Introduction to interfacial waves
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Lecture - 40
Capillary-gravity waves

We were looking at the solution to the cauchy poisson problem for a delta function initial condition and we had understood what a local observer sees.

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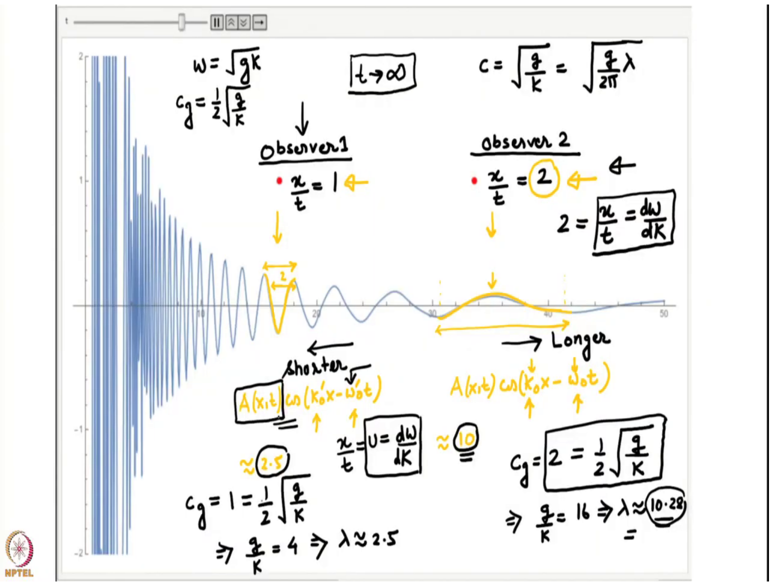

In particular, we had understood that locally the wave profile can be represented as something of this form A of x t into some cos k naught x minus omega naught t, but here this k naught and omega naught are to be determined from the requirement that whatever is the speed of the observer which is given by x by t is equal to constant.

So, let us say the speed of the observer is some u, this is to be equated to d omega by d k and from the solution of this equation, one get obtains a k naught that is the local wave number that the observer sees and this is the frequency corresponding to that wave number obtained from the dispersion relation.

It is clear that this is going to depend on the speed of the observer. So, every observer sees a different wave number. Now, you can also see that the amplitude of the waveform in general is a function of space and time. And we will later towards the end of this course, we will ask the question what is the equation which governs this amplitude.

So, this summarizes what we have learnt so far about the solution to the cauchy poisson problem which in general solves and asks what does the waveform look like when we have an infinite number of wave numbers excited through a delta function initial condition, and what is the local description of the wave packet.

Obviously, this is an approximation because we have not yet put non-linear terms into our equation. Despite that it is clear that the description even within a linearized framework becomes quite complicated when an infinite number of wave numbers are present.

Now, let us move further and let us now ask questions related to other effects which we have neglected until now. In particular, we have ignored the fact that the depth of the pool on which these waves propagate could be finite. We have also ignored the effect of surface tension.

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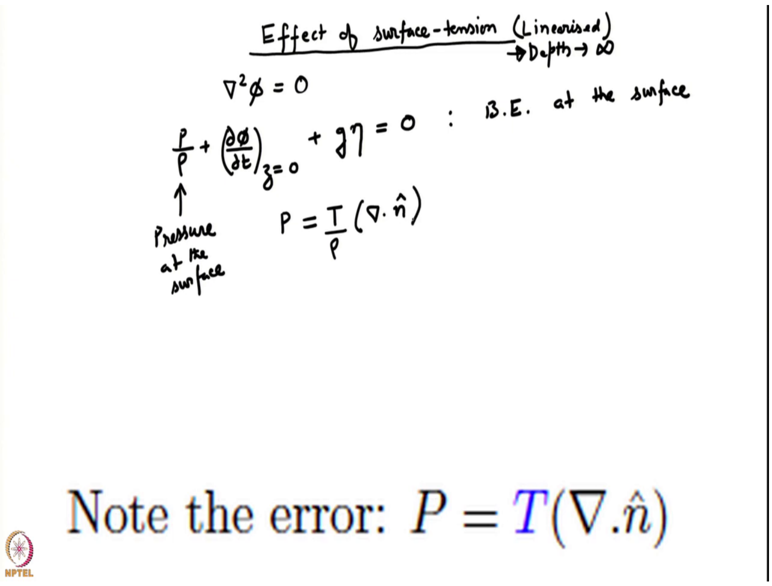

So, let us first start with the effect of surface tension; effect of surface tension. We will do this in exactly the same manner that we have done now. So, we are going to do this again in a linearized approximation and we are going to assume the depth of the pool to be infinite as we have done until now. We will first put in the effect of surface tension and then, we will relax the assumption of infinite depth and that will bring us to finite depth.

Let us first do the case of surface tension. So, as we know our equations are the same; the governing equation is the Laplace equation. Then we have a Bernoulli equation which determines pressure. In this case, I am straight away writing the linearized Bernoulli equation which we would have obtained at order epsilon. Now, this is the Bernoulli equation at the surface at the free surface.

So, this is the pressure at the surface. Until now we have assumed this pressure to be 0 because we have ignored the gas gaseous medium above. Now, however because of the presence of surface tension, we know that there is going to be a jump in pressure.

So, let us calculate an expression for the jump in pressure and we will find that it is just a modification of the of one boundary condition and once we put that into account, the rest of the analysis remains almost nearly the same.

So, our pressure earlier at the interface was 0. Now, we will say that the pressure is given by the jump condition at the interface. As we know that if the interface has curvature, then surface tension causes a pressure jump in this case in the base state the interface is flat.

So, there is no pressure jump across the interface in the base state, however the moment we excite some waves of the interface a local curvature develops and surface tension causes a pressure jump.

The expression for the pressure jump is given by this expression; P is equal to surface tension coefficient. So, this is really a T prime. So, this is let me call it T by rho into the divergence of a unit normal at the surface.

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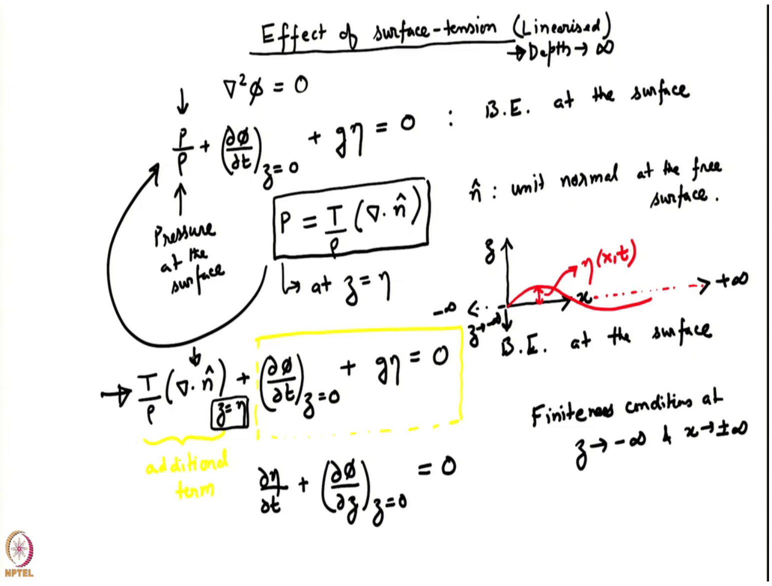

So, n is a unit normal at the free surface. The fact that curvature is given by this formula can be found in many books. I will give you a reference and you can look up that reference at the end of the video. So, this is our boundary condition and this is pressure not anywhere, but pressure at z is equal to eta; eta is the free surface. So, recall once again that our coordinate system looks like this.

This is x, this is z. We are doing going a two dimensional approximation. So, that is our free surface and this distance displacement from the base state is eta of x comma t in the base state the interface is flat and because this is linearized. So, all quantities are computed at z is equal to 0, we have seen that earlier.

Now, let us proceed with that our, so this has to be plugged in into the Bernoulli equation. So, this condition has to be plugged in into the Bernoulli equation and this pressure is the pressure at the surface. So, our modified Bernoulli equation boundary condition becomes so this thing we will write it that it has to be calculated in general at z is equal to eta.

The perturbed surface later we will see that in the expression for del dot n in cartesian coordinates z does not appear. So, we will not worry about where it has to be applied. But let us retain the fact that it is applied at the disturbed interface plus g eta is equal to 0. This is my modified Bernoulli equation at the surface, this is a boundary condition.

Earlier we had used only this much. Now we have an additional term and that term comes because of surface tension. You can see that if I set the surface tension coefficient T to 0, then I obtain my previous boundary condition the kinematic boundary condition is unaffected by surface tension. It is a statement of mass conservation as we saw earlier and so, that remains the same. I am straight away writing the linearized version that we had already obtained earlier.

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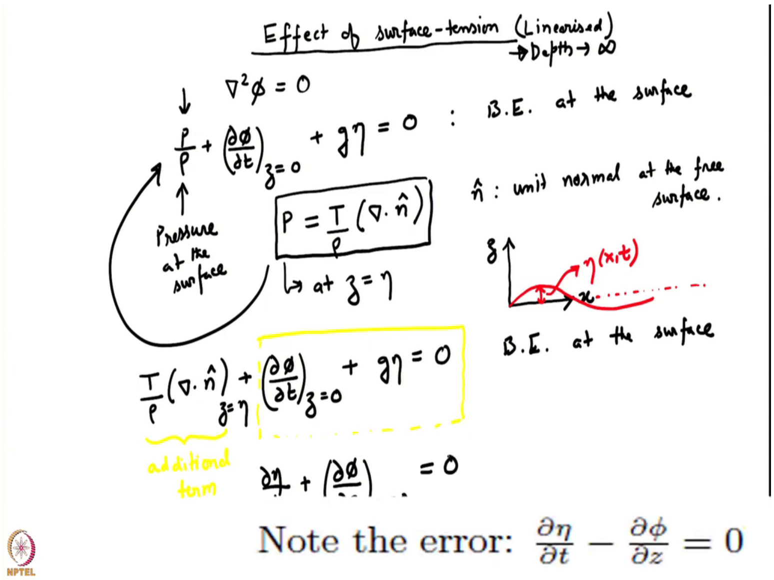

And of course, we have to have finiteness conditions at z goes to minus infinity and x goes to plus minus infinity. So, again it is horizontally unbounded. So, this is plus infinity on this side, it goes to minus infinity and below z goes to minus infinity. So, it is a infinitely deep pool as we have been doing until now.

The only thing that we have put in extra is the surface tension factor. So, let now let us find out what is the how does one do this. So, one proceeds in exactly the same manner because we have done this already.

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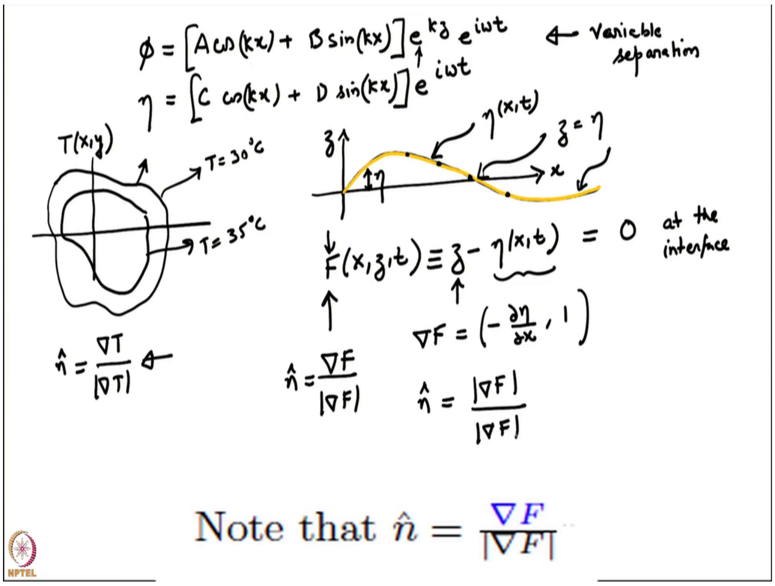

So, I am going to not go into the details of this. We have seen earlier that we can write phi as A cos k x plus B sin k x, where in general A and B are complex constants into e to the power k z. And now we are doing a normal mode approximation on this. So, this is e to the power i omega t.

Similarly, eta is some C cos k x plus D sin k x into e to the power i omega t. Now as usual we will plug this in into our equations. We have already satisfied the Laplace equation. This came from variable separation we have seen this earlier. So, I am not going into over again and so, the Laplace equation is already satisfied.

So, we only need to worry about satisfying the boundary conditions and there are two of them; the finiteness condition has also been satisfied. There were two exponentials. We have eliminated one of them by setting the prefactor to 0. So, this exponential decays as we go to minus infinity, so the boundedness constraints have also been satisfied and we have obtained up to here.

So, the only thing that we need to do is satisfy the boundary conditions. Now before we do that, we will have to worry about how does one calculate this term which has come because of surface tension. So, this additional term that has come T by rho into divergence of n and n is a unit vector to the perturbed interface because this has to be applied at z is equal to eta.

So, let us see how to calculate n. Recall that if we have a plane and let us say I have some scalar variable. Let us say I have some scalar variable temperature which is a function of x comma y. So, everywhere on the plane there is a different temperature, it could be the surface of a object. So, everywhere there is a different temperature and the temperature is varying as a function of x if I could draw lines of constant temperature. So, these lines could be contours in general.

So, let us say this is a contour on which temperature is equal to 30 degree Celsius. Then I could have another contour on which temperature is equal to let us say 35 degree celsius. If I ask you to compute normal's to each of these contours, you know from your course on vector calculus that gradient of temperature points in the direction of maximum change in temperature and if I take grad T divided by mod grad T that gives me by definition a unit normal to each of these contours.

I will use the same idea here. We have an interface which is perturbed. So, we have an interface which is perturbed. In some form, this perturbation is of the form eta of x comma t where eta is defined as this distance. This is x and this is z.

We have to compute a function if I have to use this idea to compute my unit normal, then we need to find a scalar quantity which is constant on this curve on the perturbed interface. So, I need a scalar quantity which is constant on this perturbed interface everywhere such a quantity is easy to calculate.

So, suppose we say that f of x z t and I define f of x z t as z minus eta of x comma t because the interface is given by z is equal to eta. It is clear that f is 0 at the interface or the free surface. So, f this quantity f is 0 no matter what be the value of x.

So, I have to be on the interface. So, everywhere on the interface the value of f is z by construction. So, using the same idea is here if I compute grad f and divide it by mod grad f, then I get a unit normal to the interface because the interface is a curve of constant f.

So, this gives me the formula for computing the unit normal. So, you can see that grad F in cartesian coordinates. I just have to take the del by del x. So, del by del x is minus del eta by del x del by del z is just 1. It is the just the differentiation of this. This is not a function of z. So, grad f has two components minus del eta by del x and 1. I am interested in n hat. So, I will have grad F by mod grad F.

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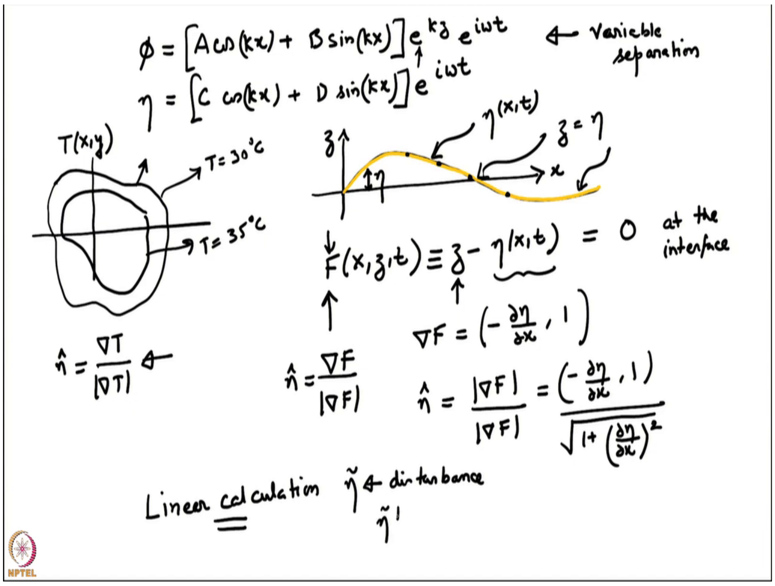

Or in other words, this is the vector whose components are minus del eta by del x comma 1 divided by square root 1 plus del eta by del x whole square. We are doing a linear calculation. Eta is the disturbance to the interface and so, we are not allowed to retain terms which are bigger than eta to the power 1.

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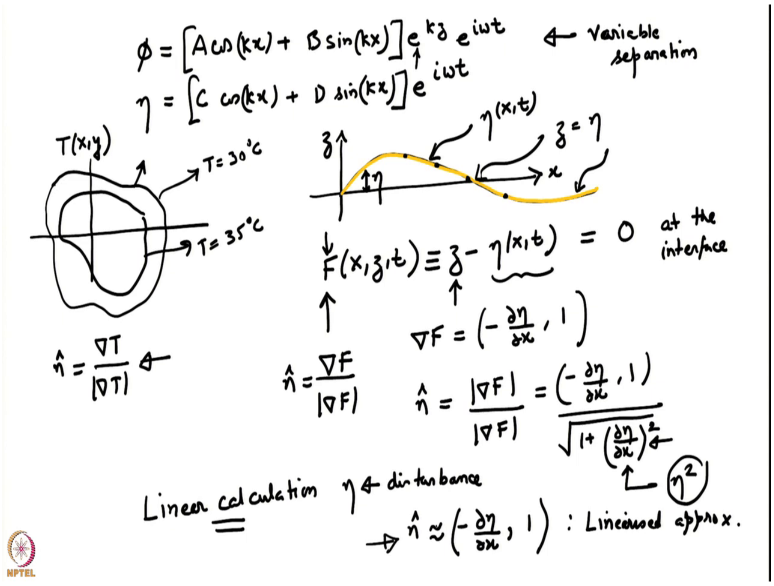

So, there is no. So you can immediately see that this term is going to be eta square and so, this is going to be a non-linear term. So, a linearized approximation to the unit normal is just this. This is an approximation is just this vector whose x component it is minus del eta by del x and whose z component is 1.

This is an approximation. This is not the exact unit vector. You can see it has not been normalized, but that is because under the linear approximation, the length of this vector is just 1 because this is a quadratic contribution.

So, this is a linearized approximation. One can do this more formally by expressing it as an expansion in perturbation series and so on. But by now we have developed some amount of experience in doing this and so, we can do it intuitively.

So, we are saying that this is a this is the linearized approximation to the unit normal. Now, what we need to do? In order to incorporate this into the surface tension term is we need to calculate the divergence of the of this unit normal. So, let us calculate the divergence of this unit normal.

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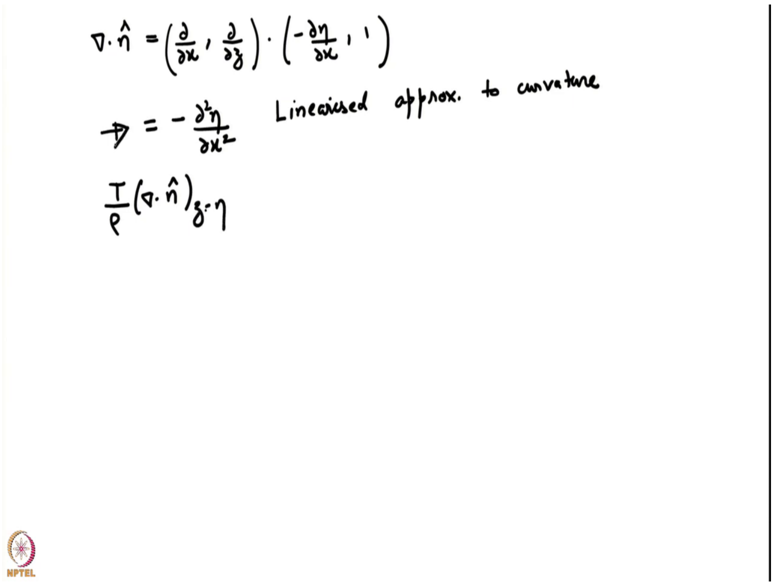

So, divergence of unit normal is our divergence operator is del by del x and del by del z. This operating on unit normal whose first term is del eta by del x and second term is 1, you can immediately see that the del by del z does not do anything and we just get minus del square eta by del x square.

This is the linearized approximation to the interfacial curvature. We have to go back and plug this into the Bernoulli equation. The Bernoulli equation said that we are going to write it again del dot n. Now, this was applied at z is equal to eta, but my expression for del dot n does not have any z.

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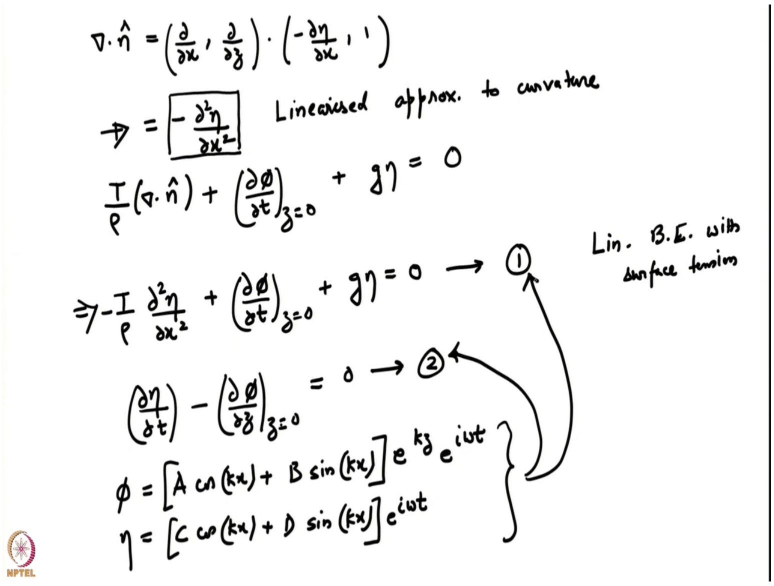

So, I can drop this z is equal to eta plus del phi by del t this like before was applied at z is equal to 0 and we have a gravity contribution which is g z g eta. Now if I plug this form, the linearized approximation to del dot n hat, then this is just minus T by rho del square eta by del x square plus del phi by del T at z is equal to 0 plus g eta is equal to 0. This is my equation 1. This is linearized Bernoulli equation with surface tension.

Similarly we also have the linearized kinematic boundary condition del eta by del t minus del phi by del z at z is equal to 0 is 0. This is equation 2. We have already decided that the forms of phi is A cos k x plus B sin k x into exponential of k z e to the power i omega t. Similarly eta is C cos k x plus D sin k x into e to the power i omega t.

So, now if one takes this and plugs it into equation 1 and equation 2, then we obtain the analysis is very similar to what we had obtained earlier. So, I am straight away going to write down the answer. So, one collects all terms which are coefficients of cos x.

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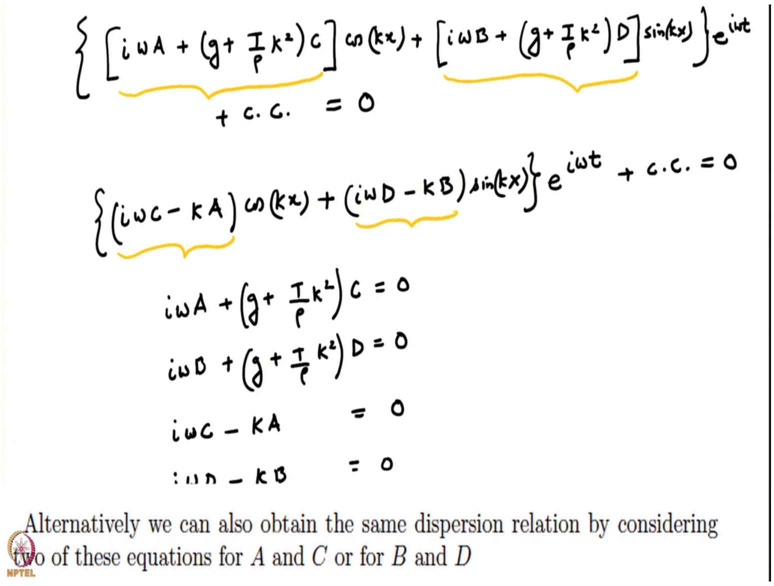

So, one will obtain i omega A plus g plus t by rho into k square into C into cos k x plus i omega B plus g plus t by rho k square into D into sin k x. This whole thing multiplied by e to the power i omega t plus of course there is a complex conjugate equal to 0. This comes out from one of the equations, the linearized Bernoulli equation.

Similarly, the kinematic boundary condition implies i omega C minus k A into cos k x plus i omega D minus k B into sin k x. This whole thing into e to the power i omega t plus complex conjugate is equal to 0. That comes out from the kinematic boundary condition; like before cos and sin are linearly independent.

So, we set the prefactors to 0. So, we obtain four equations in four unknowns, the equations are i omega A plus g plus T by rho k square into C is equal to 0 i omega B plus g plus T by rho k square into D is equal to 0 and then, we have i omega C minus k A is equal to 0 and i omega D minus k B is equal to 0.

So, all I am doing is just setting the coefficients to 0. So, I will highlight the coefficients. So, I am setting this part to 0 because this is the coefficient of cos k x. I am setting this part to 0 because this is the coefficient of sin k x. Similarly I am setting this part to 0 and I am setting this part to 0 ok. That gives me four equations in four unknowns A B C and D are my unknowns. So, one can write this is a linear set of four algebraic equations.

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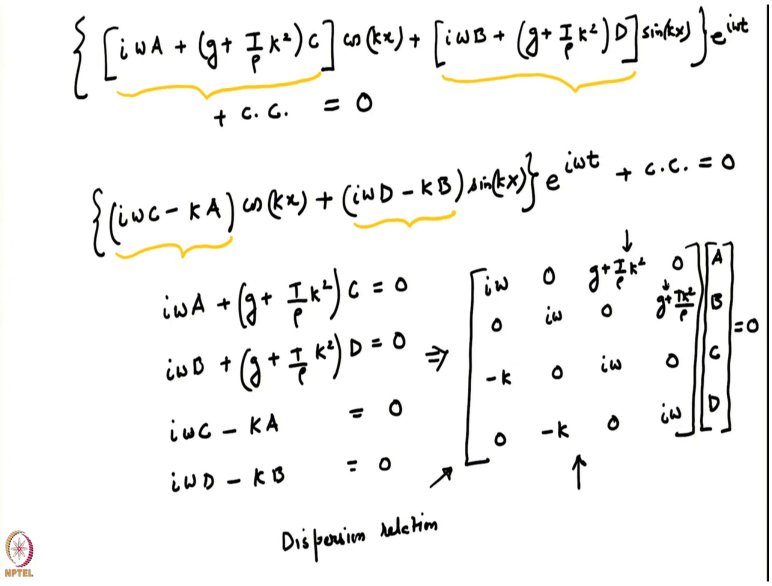

Once again you can write it as a matrix i omega 0 g plus t by rho into k square 0 0 i omega 0 g plus t k square by rho minus k 0 i omega 0 0 minus k 0 i omega. This multiplies A B C D, A B C D and it is a homogeneous set of equations. So, this is equal to the zero column vector. I am not writing that vector you can. So, this is the zero column vector, ok.

So, like before we obtain we expect non-trivial solutions because A B C and D, all of them cannot be 0. That is of course a solution, but we do not want trivial solutions. So, for non-trivial solutions the determinant of the matrix has to be 0. That determines our dispersion relation.

You can see that we had obtained the same matrix. Earlier you can go back and check when we had done this exercise for pure gravity waves, you will see that just setting T to 0 in these two terms, you will recover the matrix that we had got earlier.

So, this is just a slightly more complicated version of the same matrix. We had got a fourth order relation in omega a quartic in omega which could be factorized very easily, even this one factorizes very easily.

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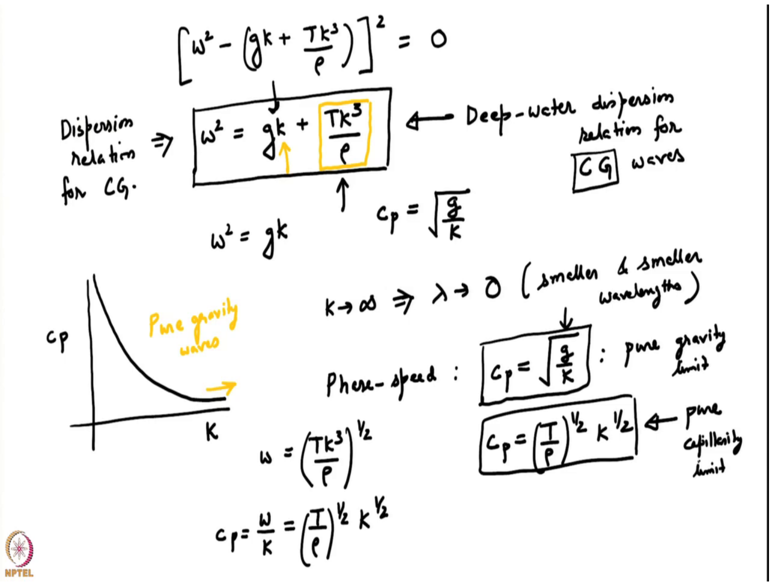

And you can immediately show once you work out the determinant, you can show that this is omega square minus g k plus T k cube by rho whole square. So, that is a quartic in omega and this just tells us omega square is equal to g k plus T k cube by rho. So, this is the deep water dispersion relation for capillary gravity waves.

We have obtained a special case of this earlier omega square is equal to g k. We know that if we plot let us plot the phase speed. So, the phase speed of pure gravity waves we have seen is g by k. So, if I plot phase speed for pure gravity waves. We know that it will go like this k.

Now, you can see that as k goes to infinity, we are looking at smaller and smaller wavelengths. So, k going to infinity implies lambda going to 0. So, smaller and smaller wavelengths. So, you can see that this is where this term provides a correction. This extra term is will provide a correction because as k gets larger and larger, k cube which occurs here in the second term is much larger than k.

So, we intuitively expect that as k becomes larger and larger, this behavior for pure gravity waves will be corrected and the phase speed for capillary gravity waves will look different. So, how does how does it look? So, we can see firstly that for sufficiently large k, you can immediately see that for sufficiently large k this term dominates. So, if I plot the phase speed of capillary gravity waves phase speed, so let us look at the various limits.

So, the pure gravity limit is g by k; the pure. So, this is the pure gravity limit. This predicts that as k becomes larger and larger, the phase speed becomes smaller and smaller. Let us look at the pure capillary limit. So, we said this term the first term to 0 and we keep only the t k cube by rho term. So, we have omega square is equal to T k cube by rho or omega is equal to T k cube by rho to the power half.

What is omega by k? Omega by k is T by rho to the power half into k to the power 3 half minus 1 which is one-half. So, you can immediately see that this phase speed which is the phase speed for pure capillary waves has a very different behavior. This is T by rho to the power half k to the power half.

This decreases at large k, this increases at large k and I expect this pure capillary limit to dominate at large k because large k implies smaller and smaller wavelengths and as the length scales become smaller and smaller, surface tension becomes dominant over gravity. So, there are two distinct limits.

One is very large wavelengths when it becomes mostly the waves behaves like a pure gravity wave. One is very small wavelengths when the waves behave like pure capillary waves and an intermediate regime, where both gravity and capillarity are both important and they are called as capillary gravity waves.

Let us plot this dispersion relation that we have obtained. So, this is the dispersion relation for capillary gravity waves. So, let us plot the dispersion relation and let us compare it with the two limits. This is the pure gravity limit and the pure capillarity limit and we will see some very interesting behavior in these waves.

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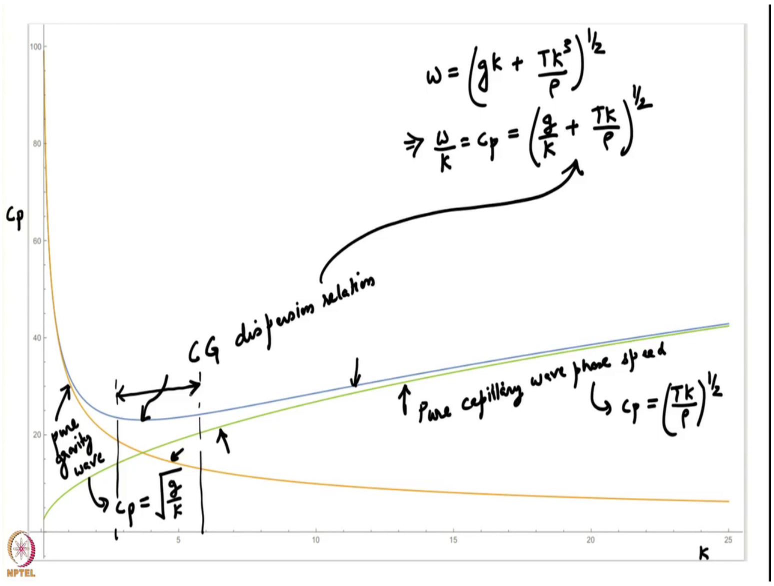

So, here I have plotted the phase speed as a function of k. Now at large k you can see. So, the blue curve is the capillarity gravity dispersion relation or c p as a function of k, ok So, we have got omega is equal to g k plus T k cube by rho to the power half. This is the full capillary gravity dispersion relation. This implies omega by k which is basically our definition of cp is I will get a g by k here and I will get a T k by rho in the inside. So, you can see that.

So, this is the capillary gravity dispersion relation, the full one. You can see that at large k, the blue curve becomes asymptote to the green curve, the green curve is the pure capillary wave phase speed. We have seen that is c p is T k by rho to the power half at very small k.

The blue curve once again becomes asymptote to the orange curve, the orange curve is the pure gravity wave which basically says c p is equal to square root g by k.

And in an intermediate regime, neither the green curve nor the orange curve are good approximations to the blue curve. This is the regime where one cannot in general drop capillary effects or gravity effects. We will make an estimate of this region in the next class.

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