Introduction to interfacial waves
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Lecture - 38
Cauchy – Poisson problem for delta function initial condition

We were looking at the solution to the Cauchy-Poisson problem for a delta function initial condition, where the interface was perturbed with 0 initial velocity but with a displacement, which was represented as a delta function at the origin.

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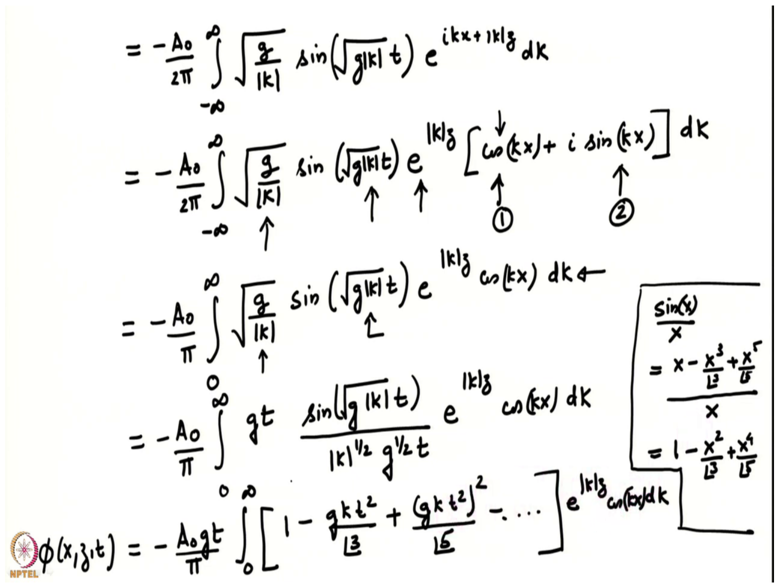

We had seen that we would take the approach, where we would write the equation for the velocity potential, the expression for the velocity potential contained a term of the form sin x by x or rather sin of some argument divided by its argument. We expanded that in a Taylor series and then we said we would integrate it out term by term, let us continue from there.

So, we had the following expression for phi of x comma z comma t. And, we have expanded it out in a Taylor series for the term sin of square root g k t divided by square root g k into t.

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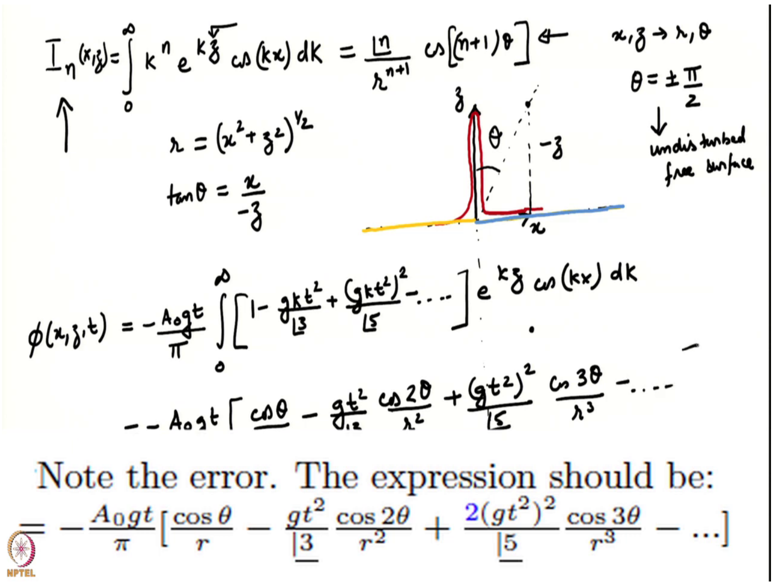

So, now, you can see that each term in this infinite series each term has the form integral 0 to infinity k to the power n e to the power k z. We are not retaining the mod on k, because now the limit of integration has been changed from 0 to infinity, cos kx dk you can easily see this if you go back.

So, you can see very easily that the first term has the structure, it is k to the power 0 e to the power k z cos k x d k. The second term would be k to the power 1 e to the power k z cos k x d k k square and so on and so forth. So, we need a formula which looks like this. Notice that z is negative here in general and so, this integral is expected to be a convergent integral.

One can so, if I call this integral I n note that this is a function of x and z, then it is possible by integration by parts I can I urge you to try this for yourself that you can work this out for each value of n. So, the general formula is easier to represent in polar coordinates ok. So, the general formula in polar coordinates is of the form factorial n by r to the power n plus 1 cos n plus 1 theta.

So, here we have written the answer in polar coordinates. How did we define r and theta? So, r is the usual x square plus z square to the power half and theta is defined slightly differently, it is x by minus y ok. So, what does this means? We are going from, so, our this is our undisturbed surface and our original coordinate system was let us say this, this was x and this was z. And, we were interested in negative values of z whenever there was a wave.

So, in general there would be some kind of a delta function perturbation here. So, infinitely tall, but I am just drawing it as a finite height and then we are interested what happens at later time t. So, now, in order to express this integral, we have gone over from x comma y to r comma theta. This should be z, this should be z, why did we take minus z, because z in this expression is negative.

We are interested only in negative values of z and so, minus z is going to be a positive quantity. So, our polar coordinates are defined like this. So, this angle is theta. Pay attention that this is not the usual way in which we define polar coordinates, usually theta is defined with respect to the horizontal axis, here we are defining with respect to the vertical axis.

So, any point here. So, if we have any point here for example, that will have a negative z and so, minus of negative z will make it a positive z ok. So, the corresponding mirror image of that point, let us say somewhere here and then tan theta would be x divided by minus z ok. So, this is this coordinate is minus z. So, you can see that this is how it is defined.

Now, it is very clear that theta is equal to plus minus pi by 2 represents the undisturbed interface, free surface or in other words this line, this line and this line. Let me use two different colors. So, plus pi by 2 would be the line in blue minus pi by 2 would be the line in yellow, this is going to simplify also some of our calculations going to polar coordinates ok.

So, we have written down the formula for the integrals and once we have done that so, let me write down the solution for phi that we had got earlier is equal to and we had found that phi is equal to minus A naught gt these things we can pull out 0 to infinity and then we would have 1 minus g k t square by factorial 3 e to the power k z, z is negative cos kx dk.

Now, we can use the formula that we have written here and integrate term by term. If, we do that then we obtain the first term is for n is equal to 0, it is just e to the power kz cos kx dk. So, this formula for n equal to 0, it is just cos theta divided by r. Similarly, the next term would be gt square by factorial 3 that is the coefficient, then we have k e to the power kz cos kx. So, that is for n is equal to 1.

So, this would be cos 2 theta by r square. Similarly, we can go on g t square whole square divided by factorial 5 cos 3 theta by r cube minus dot. So, now, we need to substitute theta is equal to plus pi by 2 or minus pi by 2. Why do we want to do that? Because, this formally is the solution to phi, it is represented as an infinite series. I will give you a close form answer also shortly.

But, whatever this function is this function is telling me what phi is. And, I can take an infinite number of terms in this function and calculate phi at any point in the liquid region. Now, we are interested more in eta, we would like to visualize how does the interface evolve in time.

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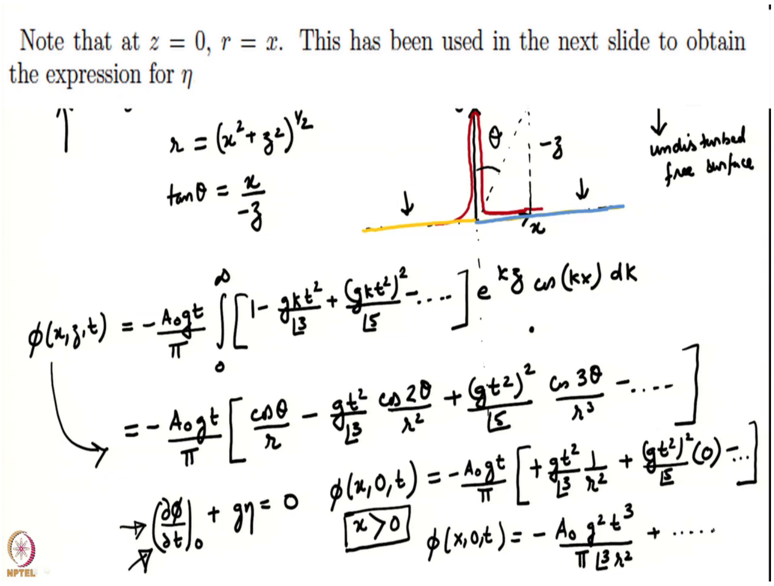

For that, we are going to use this as I told you earlier we are going to use this formula del phi by del t at z is equal to 0 plus g eta is equal to 0. What is this? This is nothing but the linearized Bernoulli equation which we have already used, while solving the Cauchy-Poisson problem.

So, you can see that if I have an expression for phi, I can differentiate it with respect to time, apply it at z is equal to 0 and then calculate eta from there ok. So, that is the prescription that I am going to do. Now, for that, I would like to set z is equal to 0 already in the expression for phi. That is because this expression is a derivative with respect to time. So, setting z is equal to 0, will not make any difference for the derivative with respect to time.

So, I can set z is equal to 0 first and then take the derivative with respect to time and that will tell me what is del phi by del t at z is equal to 0. So, you can see very clearly that if I set z is equal to 0, so, phi x, z is equal to 0. So, the z is equal to 0 represents the yellow and, the blue lines. The yellow and the blue lines have two different values of theta. The blue line has plus pi by 2 the yellow line has minus pi by 2.

Let us focus on one of them. The blue line x is greater than 0, theta is plus pi by 2. So, if I substitute theta is equal to plus pi by 2, then I am getting an expression and this expression will be true for x greater than 0 ok, because we are substituting theta is equal to pi by 2 ok. So, if we substitute theta is equal to pi by 2, then we obtain minus A 0 g t divided by pi. You can see cos pi by 2 is 0, so, the first term will not be there then we will have minus gt square by factorial 3.

And, then cos 2 into pi by 2 cos pi is minus 1 so, that will make this plus and then we will have a 1 by r square plus g t square whole square by factorial 5 and then we will have cos 3 pi by 2, which is also 0. So, we will have more terms. So, now, we can see that phi has a rather simple expression, phi has this expression, that phi of x comma 0 comma t is just minus A 0 g square t cube by pi factorial 3 r square ok plus of course, there will be more terms at higher orders we will write it only up to here.

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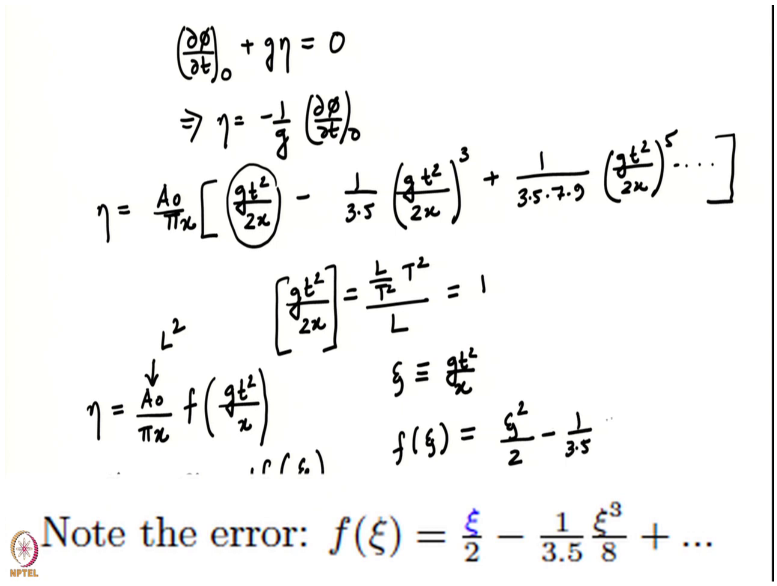

And, if you substitute it into that equation del phi by del t at 0 plus g eta is equal to 0, then this implies eta is equal to minus 1 by g del phi by del t at 0. You can show that, one obtains A 0 by pi g t square by twice x minus 1 by 3 into 5. So, these second terms you will not obtained from the expression that I wrote down, but if you go further in the infinite series you will get these terms and so on.

Now, this is the expression for eta as a function of time, notice an important thing that this quantity, so, there is a 1 by x here which I have missed. So, notice an important thing that this is a non dimensional quantity. So, g t square by twice x g is length meter per second square by length. So, this is 1, the dimension of this is non dimensional.

So, we can write this as eta is equal to A 0 by pi x into some function of g t square by twice x or rather g t square by x. We can write this in a even more non dimensional form eta tilde which is a non dimensional eta is eta into x divided by A naught, recall that A naught has the new dimensions of length square. So, this eta into x is a non dimensional quantity, eta into x divided by A naught is a non dimensional quantity.

And, so, non dimensional eta is a function of non dimensional xi where xi is defined as g t square by x. What we have done in this exercise is we have just found, what is f as a function of xi, we have found that f as a function of xi. So, there is a 1 by pi here. So, f as a function of xi is just given by this series xi square by 2 minus 1 by 3 into 5 xi square xi cube by 8 plus dot dot dot.

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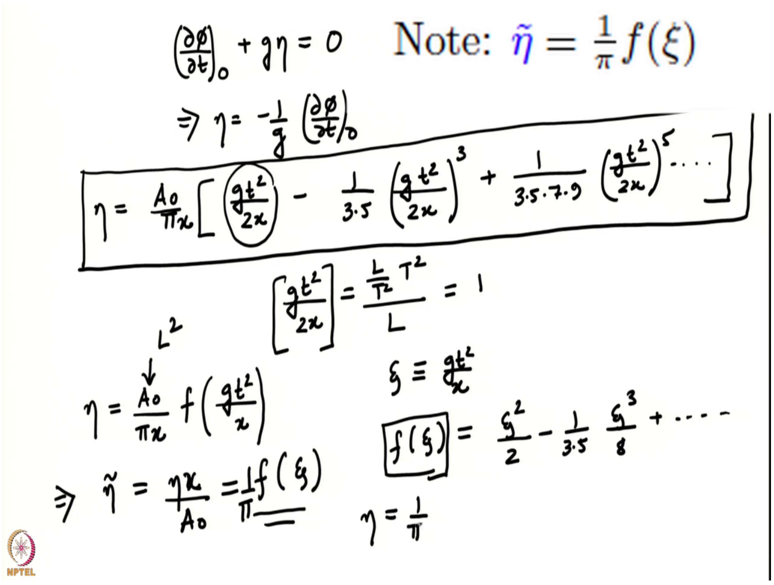

So, we have got an infinite series representation of this function that is what our solution has told us.

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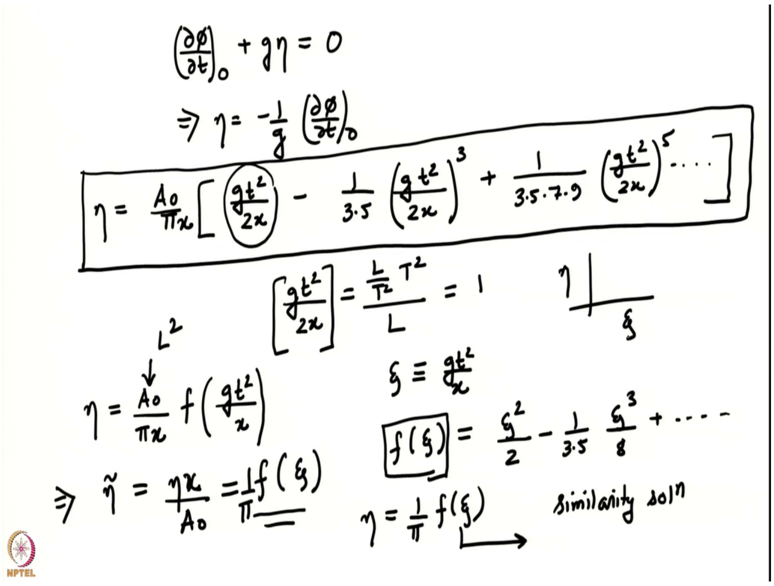

This form where eta is representable as a function of a single variable is called a similarity solution. It contains a lot of information, because it is telling us that as the interface evolves in time. If, I take any snapshot at any given instant of time and if I scale it properly, I can collapse it onto one single curve and that single curve is f of zeta.

So, if I plot eta as a function of xi, then I will have a single curve and that curve contains information of eta at all points in space and at all instance of time that is what this is telling us. Now, for practical purposes it is unwieldy to take this infinite series representation and then calculate eta as a function of xi. So, I am going to give you another form, it can be worked out easily and that form is the one that we are going to use to actually visualize, how does the interface evolve as a function of time.

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It can be shown that

$$\eta(x,t) = \frac{a_0 g^{1/2} t}{\pi x^{3/2}} \left[\left(\int_0^{(S/4)^{1/2}} \cos l^2 \, dl \right) \cos\left(\frac{gt^2}{4x}\right) + \left(\int_0^{(S/4)^{1/2}} \sin l^2 \, dl \right) \sin\left(\frac{gt^2}{4x}\right) \right]$$

Note that the a_0 on the R.H.S. is A_0

So, it can be shown that eta of x comma t this is what we are after is also given by the following expression. A non dimensional argument plus another coefficient and sign with the same non dimensional argument, this form is much more available to a numerical solution, because one does not have to take an infinite series and take more and more terms.

In particular the infinite series that was shown in our last slide can become a slowly converging series, when gt square by x becomes a relatively large number. This is a more useful representation; these integrals are easily numerically calculatable. They are related to what is known as the Fresnel integrals. You can look them up or you can look up any handbook where you will find these integrals readily tabulated.

So, you can consider these integrals as functions which are known. So, they will just become functions of xi, the upper limit of integration is xi. So, these are known functions of xi. So, what I am going to do next is, I am going to show you a video of the solution to this problem.

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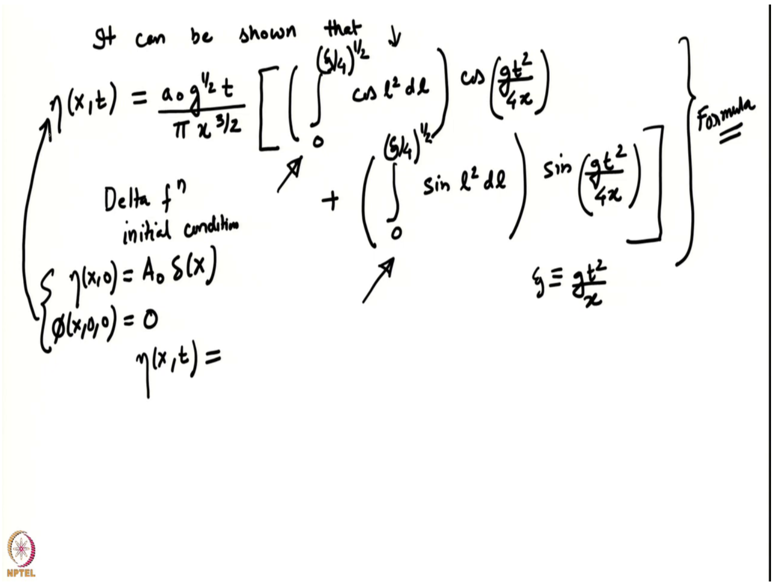

So, I am going to plot eta of x comma t given by this formula, this formula, you can do it yourself using any of the standard packages MATLAB or Mathematica will do it. This has to be still done numerically because these integrals need to be evaluated for every value of xi.

I have converted into x and t, recall that xi is defined as gt square by x. So, I have converted into x and t and I am going to plot now eta of x comma t, as a function of x as time progresses. Now, recall that this is a solution for the delta function initial condition. So, we had taken eta of x comma 0 to be A naught delta of x and phi was of course, 0. So, this is telling us that with these two initial conditions this is what the interface does at all later times t.

You will see that you will see a wave packet spreading out. The wave packet will be infinitely long. So, we will look at a portion of the space region in looking at the wave packet. And, you will see that in general the wave packet will change shape as it evolves and as it translates from left to right. So, let us look at the wave packet and let us try to understand its physical features.

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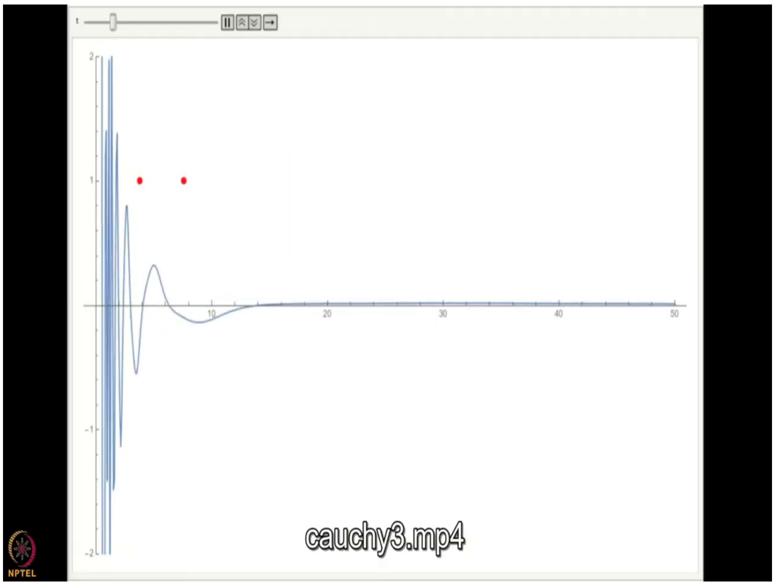

So, you can see that the curve in blue is that wave packet.

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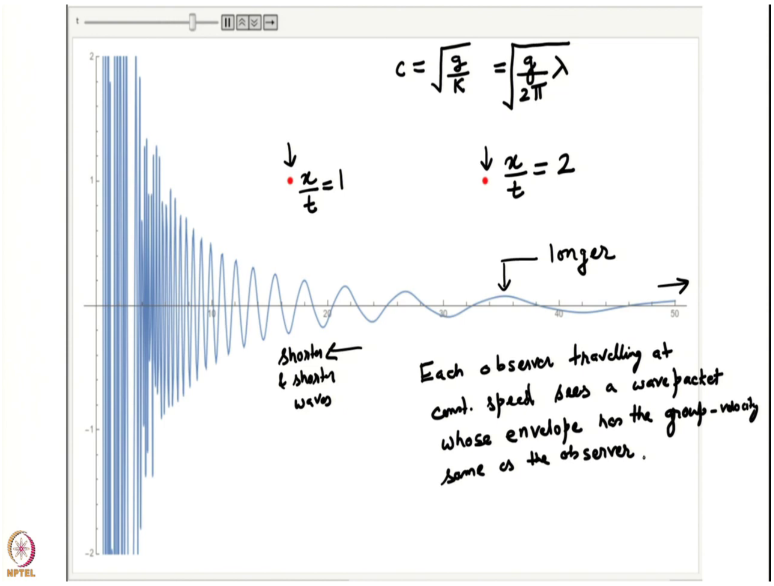

So, we will play that again. So, you can see that the wave packet is spreading out. Now, notice some interesting features. So, of course, you can see that there are these very closely packed tightly packed oscillations near the origin, that is an outcome of the fact that we had a delta function forcing at origin at time t equal to 0. Now, it is useful to look at a distances far and far from the origin.

You will see the first thing that you notice is that, that if you go far in the origin, so, if you just stop this at if you take a snapshot we will do it shortly, you will see that the waves which are farthest are also the waves which are longest. So, the longer wavelengths are the farthest out, this is consistent with our dispersion relation.

Our dispersion relation says the phase speed is given by square root g by k. Or in other words the phase speed is directly proportional to wavelength lambda. So, the greater the wavelength the faster the wave moves. What we are seeing here is initially there was a delta function. So, all wave numbers in the spectrum were excited equally. Now, each wave number will propagate with its own speed which is square root g by k. And, what we are seeing is a Fourier superposition of that at later time t.

Now, there are a number of things that we will understand in this solution. In particular we will see that if those 2 red dots that you see seeing are what is being observed if you move with a constant velocity. So, if x by t is a constant or in other words I am just x is linearly varying as a function of time.

So, I am moving with a constant velocity. So, there are two observers who are moving with two different constant velocities. What will they observe? What part of the wave packet will they observe? We will see that what they observe has got to do with the group velocity.

So, let us understand this in a little bit more detail. So, I am going to stop this video and go back and take a snapshot of this and try to understand what is the physical meaning of this solution to the Cauchy-Poisson problem for a delta function initial condition.

So, this was this is a typical instantaneous snapshot. As I said earlier, you can see that the waves outward are longer. As you go inward at any instant of time the waves are shorter that is because our dispersion relation was gave us a phase speed which is root g by k.

So, this is equal to g by 2 pi into lambda. So, the longer the wave the faster it moves ok. So, the longest waves are outward. You can see that you can go more and more further and you will always have a wave which is even longer. And, if you go this way you will get shorter and shorter waves. We also saw that this profile changes shape as it alters in time as it propagates in time.

So, as we go along if we move at constant speed, so, those 2 red dots are moving at constant speed ok. This dot is moving at x by t is equal to 2. So, it is moving at 2 d x by d t is 2, this red dot is moving at x by t is equal to 1. So, the gap between them will keep increasing as a function of time. What will we see, what will an observer see, if he or she moves with some constant speed?

We will see the answer to this question. And, we will see that this is related to the group velocity of the wave. And, what this observer, what each observer traveling at constant speed, so, I will write this down; each observer travelling at constant speed sees a wave packet whose envelope has the group speed or the group velocity same as the observer.

So, we will try to understand the meaning of this in the next video.