Introduction to interfacial waves
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Lecture - 39
Cauchy–Poisson problem for delta function initial condition (contd..)

We were looking at the solution to the Cauchy Poisson problem for a delta function initial condition, where the interface at time t equal to 0 was perturbed as a delta function at x is equal to 0. I showed you a movie of the solution as a function of x and how the interface propagates in time. We saw that it evolves in time this is a snapshot of the interface.

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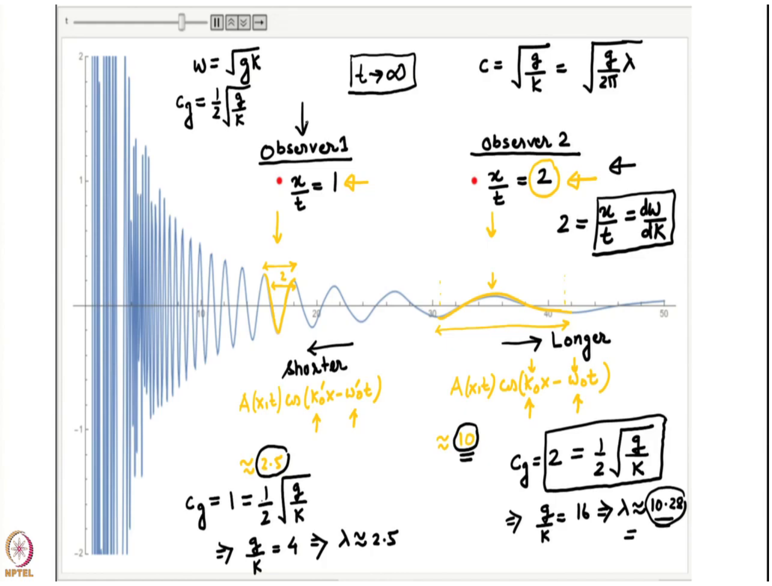

In particular I had mentioned that, the dispersion relation predicts that longer waves move faster than shorter waves. So, at any instant of time, if you take a snapshot we see that the longer waves are the most outward ones are the longer waves, the inward ones which are closer to the origin are shorter waves. So, we get longer and longer waves as we go to the right, and we get shorter and shorter waves as we go to the left, and this is true at any given instant of time.

Now, I also said that what will a local observer, who moves at a constant speed observe. So, for reference we had two observers indicated by the two red dots. Both of them were moving at two different speeds. Observer 1 was moving at unit speed and observer 2 was moving at twice the speed of observer 1. We were asking, what does the observer observe locally? So, let us see what the observer observes locally. So, as an example, let us focus on observer 2, who is moving at a speed of 2 units.

So, you can see that this observer; observes a local wave packet. So, I am plotting the wave packet from trough to trough. So, this is approximately the wave packet. It is we can think of this as approximately a Fourier mode; obviously, its amplitude is not exactly constant, but you can see that I can think of this as a local Fourier mode of a given wavelength. You I urge you to go back to the video that I had showed you from which the snapshot has been taken and follow this observer as the observer moves outwards.

You will find that the observer does not observe a point of constant phase, or in other words if you look at the crest of this wave packet at this instant. So, if you look at this point. After sometime you will see that the observer will move to the right, but the crest will move also to the right, but faster than the observer.

However, as the observer moves, the observer sees approximately this wavelength. So, this wavelength can be expressed as some amplitude which is not a constant, but we will modulate as a function of both space and time into some local wave number cos k naught x in minus some local frequency. These local wave numbers and frequencies as we will soon see will also be turned out to be a will also turn out to be a function of space and time.

So, let us make an estimate of this wavelength from this picture and let us try to understand, what determines this wavelength? So, you can see that this wavelength is so, this point is slightly more than 30 and this point is about 42. So, approximately this wavelength I will take it to be 10, it is slightly more than 10. So, this in S I units this is let us say 10 meter.

Now, similarly let us say what observer 1 observes. So, again you can see that the same structure is observed; here also I can think of this as a local waveform with an amplitude which is modulated. So, A is a function of space and time. So, once again some local A x t into cos and this will be a different k naught. So, I will call it a k naught prime x minus a different omega naught, clearly this k naught and omega naught prime are different from that k naught and omega naught prime.

So, now, let us estimate the wavelength here, you can see that this is approximately 2.5 a single the distance between two things. So, this distance is 2. So, it is slightly more than 2. So, we will take it to be approximately 2.5. Now I claim that this wavelength would be observed this is the wavelength that moves whose group velocity is 2, for observer 2. Similarly this is the wavelength whose group velocity is 1.

We will see how we obtain this conclusion? But let us first verify this. So, we know the dispersion relation omega is equal to omega is equal to square root gk. We have seen that the group velocity in this case is half the phase velocity, the phase velocity was square root g by k.

So, cg is half square root g by k. Let us say, that my claim is correct, which means that if the group velocity in this case the observer was moving with speed 2. So, if the group velocity is 2, then what is the wave number, whose group velocity is 2? So, that is given by this simple relation. So, this implies g by k is equal to 16. From this if you determine lambda you will find that lambda turns out to be approximately 10.28, this is close to our estimate.

We can try this exercise for observer 1 as well. For observer 1 we do the same exercise, we say that the wavelength that observer 1 sees is the wavelength whose group velocity is the same as that of the observer. Since, this observer moves at a speed of 1. So, the relevant wavelength would be given by the equation this. And, so, we would observe g by k is equal to 4 or in other words one-fourth of this value.

So, approximately lambda will be approximately 2.5. You can see that this is very close to this estimate and this is very close to that estimate. How do we know this? For that, we need to go back to our expressions and we need to work on this a little bit more. Pay attention that these conclusions that, what the observer observes is valid as time gets larger and larger. So, let us work on those, how did we reach these conclusions. For that let us do a simple exercise.

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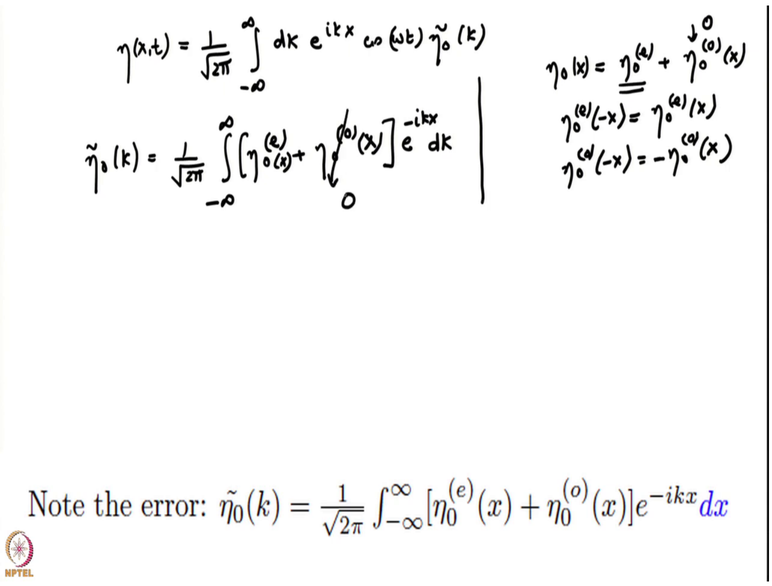

So, our solution to the Cauchy Poisson problem is always expressible as a inverse Fourier integral of this form we have seen this before. Now, in this particular case eta 0 tilde of k was a constant because our initial condition was a delta function for eta 0. However, we can make some general conclusions about the structure of these integrals.

So, let us see how? So, let us assume that we can split because we have taken eta 0 of x to be a delta function and a delta function is an even function. So, in general we can split eta 0 of x into an even part and an odd part. By definition the even part satisfies, this relation and the odd part satisfies this relation. Because, we are dealing with the delta function initial condition we will assume that our initial condition is even.

So, eta 0 only has an even part the odd part is 0, then it can be shown. So, eta 0 tilde of k by definition is just an even part plus an odd part, which we are going to take to be 0. This is the definition of eta 0 tilde of k now if I set this part equal to 0.

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$$\eta(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} \cos(\omega t)\, \tilde{\eta}_0(k)$$

$$\tilde{\eta}_0(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\eta_0^{(e)}(x) + \eta_0^{(o)}(x)\right] e^{-ikx}\, dk \quad (\eta_0^{(o)} \to 0)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \eta_0^{(e)}(x)\, e^{-ikx}\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \eta_0^{(e)}(x) \left[\cos(kx) + i \sin(kx)\right] dk$$

$$\eta_0(x) = \eta_0^{(e)} + \eta_0^{(o)}(x) \quad (\eta_0^{(o)} \to 0)$$

$$\eta_0^{(e)}(-x) = \eta_0^{(e)}(x)$$

$$\eta_0^{(o)}(-x) = -\eta_0^{(o)}(x)$$

Note the error: $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \eta_0^e(x)[\cos(kx) - i\sin(kx)]dx$

Then, it is easily seen that what is left behind is just the even part into e to the power minus ikx dk. Now, I can write the e to the power minus i kx as cos kx plus sin i sin kx. You can easily see that the second term which is a product of sin kx into eta 0 even of x is an odd function. So, let me write this cos kx plus i sin kx dk. The second part which involves the product of this and that you can see is an odd function why?

Because, at x when I replace x with minus x, this part will not change sin whereas, this will change sin because sin is an odd function. So, you can see that the product of the two is going to be an odd function. When we are integrating it from minus infinity to infinity the second part will go to 0, the first part will only survive.

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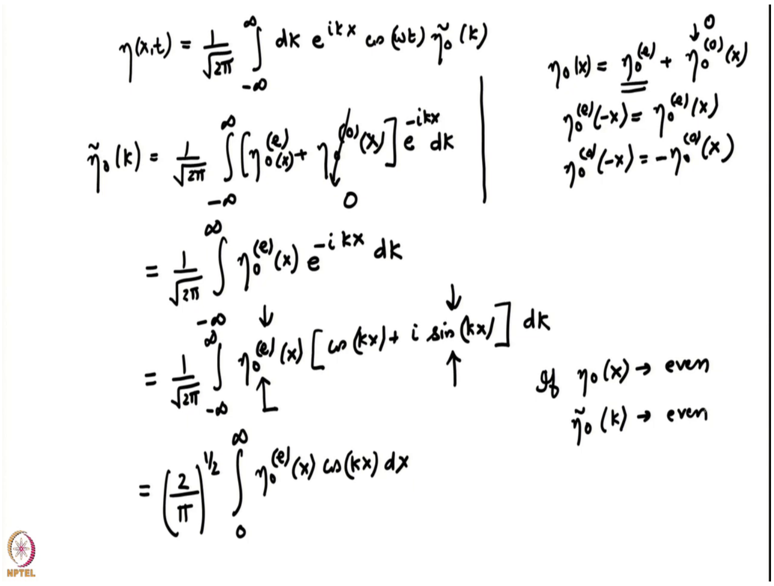

Because, the first part is an even function I can do the same thing as before, I can write it as 2 times the same integral from 0 to pi instead of integrating from minus infinity to plus infinity we integrate it from 0 to infinity. So, I write this as. So, I pull the factor of 2 outside. And, the 2 and the square root 2 got cancelled and made the pre factor square root 2 by pi. And, so, we just have eta 0 even of x into cos kx dx.

This is just telling us that if eta 0 of x is even its Fourier transform eta 0 tilde of k is also even. So, now, let us use that fact and let us write down the Fourier integral.

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$$\eta(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx}\, \tilde{\eta}_0^{(e)}(k) \cos \omega t \qquad \omega = \sqrt{g|k|}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} dk\, \cos(kx)\, \tilde{\eta}_0^{(e)}(k) \cos(\omega t)$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} dk\, \tilde{\eta}_0^{(e)}(k) \left[\cos(kx-\omega t) + \cos(kx+\omega t)\right]$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} dk\, \tilde{\eta}_0^{(e)}(k)\, \mathrm{Re}\left[e^{i(} \right.$$

Note the error: $= (\frac{2}{\pi})^{1/2} \int_0^{\infty} dk\ \tilde{\eta}_0^{(e)}(k)[\frac{\cos(kx-\omega t)+\cos(kx+\omega t)}{2}]$. This missed factor of half will not influence further analysis

So, we have seen before that eta of x at any time t is 1 by square root 2 pi into and now, we are assuming that the Fourier transform of the initial condition is even, because my initial condition is even. So, it is just an even function of k cos omega t. So, you can readily see, that this is an even function; this is also an even function because omega is equal to square root g into mod k. So, when k becomes minus k this remains the same. And, this has 2 parts to it a cosine and a sine part.

The cos part multiplied by the other two makes the entire integrand even whereas, the sin part makes the entire integrand odd. So, once again by the same argument, it is easy to show that this can be written as just this. For even initial conditions, we get this. And, now we once again have 2 by pi to the power half 0 to infinity dk and now I am going to keep the eta 0 tilde even as k as some arbitrary function, some arbitrary even function and I am going to write cos kx cos omega t as a sum of two traveling waves. So, I can write it as cos kx minus omega t plus cos kx plus omega t.

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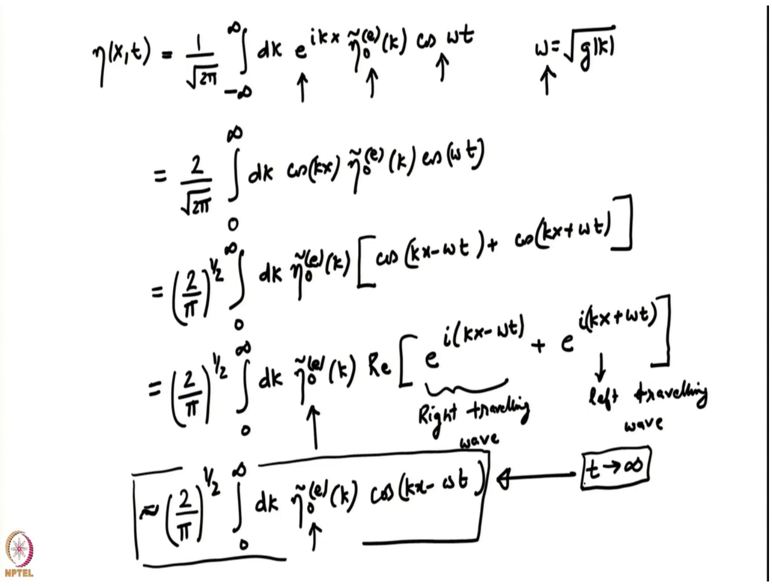

And, then we can write this as 0 to infinity dk eta 0 tilde of k into the real part of e to the power i kx minus omega t plus e to the power i kx plus omega t. Now, we can plug in our initial condition and work on these integrals. You can see that, this is a right traveling wave and this is a left traveling wave, we have seen this earlier. If, the initial condition is perfectly symmetric as it is in this case because we have assumed it to be even.

So, what will happen is one part will propagate to the right, another part will propagate to the left, they will just be mirror images of each other. So, we can focus on just one part. Let us say that, we are propagating on the we are focusing on the right travelling part. So, we are essentially interested in this part. So, the first term so, this is not equal to so, the first term is of the form dk into real part of or rather cos kx minus omega t.

So, we are interested in integrals of this form and as we said we are interested in what happens to integrals of this form. For even eta 0 tilde of k at large times, this if we understand what is the behaviour of these integrals then, we would have understood what we saw in the last slide, where we found or rather we claimed that, the every observer moving at constant speed sees a local wave number, whose group velocity matches the speed of the observer.

Looking at these integrals in the limit of t going to infinity will lead us to the same conclusion. So, let us look at it how. Let me write the integral using still complex exponential notation but we will take the real part of it.

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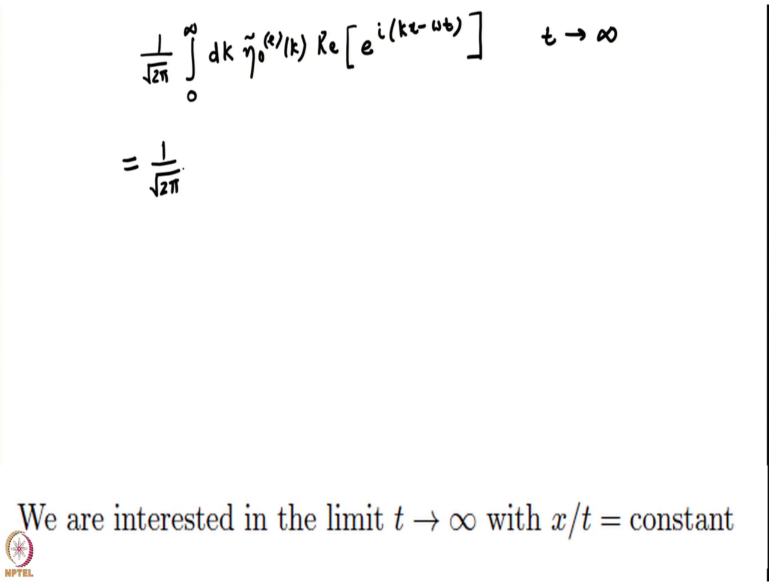

So, we are focusing on the right traveling wave. So, 0 to infinity dk into eta 0 tilde, which has been assumed to be an even function of k into real part of exponential i rather e to the power i kx minus omega t. And, we are interested in this integral as t goes to infinity.

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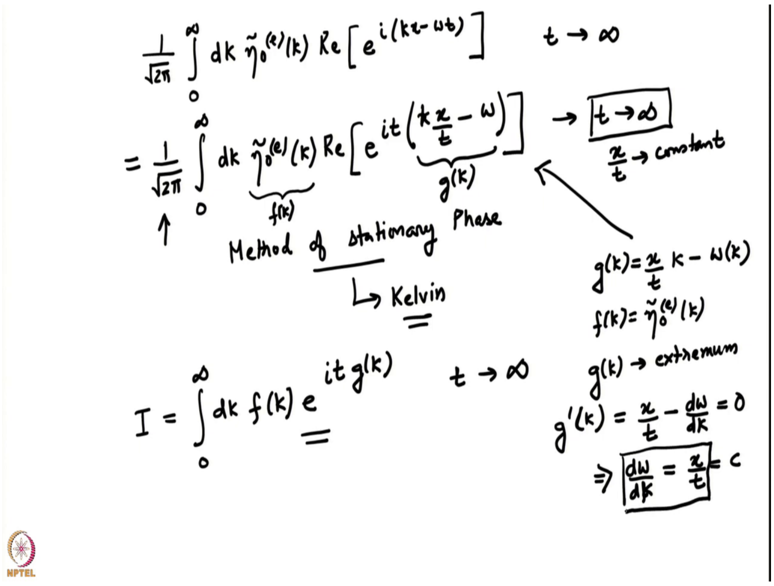

Now, so, you can see that I can write this integral as I can pull the t out and I can write it as k into x by t minus omega. Now, we are interested in the form of this integral as t goes to infinity. Now, I will show you one particular method, that method is called the method of stationary phase. This method is an is a perturbative method, which allows us to evaluate the leading order contribution or the leading order value of these integrals as a parameter, which in this case is time go becomes larger and larger.

This method was used by Kelvin to evaluate integrals of this form in the context of water waves. So, let us understand the basic idea of the method and then it will become immediately clear, that when the observer moves with a certain speed the wavelength that the observer sees at any instant of time is the same wavelength whose group velocity matches the observer. So, how do we see this?

Let us say that we have to evaluate an integral of this form. Some so, we essentially need integrals of the form 0 to infinity I will I am ignoring this factor this is not so, important for the analysis. And, then we are we have some function here this is what I am calling f of k. And, then we have e to the power i t into some function g of k.

Now, so, what whatever is written here is my function g of k, in this case g of k is a function of x and time as well. So, this is the kind of integral that we would like to evaluate for time going to infinity. Now, it is useful to think of these integrals, because I have to take the real part of this exponential at the end, it is useful to think of these integrals as an area under the curve. So, let us get take an simple example.

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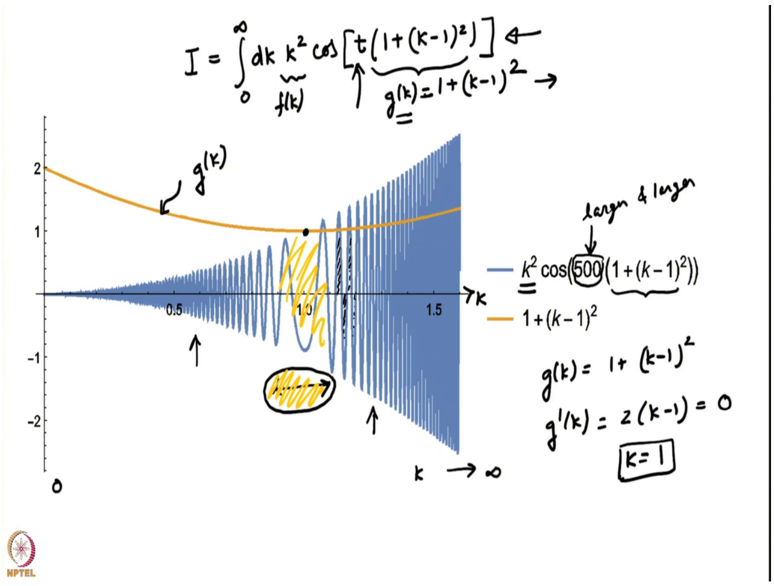

So, here I have plotted a simple example. So, what I have done is I have plotted this quantity. So, let me write the integral here. So, I in this case is 0 to infinity dk my f of k is k square. So, I have just chosen it to be k square it need not be k square it can be any other function, any other well behaved function of k. And, then I need to take the real part of the exponential so, I have just taken cos and time so, cos t into some function g of k which is this.

So, t into 1 plus k minus 1 whole square, I will tell you why I have chosen this. So, in this case f of k is k square and g of k is 1 plus k minus 1 whole square. This curve is a plot of g of k as a function of k. You can immediately see, that g of k has a minimum at k equal to 1. Now, if suppose we want to evaluate this integral, for larger and larger values of the parameter t; t will turn out to be time in our earlier example, but we can think of time as a parameter, so, we want to evaluate this integral for larger and larger values of t.

So, I have plotted for example, t is equal to 500 here. You can immediately see that if I did this integration and if you think of this integration as an area under the curve. So, I have to integrate all the way to from 0 to infinity from 0 to k equal to infinity. You can see that this integrand is extremely oscillatory and the more the value of t, in this case it is 500 suppose you make this 600 or if you make this 1000, these oscillations will get faster and faster in k.

So, you will see that everywhere almost in the entire domain as t becomes more and more these oscillations become more and more intense. So, you will have very closely spaced oscillations. So, you can see that, if I think of this as an area under the curve, then successive oscillations will just cancel each other. So, this is a positive area, its cancelled by the immediately negative area, then the next is again a positive area, it is again getting cancelled nearly by the negative area.

So, in almost this cancellation will happen more and more as this number becomes larger and larger. And, so, this entire integral the dominant contribution to this integral as you will see, will come from this region, because this is the region where the oscillations are the slowest. Now, what is this region? This is the region where g of k has an extremum, in this case g of k has a minimum. So, you can see that g of k is 1 plus k minus 1 whole square.

So, g prime of k is twice k minus 1 and if I set this equal to 0 then k is 1, you can check that k equal to 1 is a minimum by taking the second derivative. Now, you can see that, the place where g of k has an extremum the integrand oscillates very very slowly around that point. And, so, if I integrate this area under the curve and if I obtain the area under the curve, the most of the contribution to that entire area is going to be from this region.

So, I am just going to highlight this region in yellow from this region ok or from this region ok. Now, that forms the essential idea of how to estimate these integrals. What one does is one calculates points, where g of k has an extremum. And, then one says that as time goes to infinity the dominant contribution to these integrals, come from those places where g of k has extremas. Now, let us remember this and let us go back to our original integral.

So, our original integral was of this form. The original integral which we are looking at was of this form. And, we want to now you can see that our g of k is defined as x by t into k minus omega. And, omega we have to remember is itself a function of k. Now, given this form so, our f of k is the Fourier transform of the initial condition, we have assumed it to be even so, the Fourier transform is also even.

So, you can immediately see that the dominant contribution to this integral as time goes to infinity is going to come from the place, where g of k has an extremum a minimum or a maximum. For that to find that out, we will have to take the derivative of g of k with respect to k. If we do that then what do we get we just get x by t minus d omega by dk is equal to 0.

This is telling us that the dominant contribution to these integrals is going to be from the place, where d omega by dk matches x by t. I have to mention that this limit is taken in such a way, that x by t is held constant. So, t going to infinity x by t held constant. So, x by t is some constant. This integral has a dominant contribution when d omega by dk matches x by t.

Now, we understand why our what we did in the previous slide works. We had two observers; one was moving at x by t is equal to 2, we had the other observer who was moving at x by t is equal to 1. What the observer 2 would observe is a local wave number, which satisfies that relation x by t is equal to omega by d omega by dk. In this case x by t is 2. So, cg is 2. So, we have this relation and from this one can work out what is it that the local observer is going to see.

So, in the neighborhood of this observer the dominant contribution to the integral comes from the point where x by t is equal to 2. In the neighborhood of this observer the dominant contribution from to the integral comes from the point x by t is equal to 1. Or in other words x by t is equal to the local group velocity is equal to 1 and the local group velocity is equal to 2.

And, so, that in turn determines what each observer sees locally. One can in principle reconstruct the entire profile by asking by putting a series of such observers moving each of them at different speeds and asking what does each observer see locally.

You can clearly see that each observer will locally see a different wave number. And as the observer moves the wave number will remain constant, but the observer and the wave number is related by the fact, that the observer speed matches the group velocity corresponding to the wave number.

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- The delta function initial-value problem solution to the Cauchy-Poisson problem may be found in greater detail in the following book:

  Hydrodynamics, H. Lamb, Chapter 9 (Surface waves), article 238, Cambridge Univ. Press

- The application of the method of stationary phase to the Fourier integrals of the form discussed here is discussed in the following books:

- Theory and applications of ocean surface waves, Part 1 - Linear aspects, C. C. Mei, M. Stiassnie and D. K.-P Yue, World Scientific, Advanced Series on Ocean Engineering (see Chapter 2, section 2.1.1)

- Nonlinear water waves, L. Debnath, Academic Press Inc., Chapter 3 (section 3.2)