Introduction to interfacial waves
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Lecture - 41
Waves on a pool of finite depth

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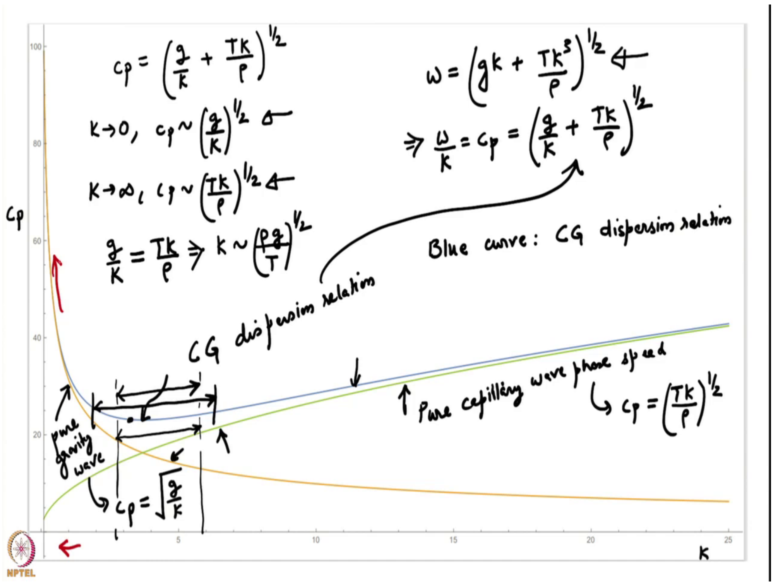

We were looking at the dispersion relation for capillary gravity waves in deep water. We had found that the dispersion relation was given by this relation gk plus Tk cube by rho the whole thing to the power half. We had plotted the phase speed as a function of k and we had found that there is a qualitative difference due to the inclusion of surface tension.

In particular, the phase speed for a pure gravity wave is a monotonic curve whereas, the phase speed for a capillary gravity dispersion relation has decreases at small k, but then again starts increasing at large K. So, there is a minimum. So, this is the minimum. We had also found that for sufficiently small k the dispersion relation can be well approximated as if it is a pure gravity wave.

For sufficiently large K it can be again approximated as a pure capillary wave. In an intermediate regime which is in this regime not neither the green curve nor the orange curve represents the full dispersion relation accurately in that region one has to take into account the effect of both gravity and capillarity.

Let us make a quick estimate of what is the wavelength, when we should be taking full both gravity and capillarity into account and what are the wavelengths where we do not need to worry about one of them. So, as you can see the phase speed expression has two parts to it g by k plus T k by rho to the power half.

For k going to 0, c p is well approximated as g by k to the power half; for k going to infinity c p is once again well approximated by T k by rho to the power half. At intermediate values we expect both terms to be of the same size and that is the regime when both capillarity and gravity effects are of the same size and neither of them can be neglected in favour of the other.

So, we can make a very quick estimate we are saying that we are looking for that range of k where g by k is the same size as T k by rho; g, T and rho are fixed for a given set of fluid parameters, surface tension the density of the fluid and acceleration due to gravity is fixed.

This tells us the range of wave numbers or a particular cut off wave number around which capillary gravity waves are important or rather capillary gravity effects are important and far away one of these two limits is valid. So, you can immediately see that that k is given by rho g by T to the power half.

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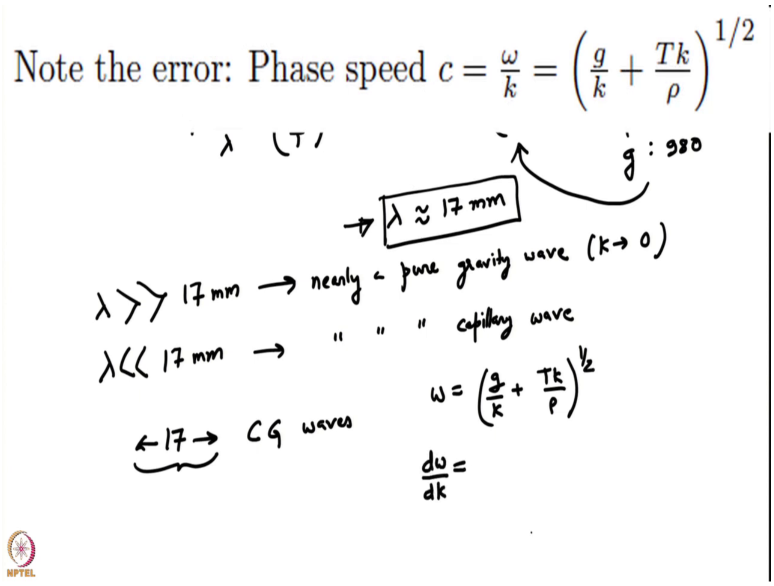

If we compute the lambda corresponding to that k so, k is given by rho g by T to the power half; k is 2 pi by lambda. So, this is 2 pi by lambda which is equal to rho g by T to the power half. So, one can compute lambda is 2 pi T by rho g to the power half. If you plug in air water values of surface tension T that is around 72 in cgs units rho is 1 for water and g is 980 in cgs units. If you plug in this and calculate you will find that lambda is approximately 17 millimeter.

So, this is telling you that for wavelengths lambda much much greater than 17 millimeter the wave is nearly a pure gravity wave. So, lambda becoming very large, this implies k going to 0. Lambda much much less than 17 millimeter this is nearly a pure capillary wave. Lambda in the window of 17 millimeter 17 plus minus some delta capillary and gravity effects are both important and this becomes a capillary gravity waves.

One cannot drop either gravity from the dispersion relation or capillarity from the dispersion relation. Again, these estimates are valid only for air and water. If we change the fluid then the surface tension value will change and so, these estimates will change. In this region when neither the green curve nor the orange curve are good approximations to the blue curve we have to be careful and we have to include both capillarity effects and gravity into the dispersion relation and the resultant waves will behave as capillary gravity waves.

Note that the phase speed has a minimum. The phase speed is not a monotonic function. In particular I encourage you to think about the groups velocity of a capillary gravity wave. You will see that the group velocity, so, if you take the full dispersion relation omega is equal to g by k plus T k by rho to the power half and if you compute d omega by d k, the group velocity for capillary gravity waves. You will find that even that has a minimum.

This has a lot of physical consequences for example, the existence of this minimum and below which there is no so, the phase speed looks like this as we have seen, a similar curve will also be there for group velocity. This has physical consequences. I encourage you to think about this particularly in relation to the Cauchy – Poisson problem.

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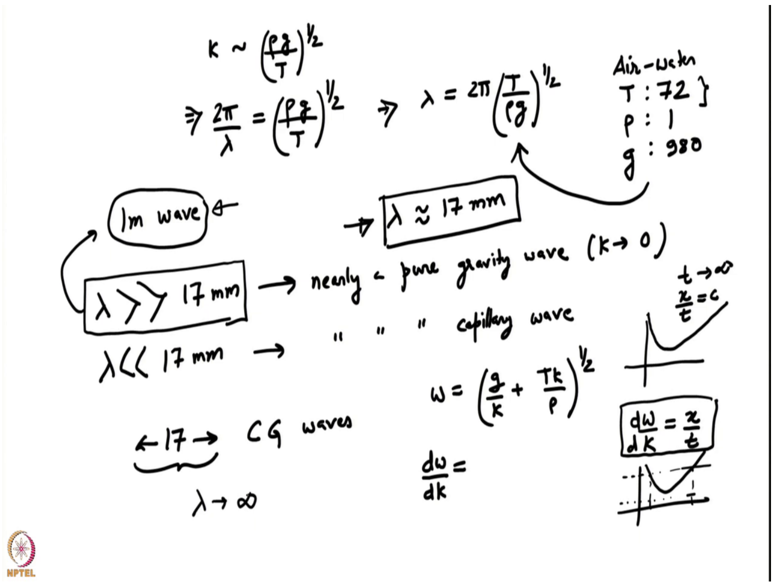

In the Cauchy – Poisson problem we had found that omega by k; so, for t going to infinity and for an observer whose x by t is constant, we had found that omega by k or rather d omega by d k is the dominant contribution is given by this equation. Now, when your group velocity curve has a minimum, then below a certain speed this equation does not have any solutions.

So, when x by t is below a certain speed, there is no solutions to this equation; above a certain speed there are two solutions. There is one k here and there is another k there; one k coming from the capillary branch of the dispersion relation, another coming from the gravity branch of the dispersion relation. The wavelengths will be very different. This has physical consequences for what the observer sees locally. I encourage you to think about it.

Now, let us look at this limit of lambda becoming much much greater than 17 millimeter. So, in the case of air water you can immediately see that if we deal with let us say a 1 meter wave which would be the case when we are in an oceanic context, then you can immediately see then in oceans waves are almost exclusively gravity waves. So, a wave of this wavelength or above is typically a gravity wave. Tsunamis are entirely gravity waves. In fact, they do not even behave like deep water waves as we will see shortly.

Now, as we take this limit lambda going to infinity it is also clear that the deep water approximation itself is going to break down. Remember that we had said that we are going to assume that the depth of the fluid is infinite. Now, in practice in a given situation the depth is never infinite and so, that approximation is valid only when the waves that we are looking at are sufficiently the wavelength of the waves is sufficiently small compared to the depth of the fluid.

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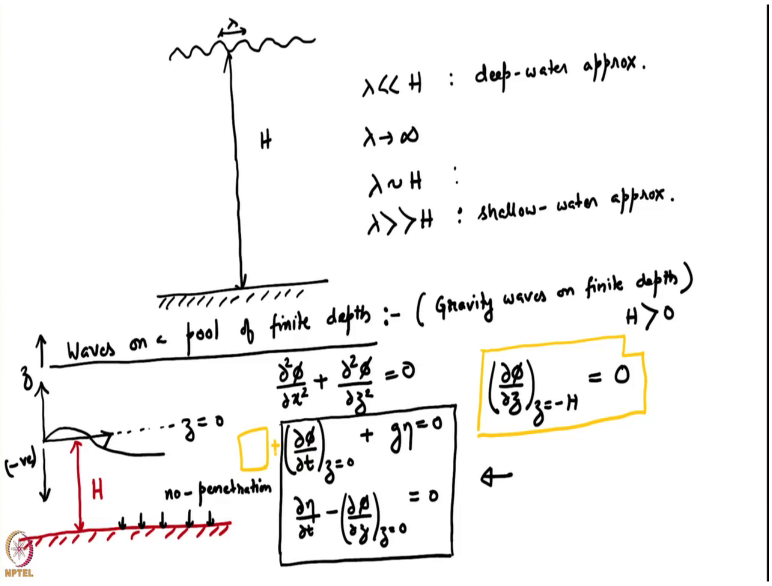

So, if I have a situation where the wavelength of the fluid of the wave is much much smaller than the depth the undisturbed depth H. So, this is the wavelength then I expect this to be a situation. So, lambda is much much less than H. So, I expect this to be a situation where the deep water approximation should hold good.

However, as I make lambda bigger bigger and bigger, you can see that the deep water approximation is going to fail because as lambda becomes of the same size of H, the wave will start feeling the presence of a bottom wall. In particular, when lambda becomes much much greater than H then we have the reverse limit which is called the shallow water approximation.

We will look at the shallow water approximation in some detail later in the course. Let us first work out what is the correction to the dispersion relation when we have a pool of finite depth. So, waves on a pool of finite depth. Until now we have only done deep water. We have looked at capillary waves, we have looked at gravity waves, we have looked at capillary gravity waves.

Now, what we are going to do is we are going to make the depth of the pool finite. We will call it H. Our coordinate system remains exactly the same as before. So, this is my interface, this is my undisturbed surface at z is equal to 0. Earlier z was going all the way to minus infinity, now it will go to some lower level capital H.

So, I will put a bottom wall and the depth is a constant H. So, this is a wall and so, we will have to repeat our analysis you can see that this is just a small modification to the boundary condition. However, the expressions will become slightly different now because earlier we had boundedness constraints at z is equal to minus infinity we use that to eliminate one of the exponentials in z.

Now, both the exponentials will survive that is the essential difference which will be there as far as the mathematical calculation is concerned. So, let us do it. So, we have the Laplace equation del square phi by del x square plus del square phi by del z square is equal to 0 this is like before then we have. So, I am going to do it only for gravity waves.

And, the homework for you will be to try this by putting in surface tension also. It is just a modification to the pressure boundary condition. One has to just replace pressure with T by rho times divergence of n as I did in the last example. I will right now assume pressure to be 0 as we have done so far. So, I will do it for gravity waves on finite depth.

So, let us write down the relations. So, the Bernoulli equation pressure is 0. So, there is no pressure term in the Bernoulli equation. This is Bernoulli equation applied at the surface like before plus g eta is equal to 0. Then we have the kinematic boundary condition linearized minus del phi by del z again at z is equal to 0 is 0. And, then we also have a no penetration condition. So, no fluid goes through the wall.

So, no penetration we have to remember that we cannot impose no slip, this is an inviscid analysis. We cannot impose two boundary conditions at the wall, we can only impose one where; if you do a viscous analysis then you will be able to impose a no slip condition in order to in addition to the no penetration condition. So, we will have a no penetration condition which is del phi by del z which tells me the vertical velocity at the wall is 0.

So, del phi by del z at z is equal to minus H. Recall that the positive z direction is upwards. So, the negative z direction is downwards and this H is a capital is a positive quantity. So, H is greater than 0 is 0. So, this is the only difference rest of the equations remain the same like before and we have to ask ourselves how does this modify the analysis.

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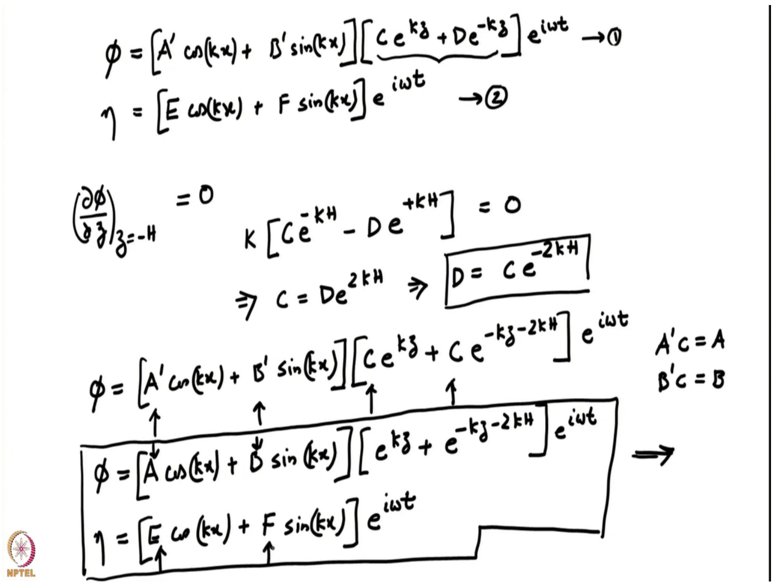

So, let us look at it. So, by variable separation arguments once again it is a linear combination of cos k x and sin k x. I will put a prime here because these constants will get multiplied by something and then I will eliminate the prime. So, cos k x plus B prime sin k x in the vertical direction earlier we were only a putting exponential k z into a constant.

Now, we will have to put both the exponentials because the domain is finite in the vertical direction. So, in general in this direction in the z direction it is C e to the power k z plus D e to the power minus k z. This second term was not there earlier, now it is going to be there e to the power i omega t, eta is equal to like before some E cos k x plus F sin k x into e to the power i omega t. This is let us call it 1 and 2; we can go and plug it into the boundary conditions.

Now, if let us first eliminate the no penetration condition or let us first satisfy the no penetration condition. The no penetration condition says del phi by del z at z is equal to minus H is 0. This is the only part which depends on z, the rest is z independent. So, I can differentiate this with respect to z, at z is equal to minus H and set the resultant expression to 0 that gives me one equation which says K times C e to the power k H or k minus H minus D because there is a derivative with respect to z.

So, minus k will come here e to the power plus k H is equal to 0. This is the equation that we obtain. This allows me to determine C in terms of D or rather D in terms of C. So, you can see that C is equal to De to the power twice k H or let me express D in terms of C. So, D is C e to the power minus 2 k H. I am going to use this to get rid of D in all my expressions.

So, this tells me that phi is A prime cos k x plus B prime sin k x into you can see that I am going to get rid of D and express it in terms of C. So, C e to the power k z plus C e to the power minus k z minus twice k H into e to the power i omega t. I am going to pull out this C and multiply it with A prime and B prime these are in general complex constants and A prime into C is going to be A and B prime into C is going to be B. This is why I had put a prime so that I could use the symbols A and B later.

So, A cos k x plus B sin k x multiplied by e to the power k z plus e to the power minus k z minus twice k H into e to the power i omega t. Eta like before remains the same E cos k x plus F sin k x multiplied by e to the power i omega t. So, this is our normal mode form and now, we have already taken into account the no penetration boundary condition.

Now, all we need to do is just like what we have done until now, take these two expressions, substitute them into the two boundary conditions that we have and obtain the dispersion relation. Once again, the algebra is straight forward. You can do it yourself I am just going to write down the final answers. So, once again we are going to get four algebraic equations in four unknowns.

They are coming from the coefficients of cos k x and sin k x and of course, there is a complex conjugate part like before. So, our equations are going to be. So, I am going to skip the algebra. I encourage you to try it for yourselves. You just have to take these expressions that I have written. The expression for phi so, these two expressions take this and plug it into the two boundary conditions, these two.

Note that we are not putting surface tension. So, we do not have to worry about calculation of divergence of unit normal and so on and so forth. Once you have done this calculation then you can restore surface tension back into the Bernoulli equation, add that term and calculate what is the modified dispersion relation for capillary gravity waves on a pool of finite depth. Right now we are just doing surface gravity waves on a pool of finite depth there is no contribution from surface tension.

So, we have to take the expressions that I have put in this rectangular box here and we have to substitute it into these two boundary conditions. And, then we have to eliminate we have to set the coefficients of cos k x and sin k x in both the equations to 0 that will give us four equations in four unknowns. Our unknowns are going to be A, B, E and F. So, A, B, E and F in general these are complex constants as usual.

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$$i\omega(1+e^{-2kH})A + gE = 0$$
$$i\omega(1+e^{-2kH})B + gF = 0$$
$$i\omega E - KA(1-e^{-2kH}) = 0$$
$$i\omega F - kB(1-e^{-2kH}) = 0$$

$H \to \infty$

Alternatively we can also obtain the dispersion relation by considering two equations: either for A and E or for B and F

So, I am going to straight away write down the equations that one would obtain. You can substitute and cross check that these equations are correct. So, one equation is i omega 1 plus e to the power minus twice k H A plus g E is equal to 0. The next equation is this is all coefficients of cos x and sin x cos k x and sin k x then one two more. So, two of them will come from the Bernoulli equation the other two will come from the kinematic boundary condition.

So, i omega E minus KA into 1 minus e minus 2 k H is equal to 0 and then i omega F minus KB 1 minus e to the power minus twice k H is equal to 0. So, like before we have four equations in four unknowns. How do you check the validity of these equations? You can take the limit H going to infinity, all the exponential terms will drop because they will go to 0.

And in that limit you should recover exactly the same equation that we had earlier found for surface gravity waves in deep water. I encourage you to try this on your own and compare with the results that we had earlier and make sure that the equation that you recover in this limit of H going to infinity deep water limit is gives you the same equations as before.

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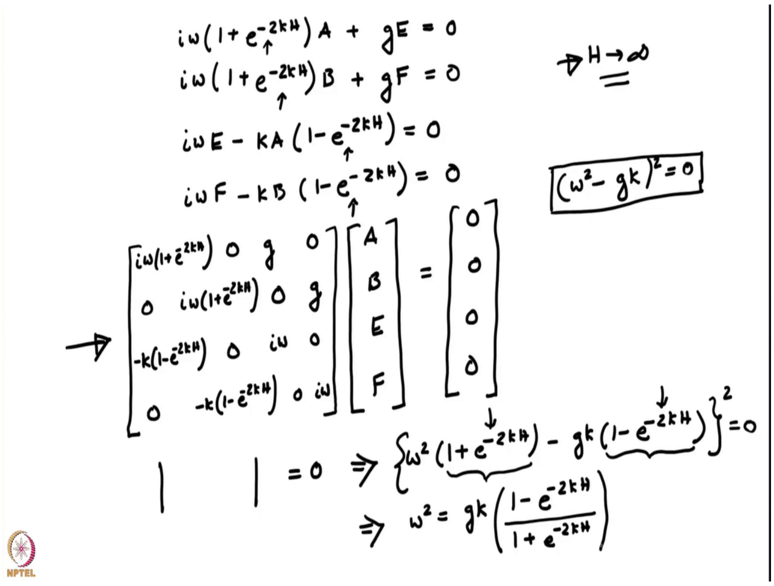

Let us continue, so, the matrix. So, this is like usual this is a matrix for A B E and F is equal to homogeneous set of equations 0 0 0 0 and we are looking for a non-trivial solution. So, the elements are i omega 1 plus exponential minus 2 k H that is the coefficient of A, then 0 is the coefficient of B, the coefficient of E is g and then 0.

Similarly, 0 the second element coefficient of B is i omega plus e to the power minus twice k H, the third element is 0 and the fourth element is g, then the coefficient of A is minus k 1 minus e to the power minus twice k H, second element is 0, coefficient of E is i omega then 0 and then this is 0 minus k coefficient of B 0 i omega.

If you take the determinant of that matrix to 0 this gives you the dispersion relation once again you will get a quartic, the quartic is factorizable as a quadratic. So, you will get the quartic omega square into 1 plus e to the power minus 2 k H minus gk into 1 minus exponential of minus 2 k H whole square is equal to 0.

I have written it in such a form that if you said this term if you take this limit H going to infinity then this term and this term will go to 0 and you will recover your old quartic which was omega square minus g k whole square is equal to 0. We had done this earlier when we did surface gravity waves in deep water. So, this just generalizes that bringing in the effect of finite depth. So, those factors that we are getting represent the effect of finite depth.

So, let us work on this. So, we obtain omega square is equal to g k into 1 minus exponential of minus 2 k H divided by 1 plus exponential of minus 2 k H I can simplify this a little bit by multiplying numerator and denominator by exponential k H.

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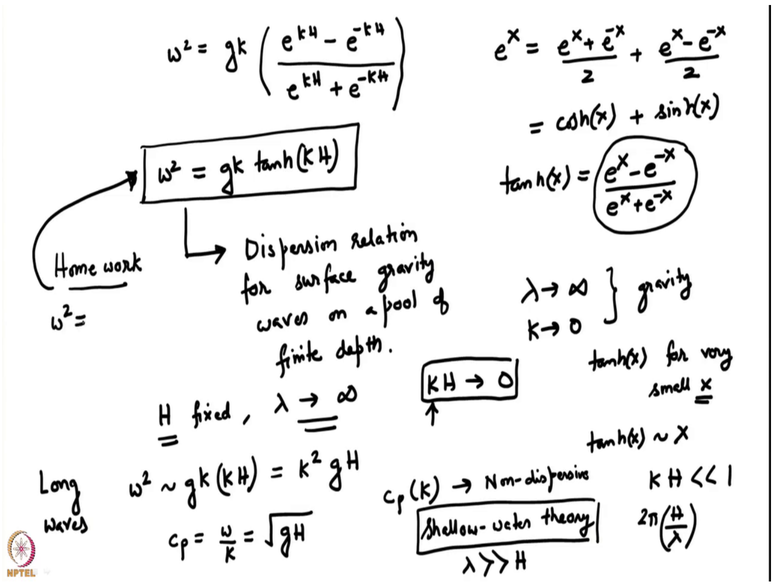

If I do that then this becomes omega square is equal to g k exponential of k H minus exponential of minus k H divided by exponential of k H plus exponential of minus k H. Recall that this is related to hyperbolic functions; hyperbolic functions are defined as by taking exponential and splitting it into odd and even parts. So, the even part is e to the power X plus e to the power minus X by 2 and the odd part is e to the power minus X by 2. This is cos hyperbolic of X and this is sin hyperbolic of X.

Sin hyperbolic by cos hyperbolic is tan hyperbolic of X and tan hyperbolic of X looks like this. So, this is exactly of the tan hyperbolic form. So, I can write this as g k tan hyperbolic k H and so, comes out our dispersion relation for surface gravity waves on finite depth. Dispersion relation for surface gravity waves on a pool of finite depth.

What is this telling us physically? Recall that we had said that as lambda becomes bigger and bigger. So, lambda goes to infinity or k goes to 0, we get gravity waves. Now, earlier we had found that the phase speed of gravity waves actually increases without bound in deep water. So, if you recall our earlier dispersion relation for the capillary gravity waves in deep water.

You can see that as k goes to as k goes to 0, the blue curve increases without bound. So, the phase speed is becoming bigger and bigger and bigger as the wavelength is becoming longer and longer. In the Cauchy – Poisson problem we have seen this behaviour for gravity waves. The further we go outward the longer is the wave and the faster it travels.

So, this behaviour gets rectified once we put in finite depth. As I said earlier the effect of finite depth starts to be felt when the wavelength of the waves starts getting comparable to the depth of the pool on which the wave is propagating. So, if we start with the pool of finite depth and if we have waves whose wavelengths are very short compared to the depth, then these waves behave as if they are propagating on deep water.

However, if you take waves which are much which are whose wavelength is comparable to the depth, then they start feeling the bottom and then the correction to the dispersion relation is becomes necessary. In particular, if you take the limit of lambda going to infinity so, lambda going to infinity so, holding H fixed so I hold the depth of the pool fixed and I take lambda going to infinity.

This is equivalent to the limit k H going to 0 because when lambda goes to infinity H is anyway fixed so, k goes to 0 so, k H goes to 0. What happens to this dispersion relation? This dispersion relation suffers a qualitative change. So, you can see what is the behaviour of tan hyperbolic X for very small X that is because our tan hyperbolic has argument k H and I want this limit k H going to 0.

So, tan hyperbolic X for very small X. You can see the tan hyperbolic X for very small X is just X. You can do a Taylor series expansion on this and convince yourself the tan hyperbolic X for very small X is just X the first term is X. And, so, for sufficiently long waves so, long waves omega square is equal to g k tan hyperbolic of k H for sufficiently small k is just k H it is just X.

This is so, this is an approximation for sufficiently long waves I am just writing the first term in the Taylor series and this is equal to k square into gH. Now, notice what has happened. For sufficiently long waves the phase speed which is omega by k is just square root gH. This is unlike deep water, this is unlike other all the other examples that we have seen until now where the phase speed until now was a function of k.

Now, once the waves become sufficiently long then they effectively behave as pure gravity waves and pure gravity waves on a pool whose depth is much smaller than their wavelength behave like long waves and they are non dispersive or in other words their phase speed is independent of the wavelength. This has a lot of consequences for the physics of how these waves propagate especially when there are many of them travelling. We will see some of this later on when we look at shallow water theory.

So, shallow water theory is basically a long wave theory. So, you can either say that the wave is long compared to the depth or you can say that the water is shallow compared to the wavelength. Either ways we are meaning the same thing. There is a non dimensional parameter which represents the ratio of wavelength to depth. k H is a non dimensional parameter. Shallow water implies that k H is much much less than 1.

So, small k limit or large wavelength limit, k is inversely proportional to lambda. So, this is twice pi H by lambda. So, shallow water implies lambda is much much greater than H. In that limit, surface gravity waves become non-dispersive. One can calculate for finite depth instead of taking the shallow water limit one can stick to a pool of finite depth where the wavelength is comparable to the depth and one can incorporate the effect of surface tension.

I leave it to you as a homework problem to work out the correction of surface tension on this dispersion relation. So, you have to do is one has to go back to the equations to the boundary conditions and modify the Bernoulli equation. So, you have to modify the Bernoulli equation and you will have you will have one more term here. So, let me write it in another color.

So, there will be an additional term here which will be coming in because pressure is now not 0. It will be T by rho times divergence of n and then we already know how to calculate that term. So, you have to include this and find your matrix and that determinant of that will give you your dispersion relation.

So, this is the this is the dispersion relation for waves surface gravity waves you know on a pool of finite depth. Obviously, when we go to the deep water limit or in other words when k is very large or for a fixed depth k is very large then we recover the known behaviour. So, tan hyperbolic.

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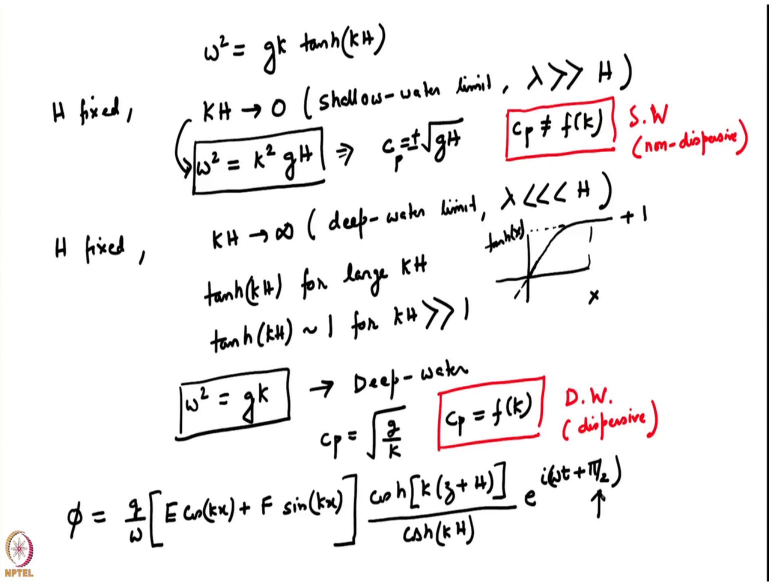

So, we have looked at two limits. So, our general relation is gk tan hyperbolic k H we have looked at the limit for fixed H. So, H fixed we have looked at the limit k H going to 0 this is the shallow water limit when the wavelength becomes much much greater than the depth.

In that limit these waves become so, in the shallow water limit these waves become non-dispersive. So, we have gk into gH. So, tan hyperbolic k H is just k H. So, that gives me a k square into gH. So, this implies C is equal to square root gH plus minus the phase speed C p keeping H fixed if we take the other limit k H going to infinity.

This is of course, is the reverse limit. This is the deep water limit. In this limit it is not consistent to ignore surface tension because my wavelengths are getting shorter and shorter compared to the depth. So, my wavelength is much much shorter compared to the depth, but if you ignore surface tension and just look at this dispersion relation.

Then we have to find out what is tan hyperbolic of k H for large k H. If you plot tan hyperbolic of X as a function of X you will find that it is asymptote to the so, tan hyperbolic of X as a function of X. You will find that is asymptote to the line y is equal to X for very small argument and then it plateaus at plus 1. So, for very large KH tan hyperbolic k H is just one for sufficiently large X.

So, this is for k H much much bigger than 1. If you plug that in into the dispersion relation you can see what do we get we recover our deep water results. So, this is the shallow water result, this is the deep water result. Very different because in the shallow water limit the waves are non-dispersive.

C p is not a function of k all ks travel with the same speed whereas, in the deep water limit C p as we have seen is square root g by k. So, every C p, every Fourier mode has its own speed. So, you can compare this with that. So, this is the shallow water limit and this is the deep water limit. You can see then in this case this is non-dispersive, in this case it is dispersive.

Later, we will see that non-linearity and dispersion can compete with each other and lead to a lot of complications which causes a lot of interesting phenomena in the evolution of interfacial waves. Just as we have worked out the formula for the dispersion relation you can take this and also work out what is the expression for phi.

I leave it to you to show that phi can be written as g by omega E cos k x plus F sin k x into cos hyperbolic divided by cos hyperbolic k H into e to the power i omega t plus pi by 2. Similarly, there is a similar expression for eta.

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$$\eta = \left[E\cos(kx) + F\sin(kx)\right] e^{i\omega t}$$

Eta is equal to E cos k x plus F sin k x into e to the power i omega t you can see that there is a phase difference of pi by 2 between phi and eta So, that i was absorbed as a phase factor of pi by 2. This expressions can be obtained easily. All that you have to go and do is follow whatever we have done until now and then try to rearrange so that you get these cos hyperbolic factors.

Recall the definition of cos hyperbolic which was e to the power X plus e to the power minus X by 2. You have to use that definition and you will be able to obtain these expressions very easily.