 tilde of K. So, I have plotted this eta 0 tilde of K as a function of K.

As I mentioned earlier, you can see that there is a cutoff. This is a K max and the Hankel transform is effectively 0 beyond this K, ok. So, all wave numbers in this range will be excited. But wave numbers beyond this will not be excited at time t equal to 0. Because this is a linear system only wave numbers which have been excited at time t equal to 0, can be present in the system at later times.

In particular, they do not exchange energy with each other. So, whatever is the range that has been excited, the same range will be present at all later times, and interface will look like a linear superposition of these wave numbers. So, K max because K and lambda are inversely related to each other, so K max implies a lambda min.

So, there is a minimum lambda in the system. The smallest value of lambda is finite it is not 0. So, as you go, as K goes to infinity lambda goes to 0 because there is a cutoff in K, there is a cutoff in lambda. So, there is a smallest wavelength present in the system when we have this kind of an initial condition. However, everything beyond that wavelength is present because this initial condition excites everything nearly up to K equal to 0. So, K equal to 0 is lambda equal to infinity. So, from lambda min to very large wave length, everything is present in the system.

We are going to do this for surface gravity waves. In surface gravity waves, the we have seen that c p is directly proportional to lambda. We have seen the c p is equal to square root g by K which is equal to g lambda by 2 pi. So, you can see that is the greater the lambda, the faster is the phase speed.

So, you will see that the longer waves as usual are farthest from the center and the shorter waves are closest to the center. However, you will see that there is a pool of quiescent fluid, where the interface is not perturbed, which goes out readily outwards and that is because there is a there is a lambda minimum which is present in our system. So, there is a lambda; so, there is this this corresponds to a lambda minimum, and this lambda minimum will move with the minimum velocity. And there is no lambda which is smaller than this lambda minimum.

So, let us look at the solution. So, what I have done is I have plugged this form of eta 0 tilde of K into this integral, into this integral and then the resultant integral can be solved numerically. Now, instead of doing that what one can do alternatively is, one can use the method of stationary phase to simplify these integrals using the same limit that we had seen earlier t going to infinity. I will straight away give you the answer of applying the method of stationary phase on this initial condition for this integral.

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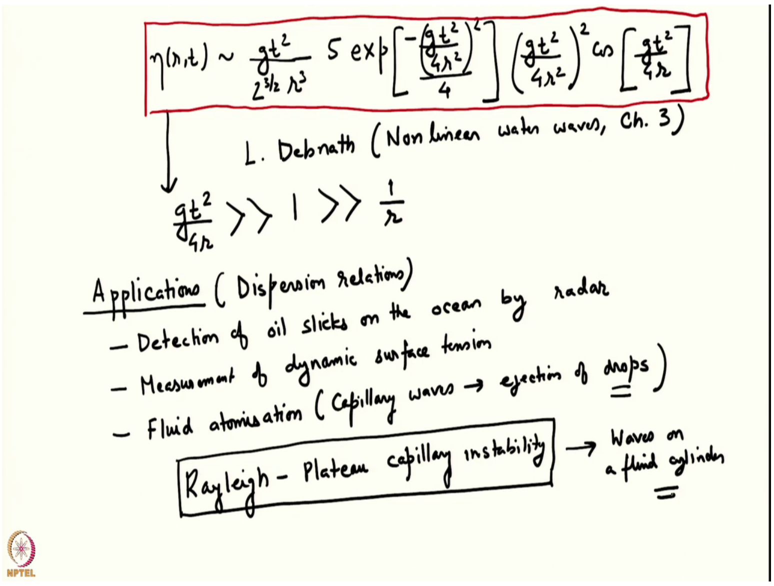

So, for this initial condition it can be shown using stationary phase that eta of r comma t goes as gt square. This form has been taken from this book, chapter 3. And this form is true because this is worked out as the asymptotic form of an integral. This is true when is much much greater than 1, is much much greater than a by r, a in this case is 1. So, this is the limit when this is true.

So, what I am going to show you is I am going to plot this formula guessing similar formulas earlier in the Cartesian case. So, I am going to plot this formula and let us understand how does the wave front propagate outwards. Here also you will see that the wave packet changes shape and that is because of the dispersive nature of the medium. We are not taking into account capillary capillarity here. We are only accounting for gravity.

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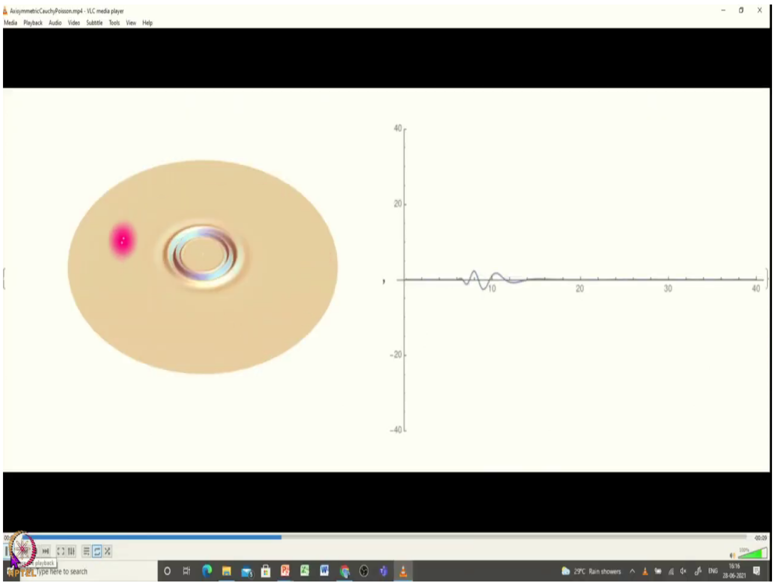

So, what is plotter on the left is a three-dimensional visualization of the integral.

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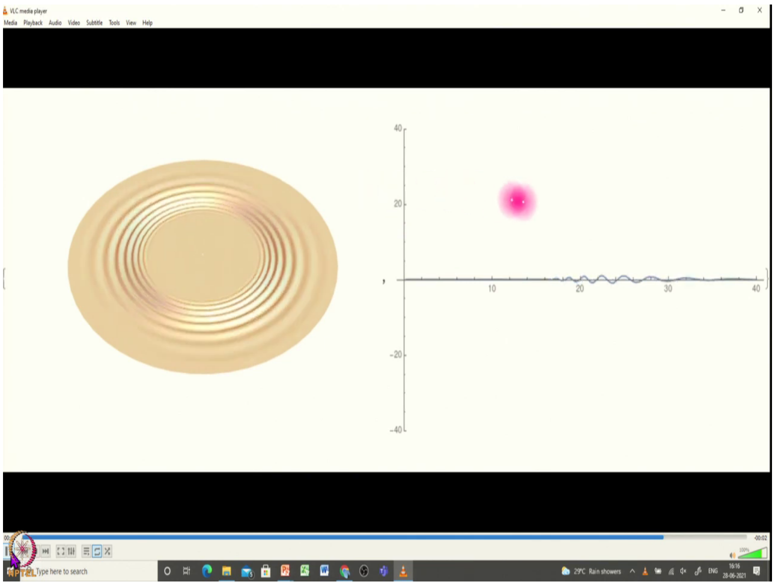

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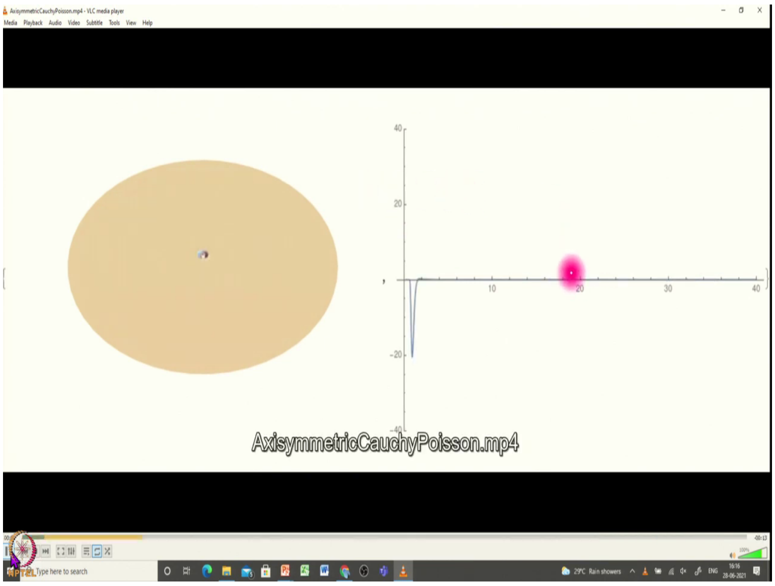

And what is plotted on the right, eta of r as a function of r at different instances of time, as time progresses how does eta of r change as a function of r.

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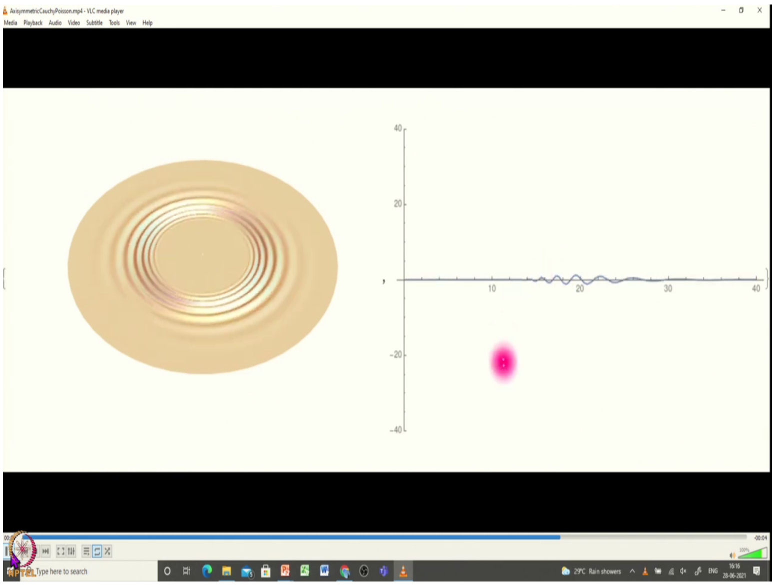

So, you can clearly see the same signatures that we had seen earlier the; so, this is being played in a loop. So, you can see it being played all over again. So, you can see that as time passes, there is a certain quiescent region where there are no waves which develop and this region becomes bigger and bigger. This is because as I told you there is a lambda min present in the system because of the initial condition that we have chosen, ok.

So, this is the wave which travels the slowest. And there is no further wave which travels smaller than this. This was different in the two-dimensional case where we had chosen a delta function initial condition. A delta function initial condition excites the entire spectrum every K from 0 to infinity.

So, there is no slowest travelling wave. So, in this case, you can see that this region becomes bigger and bigger. And you can see that the outward the waves which are at the which are travelling the fastest are also the longest wave. So, the ones which are outward have longer wavelength then the ones which are inside.

Now, this should remind you of throwing a stone into a pool of water, ok. This is the pattern that we typically see when we throw a pool, when we throw a stone into a pool of water and the resultant ripples spread outwards, ok. The whole thing is captured from this integral. So, the circular spreading out pattern that we see when we throw a stone into a pool of water is essentially governed by this integral.

We have just approximated this integral using the method of stationary phase for a particular initial condition where there is a cut off wave number and there are no wavelengths smaller than that wavelength. And so, we have reproduced qualitatively what we see when we throw a stone into a pool of water. So, with that this formally completes what we wanted to discuss about surface gravity waves in two geometries, namely cylindrical waves and waves in Cartesian geometry.

So, now let us ask what are the applications of these dispersion relations that we have learnt so far, so applications of dispersion relations. So, I am going to mention only a few applications. There are many, but I am only going to mention some which are particularly relevant in engineering.

So, one of them is the detection of oil slicks on the ocean by radar. When oil is spilled on to the ocean surface, it has a damping effect in particular on the capillary waves, that is picked up when the ocean surface is tracked by radar. In order to infer, what is seen from radar, one needs to know these dispersion relations. So, the dispersion relation for capillary gravity waves in deep water that we have learnt so far, finds a lot of applications in this particular area.

Another important application is measurement of dynamic surface tension. I will provide some references at the end of this video, where which you can read and learn more about dynamic surface tension. In order to measure dynamic surface tension, these dispersion relations find usage.

A third and important application is fluid atomization, where typically capillary waves cause ejection of droplets. Here once again it is necessary to know the dispersion relation for capillary waves in order to estimate the sizes of drops. These are only some representative examples there are many others.

Now, with that let us now move over to waves in a different kind of geometry. Right now, we have seen Cartesian and a cylindrical geometry, we will now see waves once again in a cylindrical geometry and we will do, in the next video we will do what is known as the Rayleigh-plateau capillary instability.

Now, this example will be slightly different from the example that we have seen so far. In particular, we will see that in this example some waves are unstable or in other words if you introduce a perturbation they do not oscillate, but they grow exponential in time. We have not met such kind of examples until now. We will see it when we analyze this example.

For this we will look at waves on a fluid cylinder. You will see that in the linear approximation some waves are, some perturbations are stable whereas, others grow in time. We will write down and analyze the system in some amount of detail.

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1. Determination of the dynamic surface tension of liquids from the instability of excited capillary jets and from the oscillation frequency of drops issued from such jets, M. Ronay, Proc. Roy. Soc. A, 1978, vol. 361, Iss. 1705

2. Dynamic surface tension and capillary waves, J. Adin Mann Jr., Surface and Colloid Science, vol. 13, pp 145-212, Ed. Egon Matijević & Robert J. Good, 1984