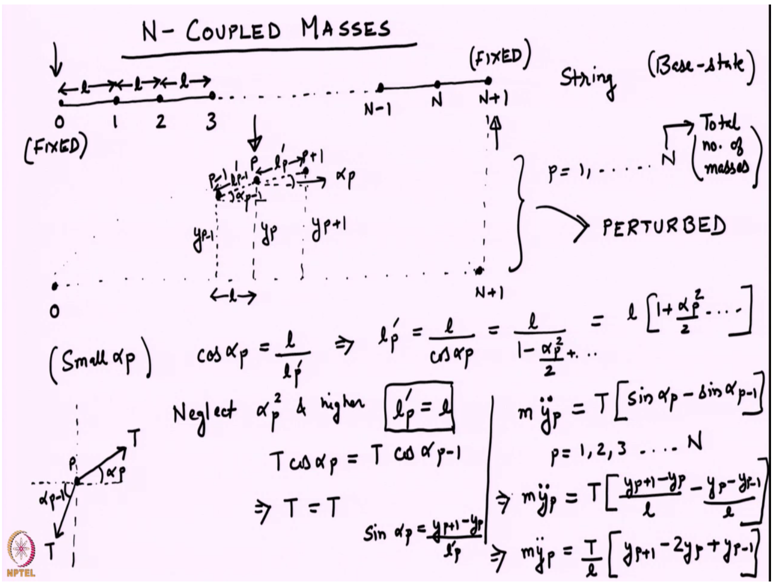
Introduction to interfacial waves
Prof. Ratul Dasgupta
Department of Chemical Engineering
Indian Institute of Technology, Bombay

Lecture – 04
Coupled, linear, spring – mass systems (continued..)

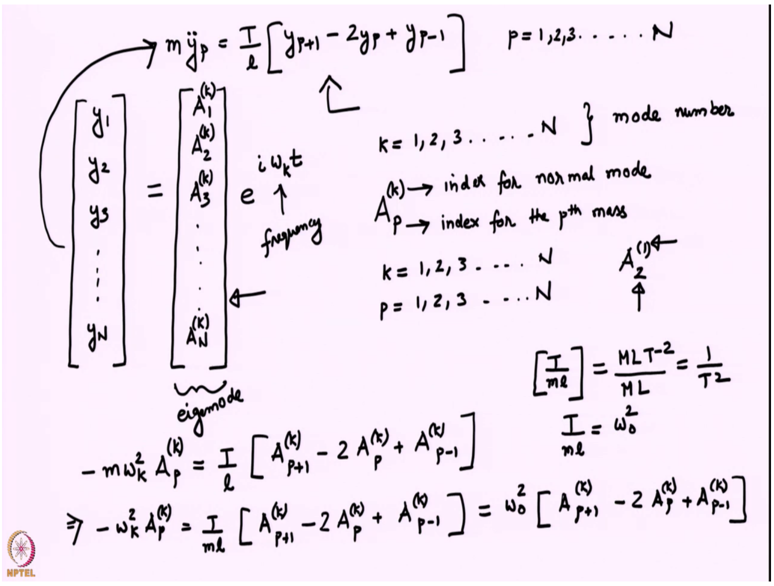
Before we proceed further, let us quickly once again summarize what we have done so far.

(Refer Slide Time: 00:18)



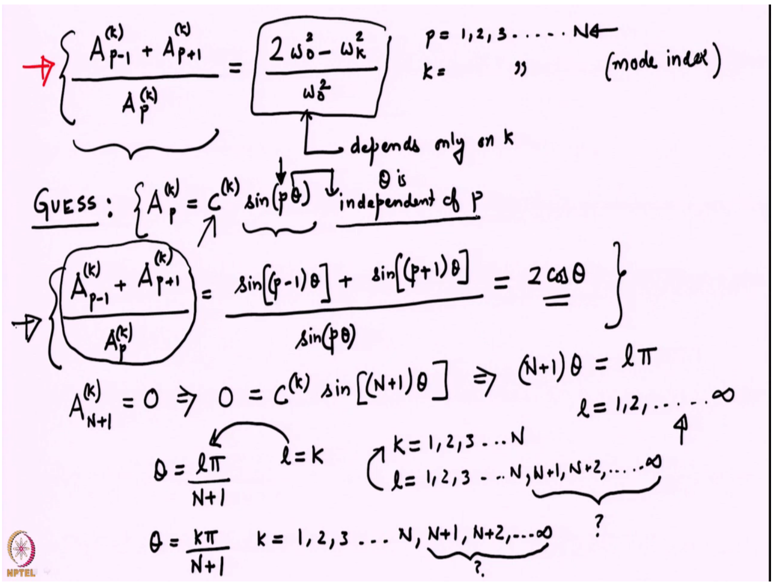
We started with a with N coupled masses in the base state they were all under tension connected through a string, then we perturbed it we wrote down the equations the equations were non-linear, we assumed small angles and we linearized about the base state and we got equations for perturbation.

(Refer Slide Time: 00:38)



These were N coupled linear ordinary differential equations on which we impose the normal mode approximation. Now, here we are trying to we are trying to solve the system and determine the eigen modes and the eigen frequencies of the system without writing down matrices which is why we have kept the total number of masses in the system as capital N which could be any integer. Now we obtained the once we substitute the normal mode approximation, we obtain algebraic equations and we are trying to find solutions to the algebraic equations.

(Refer Slide Time: 01:06)



Now, we wrote it in a kind of variable separation separable form there are two things there are two indices P and K. The way we have written it here. The left hand side depends on both K and P whereas, the right hand side depends only on K. So, that in implies that although A P K would depend both on P and K. This particular ratio would be independent of P. So, with that in mind we made a guess for the solution the guess was not completely arbitrary, it was with we had the boundary conditions in mind and which is why we chose sin P theta.

We also checked that sin P theta when we substitute into this ratio is independent of P. So, it gives us just cos theta and this tells us that this is a correct guess provided theta is independent of P. Now with that background let us proceed further. And so, first is we have to make sure that theta is independent of P and so, theta can atmost depend on K and secondly, what is theta what is the functional form of theta.

So, let us find that for that we will have to use the boundary conditions. The left most boundary condition which is the zeroth mass is fixed is automatically satisfied by our choice of sin P theta. If we put P equal to 0 this is automatically taken care of. We have to explicitly satisfy the right boundary condition which is the P is equal to capital N plus 1 when we substitute then the displacement should be 0. So, we say that A in any mode of vibration the right most point always remains stationary.

So, K for any value of K if I substitute P is equal to N plus 1 this will be equal to 0. Now we know what is A K of P this is given by this formula. So, if I substitute it here then this tells us that 0 is equal to C K sin N plus 1 theta. So, this is basically telling us that because C K is not 0 in general. So, sin N plus 1 theta is equal to 0.

So, we can immediately write that N plus 1 into theta is equal to some integral multiple of pi. Now here l is an integer we choose it to be positive, but it goes all the way from 1 to infinity. So, this implies that theta is equal to l pi by N plus 1. Now, note that we have introduced one more index l.

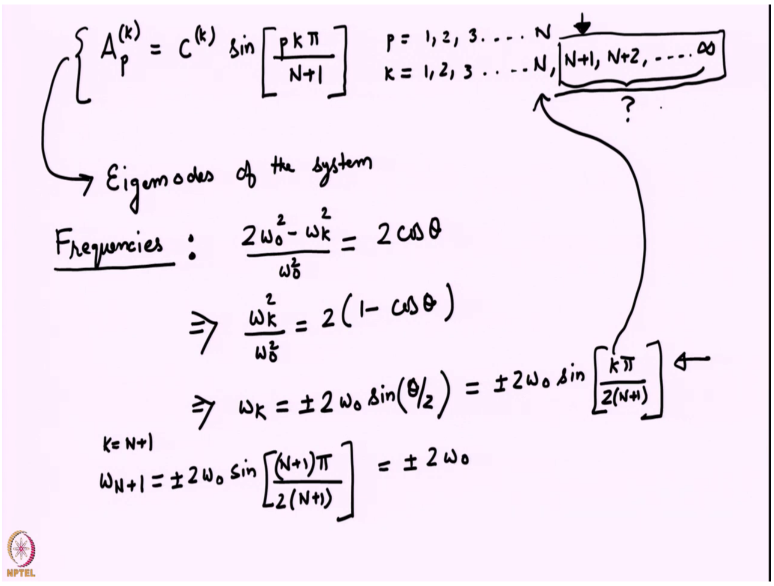
But we have also seen that A P this ratio is a function only of K that is because this ratio is the is equal to this ratio and the right hand side depends on K. So, this must only depend on K. So, as a first guess we set l is equal to K in this formula l is also an integer K is also an integer; however, we have to be careful here because K if you recall was for the mode index which normal mode is my system oscillating in. And so, K the value of K went only from 1, 2, 3 up to capital N whatever be the value of capital N; l however, goes from 1 to infinity.

So, I am going to write this here K goes from 1, 2, 3 up to capital N, l also goes from 1, 2, 3 up to capital N, but continues further with N plus 1, N plus 2 all the way to infinity and so, when we set l is equal to K we have to worry about where what are these extra things that are coming which are not admissible for the value of K. So, when we set l is equal to K. So, I am going to write theta is equal to K pi by N plus 1.

But now I have to remember that K goes from 1, 2, 3 up to N and in addition also has these additional integers which somehow do not make sense, but are there mathematically. So, we have to make sure that they do not produce anything which is physically meaningful.

So, let us continue with that. So, we have now determined theta as a function of K expectedly theta it does not depend on P as we had written down here it is only dependent on K. So, with that having done this so far we find now we can write.

(Refer Slide Time: 06:17)



That A P K is equal to C K it was sin theta so, that it just becomes P K pi by N plus 1. P goes from 1, 2, 3 up to N, there is no problem with P. K goes from 1, 2, 3 up to N like before, but because we have said l is equal to K there is also an additional value N plus 1, N plus 2 up to infinity and this is extra. So, I am putting a question mark.

So, now, you can see that we have already solved one part of the problem we have determined the eigen vectors without actually writing down matrices. So, this is represents our eigen modes of the system. We will take an example and use this formula to convince ourselves that this is indeed so, and that we have this formula actually tells us what are the eigen vectors of the system what about the frequencies?

Let us work on the frequencies. We go back to our earlier equation which is this equation indicated in red and we work on this equation. So, we write 2 omega 0 square minus omega K square by omega 0 square is equal to that ratio which involved A K P minus 1 A K P plus 1 and A K P and we have seen that with our guess it is just 2 cos theta. Now this can be written as omega K square divided by omega 0 square equal to 2 into 1 minus cos theta and we can simplify this and write omega 0 sin theta over 2.

I have used the formula for 1 minus cos theta is 2 sin square theta by 2. Now we know the value of theta. So, we can write this further as sin K pi by 2 N plus 1 and K here is subjected to that, that it goes from one all the way to infinity. So, this tells us. So, now, we have determined our formula for the frequencies of the system again notice that we did not write down any matrices we did not solve any eigen value problem.

Our guess was chosen in such a way that it was chosen in such a way that it could satisfy the boundary conditions. One of the boundary conditions was automatically satisfied by the guess the left boundary condition the right boundary condition had to be explicitly satisfied and that lead to an expression for theta and now we have an expression both for the eigen modes as well as the frequencies of the system. So, now, let us now try to answer the question that what are these extra values that we are getting.

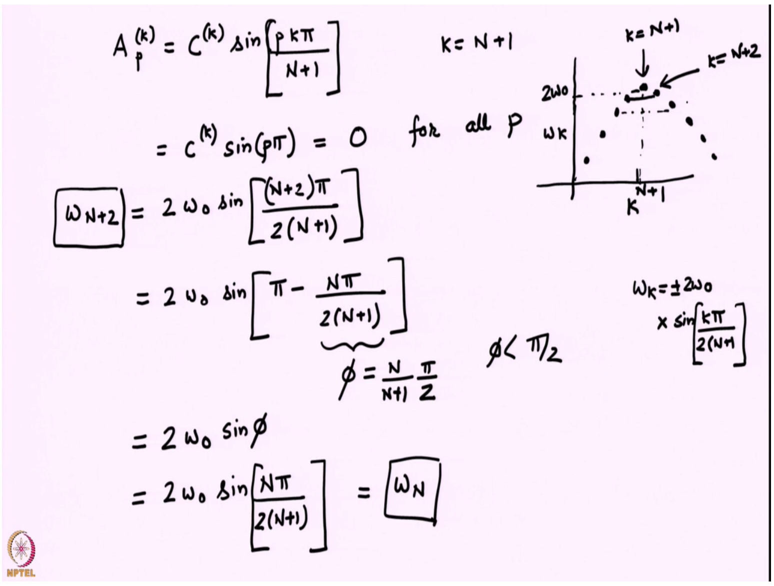
So, let us ask the question now why do we call this extra. Because remember that this is an capital N degree of freedom system whatever be the value of capital N, I would expect that many frequencies and that many eigenvectors, but no more than that; however, the value of K can actually go to N plus 1, N plus 2 up to infinity this is because in our previous slide we have set l is equal to K and l went all the way to infinity.

Now, we have to just convince ourselves that these extra values do not give us anything more which is physically meaningful. So, as a first step, let us calculate what is omega N plus 1. I would expect capital N number of modes of oscillation. So, I do not expect any frequency beyond capital N. So, let us first calculate by this formula what happens if I substitute K is equal to N plus 1. So, I am already taking the first value in this set which is under a question mark.

So, omega N plus 1 is plus minus 2 omega 0 sin N plus 1 pi by 2 into N plus 1. I am just substituting the value of K here to be N plus 1 in this formula and this is just equal to plus minus 2 omega 0 sin pi by 2 which is 1. Now this is physically meaningful this is telling us that there is a frequency which is plus let us stick to the positive frequency first. So, 2 omega 0.

So, this is definitely a physically meaningful frequency. So, it seems that even these extra things which we have under question mark is giving us something physically meaningful. Now, let us calculate the eigenvector corresponding to this frequency. So, lets substitute K is equal to N plus 1 and calculate the eigen mode corresponding to this frequency.

(Refer Slide Time: 11:55)



The eigen mode formula is; this is the formula that we had got earlier and we have to substitute K is equal to N plus 1 and if you do that you can immediately see that you get sin P pi and because P is an integer this is always equal to 0 for all P. So, this is reassuring because we had earlier found a physically meaningful frequency even for K equal to N plus 1. But this is telling us that the amplitude of oscillation is just 0.

So, although there is a frequency, but nothing really oscillates at that frequency. Now you can go back and check what happens at N plus 2. So, let us do just for the sake of demonstration let us do omega N plus 2. Omega N plus 2 would be twice omega 0 sin now this I can write it as 2 omega 0 sin pi minus N pi by 2 into N plus 1.

This is just an algebraic manipulation that we have done and then you can immediately see that this is like an angle phi. This phi is equal to N by N plus 1 into pi by 2 because N by N plus 1 is less than 1. So, this is an angle which multiplies something which is less than 1 into pi by 2. So, we know that phi is less than pi by 2. So, by our usual formula this is sin phi and this is, but if you look at the general formula for omega K, you will notice that this is nothing, but omega N.

So, we find that omega N plus 1 gives us a physically meaningful frequency, but nothing oscillates at that frequency because the corresponding eigenvector is all 0. Omega N plus 2 is basically the same as omega N. If you do this for omega N plus 3 you will find that omega N plus 3 is the same as omega N minus 1.

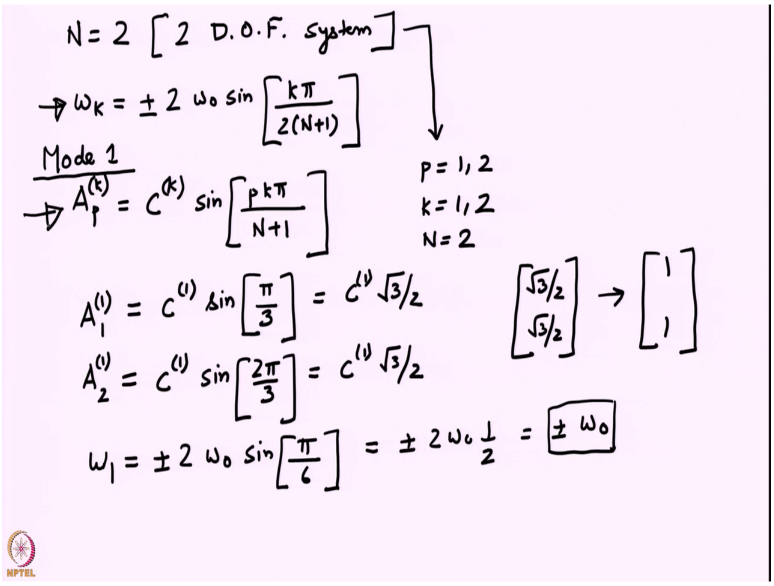
This is a consequence of the fact that we are looking at a sinusoidal variation here. So, if you plot omega K as a function of K it would look something like this. So, this is not defined for any value of K only for integer K. So, for K equal to 1 let us say it is this and then it would reach a peak at K equal to N plus 1. You can see that at K equal to N plus 1.

So, again I am writing down the general formula for omega K just for reference here plus minus 2 omega 0 multiplied by sin K pi by 2 N plus 1. So, you can see that at K equal to N plus 1 this reaches its maximum value when sin is pi by 2. So, at N plus 1 it reaches the value 2 omega naught and after that it again starts decaying.

So, this is more of a sinusoid and so, this is the value corresponding to K is equal to N plus 1, this is K is equal to N plus 2 and this obviously, is the same as K is equal to N. So, we are finding that omega N plus 2 is the same as omega N and then omega N plus 3 would be the same as omega N minus 1 and so on and so forth. I leave it to you to calculate what happens to the eigen vectors and you can go back and check if you find new eigen vectors you will not ok.

So, now let us take an example and apply this until now we have kept everything symbolic, our total number of masses capital N was arbitrary now let us choose a specific example and convince ourselves that what is the eigen modes and eigen frequencies that we are getting. So, for simplicity I will just choose.

(Refer Slide Time: 16:39)



So, capital N is equal to 2. So, 2 degree of freedom system two masses. So, let us calculate omega K for this system. I am just going to write down the formula for omega K so, that we can substitute and check.

So, for this particular system P will go from 1 to 2, K will also go from 1 to 2 because it is a two degree of freedom system. So, let us calculate what is A 1 in the first mode of oscillation what are the eigen vectors. So, this is equal to. So, K is 1, P is 1 and capital N in this case is 2. So, the denominator is 3.

So, this is C of 1 root 3 over 2. This is again in the first mode, but now the displacement of the second mass, this is 2 pi by 3 and this is also equal to root 3 by 2. So, our eigen vector is basically the C is not relevant here because it is a constant multiplying both A 1 1 and A 1 2.

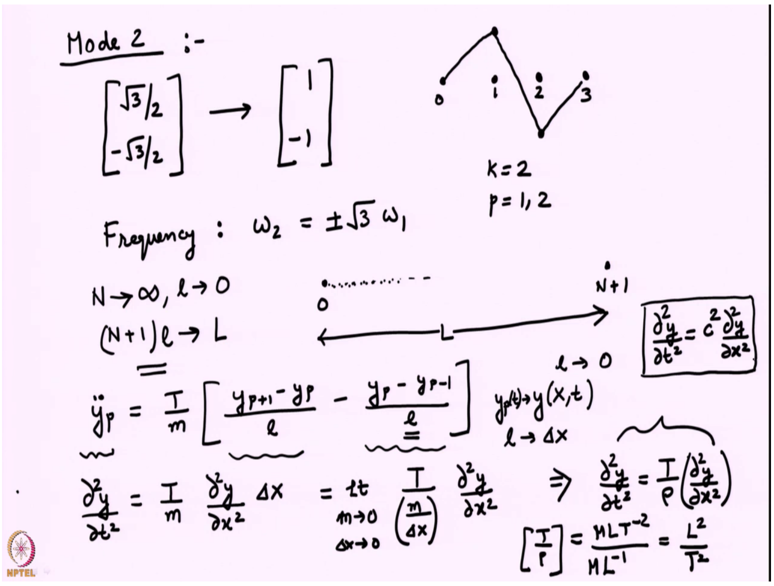
So, our eigen vector is root 3 by 2, root 3 by 2, but you can see that the root 3 by 2 is also not relevant here because I told you that you can multiply an eigenvector by any scalar and what you will obtain is still an eigenvector. So, I can multiply this by 2 by root 3 and obtain 1 1.

So, there comes our first eigenvector which we had also seen in the previous example where we are looking at horizontal oscillations these are vertical oscillations. So, this is mode 1 what is the frequency of mode 1? Omega 1 is equal to plus minus 2 omega naught sin pi over 6 is plus minus 2 pi over 6 is half.

So, we are getting plus minus omega naught. So, that is our frequency of the system. Once again it turns out to be the frequency of our old system ok the previous example that we had looked at 1 1 is the eigen vector. So, you can either take it to be 1 1 or you can take it to be minus 1 minus 1 give an equal displacement in the same direction to both the masses and it will oscillate at this frequency.

In this example also you will see that the string in between the two masses in this mode does not get stretched over and above its base state. And so, it the two masses behave as if they are uncoupled which is why you get this frequency of oscillation what about mode 2?

(Refer Slide Time: 20:14)



So, let us write mode 2. So, you can quickly show that the eigen vector. So, you can go back to the same formula I leave it to you to do it I am just going to write down the final answer it is very easy. You can find that the eigen vector is root 3 by 2 root 3 by 2 or in other words it is 1 minus 1. So, it is again 1 minus 1 or minus 1 1. So, you can either so, it the displacement.

So, if you have a fixed mass here and a fixed mass here this mode would look like give the first mass. So, in base state let us say the two masses are here. So, this is zeroth mass and this is the third mass they are both fixed and we can perturb only 1 and 2. So, this is 1 this is 2.

So, by a unit distance you perturb this mass and take it there. By a unit distance, but in the opposite direction perturb the second mass and take it there and so, the perturb configuration looks like this or you could multiply the whole thing by minus 1 and then the first mass will come down by unit distance and the second mass will come up by unit distance.

And if you set it up in this initial condition, you will excite only this mode of oscillation and the system will vibrate in a pure mode. What is the frequency? Once again you can calculate it using the formula that has given in the last page and you will just find that omega 2 is equal to plus minus root 3 times omega 1. Remember that for this calculation K is 2. P will again go from 1 to 2, but K is 2 because K is an index for the mode and mode is 2.

So, once again we find that the second frequency and the second eigen mode is identical to the previous example this is not a coincidence you can think more about why it is so; that in the in these kind of examples in the previous example we were exciting the system along its length here we are exciting the system perpendicular to its length and we are retrieving identical frequencies.

Now these formulas are very useful because once you have these formulas you can actually solve for any number of any number of masses that you have in your system. So, if once you have this these two formulas, once you have access to these two formulas that I have just written.

So, this formula and this formula you can actually write down the eigen modes and the eigen frequencies of a system where N capital N is an arbitrary number you know it could be 100, it could be 1000, as long as the boundary conditions do not change. If the boundary condition changes, you will have to redo your analysis. This example is not sufficiently general to take all kinds of boundary conditions. You can have fixed free boundary conditions and so, on. So, there this analysis has to be modified alright.

So, now let us take one step further and ask ourselves one interesting limit of this. So, let us ask what happens that we go to the continuum limit? So, I put in more and more masses in between the two fixed walls. So, I am taking capital N to infinity if I put more and more masses in between a fixed length the inter mass distance has to go to 0. So, l goes to 0.

But this going to 0 and going to infinity has to be in such a way that the gap between the two walls the distance between the 0 and the Nth plus 1 mass is fixed. So, N plus 1 into small l goes to capital L. So, this is the 0th mass which doesnt move this is the N plus 1 Nth mass which is which also does not move fixed and the gap between them I am saying is L.

So, we are free to throw in as many masses as we want this would cause small capital N to go to infinity small l the inter mass string distance to go to 0, but the gap between the zeroth mass and the N plus 1th mass would not change. So, this is just expressing that mathematically and let us see what happens to our equation that we wrote down for the P th mass.

So, we did write earlier that Y P double dot this is something the linearized equation that we had written earlier on which we had done our normal mode analysis we had written this earlier. Now I am going to do a slightly intuitive argument on this you can already see that the left hand side is a second derivative in time the right hand side appears to be like a second derivative in space.

You can see that the first term is like a first derivative, the second term is also like a first derivative as the difference of the first derivative evaluated at two different points in space. So, this is like a second derivative. And so, if I take the limit m if Iput in if I put in infinite number of masses. So, I am looking at the limit l going to 0.

So, the first thing that will happen is until now y was indexed with P it was the vertical displacement of the P th mass. So, now, because we have a mass everywhere because this point we are putting in mass all over the place. So, instead of having a discrete index y will have a continuous index which is x. So, instead of having y P. So, y P which was a function of time will go over to y which is now indexed by a continuous variable which is x and then the time dependence still stays there.

So, you can immediately see that the right hand side becomes like del square y because l also goes to 0. So, this is like del square y by del x square, but there should be an l square at the bottom and we dont have an l square. So, l is like my delta x which goes to 0. So, I have to place for dimensional reasons I have to place another delta x because this delta x and this delta x square will produce a delta x at the denominator and we only have a delta x here and the left hand side can be written as del square y by del t square.

Note that I have dropped the subscript P because P has been replaced by a continuous index x. So, now you can see that this is going towards a wave equation this is a partial differential equation, but we are not quite there as yet. As you can see that if you put in more and more masses it does not make sense to ask what is the mass of a point because there is a mass everywhere.

So, when we are going into the continuum limit it makes sense to ask what is the density of the string ok. What is the mass per unit length this is a one dimensional example so, this would be the density. So, we can see that limit m has to go to 0, you can see that if the continuum limit m has to go to 0.

And so, this would be t m over delta x delta x of course, also goes to 0 and so, m over delta x will go over to a finite number which will represent the density of the string in the base state. And so, you will have del square y by del x square. So, I can write this whole equation as del square y by del t square is equal to T by rho del square y by del x square. You can convince yourself that T by rho T is a tension. So, M L T to the power minus 2 I am writing the dimensions, this is a density a linear density mass per unit length.

So, this becomes L square by T square. So, this equation gives we recover our familiar wave equation. This is a square of a velocity, this is the velocity of a wave as we will see later. We will explore this continuum limit in slightly more detail, until now we have looked at finite degree of freedom systems. In the next class we will look at normal mode analysis conducted on a partial differential equation which represents a continuum the wave equation.

We will look at solutions of this equation in various kinds of geometries, we will do normal mode analysis and we will recover what is known as a dispersion relation. We will also see that this dispersion relation is basically the analog of the frequencies that we have obtained until now in the finite degree of freedom system.