Introduction to interfacial waves

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Lecture - 37
Group velocity and the Cauchy–Poisson problem

In the last video we had looked at the solution to the Cauchy-Poisson problem in two geometries, Cartesian and cylindrical axi-symmetry. We are going to go further into those solutions and understand the physical content that is contained in those integrals. Recall that we had written on the integrals as inverse Fourier integrals and the input into those integrals was the Fourier transform of the initial conditions.

Now, before we start looking into those solutions and trying to understand the meaning of those solutions. We will again come back to an idea which we had already discussed earlier namely, the concept of group velocity. Recall that I had given to you the mathematical formula for group velocity.

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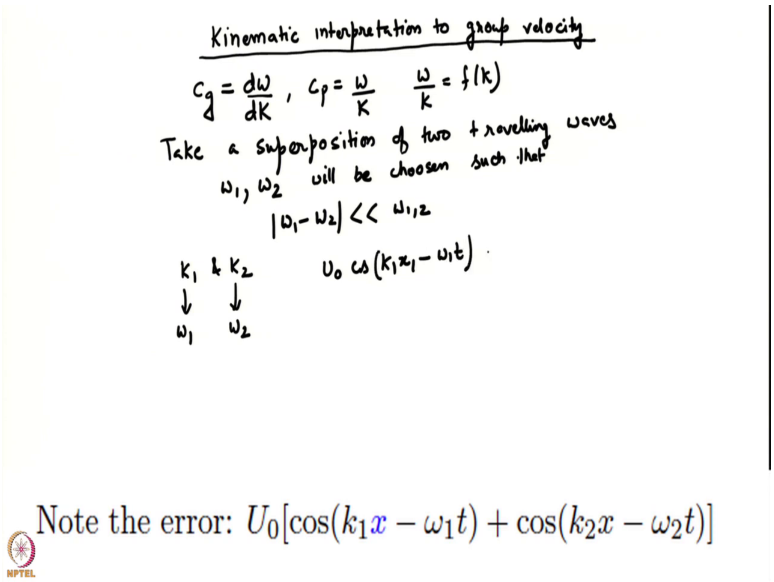

In our case the group velocity C g which was distinct from the phase velocity was written as C g was written as d omega by d k. This is in contrast to the phase velocity C p which we had defined as omega by k. Now, because our medium is dispersive medium omega by k is a function of k, therefore, d omega by d k is the local slope of the tangent whereas, omega by k is the slope of the secant which we plot from origin up to the k that we are looking at.

Now, let us understand this concept of group velocity a little bit more in detail because that will be important for physically interpreting the solutions to the Cauchy-Poisson problem. So, for simplicity I am just going to take a super position of two travelling waves.

In particular we will arrange the waves in such a manner that their frequencies or we can talk about their phase speeds also their frequencies omega 1 and omega 2 will be chosen such that omega 1 minus omega 2 the difference between their frequencies the absolute value of them is much much less than the frequencies themselves or in other words the frequencies are very close to each other. This can be arranged if we choose the wave numbers of the two travelling waves to be close to each other.

So, we will choose two wave numbers k 1 and k 2. The dispersion relation will tell us that there is a corresponding frequency omega 1 and omega 2. Let us take a super position of these two waves. For simplicity let us assume that their amplitudes are the same. So, we will have U 0 cos k 1 x 1 minus omega 1 t, U 0 is some amplitude of the wave.

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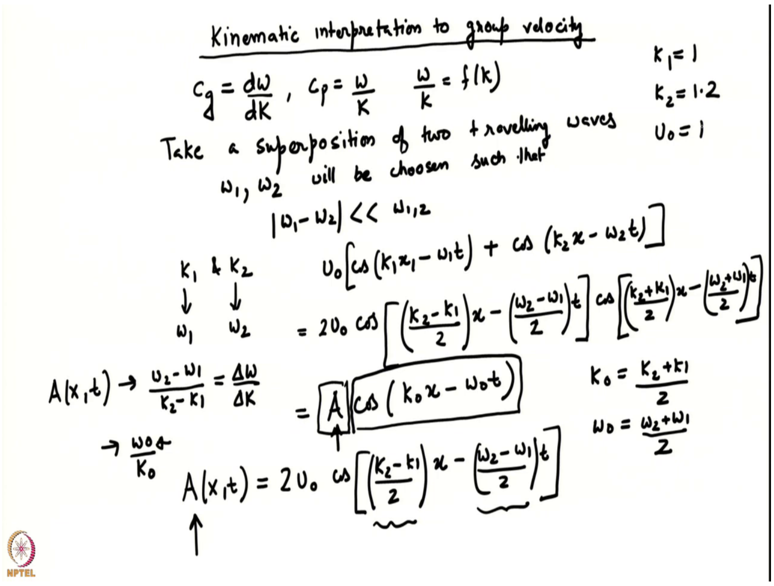

I have chosen the amplitudes to be the same and so, I am just adding them up. We know from the formula of cos a plus cos b that we can write this as now I can write this as some amplitude A into a cos wave which is k 0 x minus omega 0 t. Here k 0 would be k 2 plus k 1 by 2, the average of the two wave numbers and omega 0 would also be the average of the two frequencies.

Clearly the amplitude is not a constant, it is this the coefficient of cos k 0 x minus omega 0 t which we are calling as amplitude, the amplitude itself is a cosine mode ok. So, it is clear that A the amplitude is defined as the amplitude is also a function of space and time. So, I will write it as 2 U 0 cos k 2 minus k 1 by 2 x minus omega 2 minus omega 1 by 2 into t.

Now, you can see something very interesting. You can see that this part the amplitude part or the envelope of the wave ok, so, this is like the envelope is itself a Fourier mode is itself a travelling wave, but it has a velocity which would be given by the ratio of this to that. Recall that the ratio of the frequency to the wave number gives us the speed of motion of a Fourier this the speed of a Fourier mode.

So, the A of x comma t A of x comma t will be moving at a speed omega 2 minus omega 1 divided by k 2 minus k 1. Within that envelope there will be this part and that will be moving at a different speed. The other speed would be omega 0 by k 0. So, it is clear that the envelope moves with this speed whereas, the phases within the envelope move with that speed.

Let us visualize this in a video. So, in the video that I am going to show you I have chosen k 1 is equal to 1, k 2 is equal to 1.2. So, neighboring wave numbers and I have used the dispersion relation to determine corresponding omega 1 and omega 2. I have chosen the amplitude U 0 also to be 1. So, let us see what it looks like.

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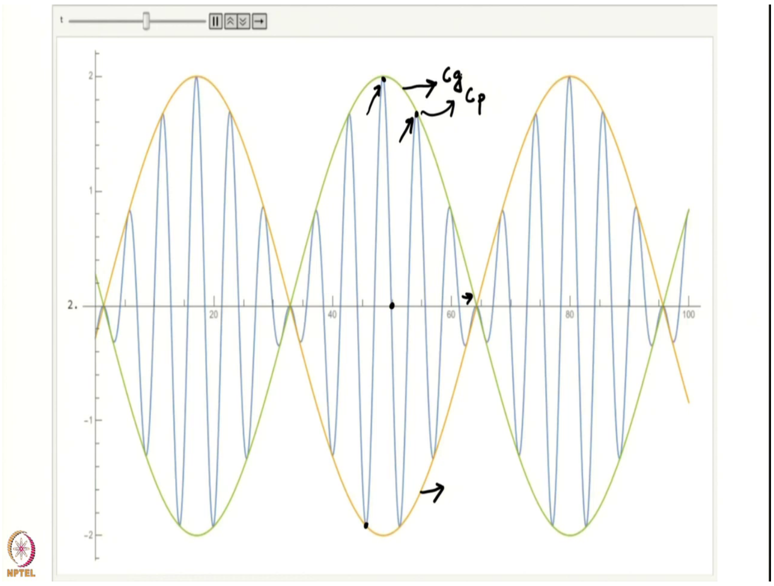

So, you can clearly see that there is an envelope of the wave ok. Now, within the envelope if you look at the blue curves you will see that the blue curves are not stationary they are also moving and moving within the envelope. So, you can see that the blue curves. If you follow the envelope the green curve and the yellow curve you will see that the green curve and the yellow curve the envelope is actually moving slower than the blue curves ok.

So, this video is looping. So, you can see a temporary pause, but you can see if you focus on any one of the crests on the blue curves you can see that the crest actually crosses the envelope and then goes past it. So, you can clearly see that the individual phases which are indicated in blue. So, you can follow let us say a 0 or a maxima or a minima in blue.

If you follow them you will see that they are moving faster than the envelope itself. This is because in this case we have used the dispersion relation of surface gravity waves. In this case the blue curves a given phase is moving with a phase velocity which is square root g by k whereas, the envelope itself is moving with the group velocity which is delta omega by delta k.

In this case because there are only two, so, it will be delta omega by delta k when we have many such wave numbers and there is a continuous superposition of them, in that limit it will become d omega by d k. One can work out d omega by d k and let us do that. So, in this case let us calculate the group velocity for surface gravity waves.

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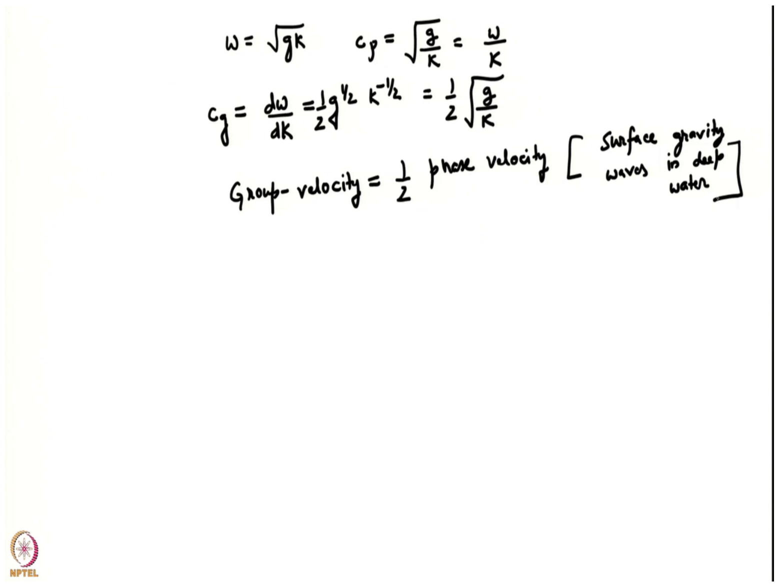

So, we have omega is equal to square root g k. So, the phase velocity is square root g by k which is basically just omega by k. The group velocity here is d omega by d k and this is g to the power half, half factor of half k to the power minus half or g by k. Now, we clearly see that the group velocity is equal to half of phase velocity. This is true for surface gravity waves in deep water.

Consequently, we now understand why we were seeing what we were seeing. So, you can see this is the picture this is a snapshot of what we were seeing and so, I said that if you focus on one such point here pause the video and go back and play the movie again you will see that if you focus on one such point here or one such point here the place where the blue curve goes to 0 or where it has a maxima or a minima.

You will see that the speed with which the envelope moves. So, this is the envelope and this is the envelope. So, you will see that there is a certain speed with which the envelope moves and there is a certain speed with which these points move. You will see that these points come from the left overtake the envelope and go through the point at which displacement is 0.

So, you will see that these points actually travel with the phase speed while the envelope actually travels with the group speed. In this case the phase speed is twice the group speed. Hence you will see the phases overtaking the envelope. Now, this is the interpretation of group velocity. One has to remember in this particular case only is the group velocity half of the phase velocity.

Later when we will look at capillary waves we will see that this is not true. Let us now try to understand how do we understand this group velocity in terms of the Cauchy-Poisson problem. We will see that when we solve the Cauchy-Poisson problem for one particular initial condition and when we try to physically understand the solution this interpretation of group velocity will be very useful.

So, let us go back and solve the Cauchy-Poisson problem for a particular set of initial conditions. So, we will return to the Cauchy-Poisson problem and now we want to solve this problem for initial conditions which are different from what we have done so far. So, far we have only taken a single mode either of single Fourier mode or a single Bessel mode and we have recovered results that we had already recovered earlier.

Now, we want to look at more complicated initial conditions. In particular we will look at an idealized initial condition where my interface is displaced as a delta function. Why do we take as a delta function? Because, recall that the delta function is like a point forcing.

So, if we understand the solution for a delta function initial condition in some sense we have understood the problem or the solution to the problem for more for a more complicated class of initial conditions where you may have we may have localized displacements of the interface. So, let us solve it for a delta function initial condition and once we have the solution to this initial condition we will see that the solution the in terms of group velocity that we have just discussed.

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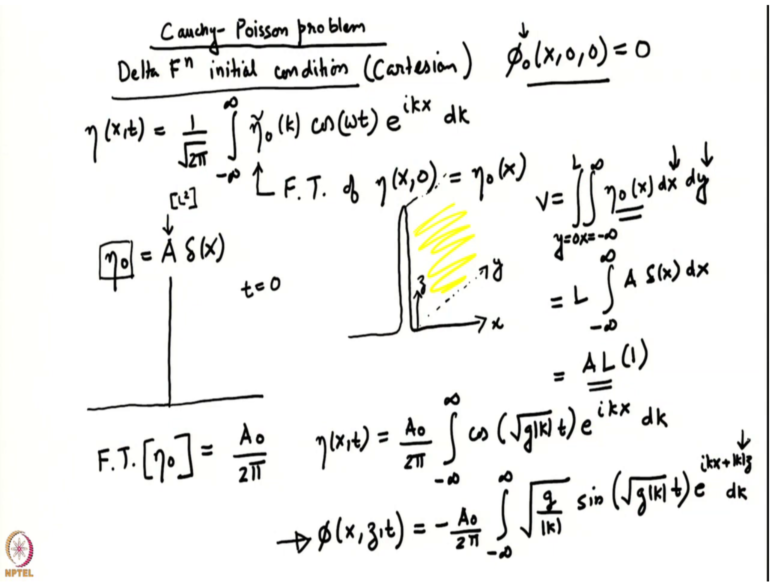

So, let us take a delta function initial condition. So, recall that we had written or solution from the for the we are doing the Cartesian case and we will assume that initial impulse is 0.

So, phi 0 of x, 0, 0 is 0. So, only a surface displacement and the surface displacement will be chosen to be a delta function at x is equal to 0.

So, let us write down for phi naught equal to 0. Let us write down the solution to the Cauchy-Poisson problem. There is only one integral now, the second term goes to 0. So, this is the solution for how the interface evolves as a function of time and as we discussed before this is the Fourier transform of eta of 0 which we have called as eta 0 as a function of x.

In particular we will choose eta 0 to be some A into delta of x. What is the physical meaning of this? We are displacing the interface like a delta function at t equal to 0. A delta function has 0 width, but it is infinitely tall. So, at t equal to 0 it is as if there is an infinitely tall column of liquid. You can see that this has this quantity A has the dimensions of length squared. Why should it be so? Delta function this is a delta function in x this is a one dimensional delta function.

So, delta function has the units of inverse of its argument. So, the dimension of the delta function is 1 by distance. Eta 0 should have units of distance. So, this A must be L square. How do we physically understand this? This implies an initial condition. You can think of an initial condition where we are taking a unit volume of fluid. So, this is a 2D thing.

So, you can imagine that we are putting a very tall column of fluid and you can extend it in the third direction which we are not solving for. So, this is my x that is my z and this is my y. So, we can extend it in the third direction and we are essentially saying that if you integrate it over say if you integrate eta 0. So, over x and over you can see that this has the dimensions of volume this is has the units of length this is length this is length.

So, you can see that with the there is no y dependency. So, this is just L and then we have minus infinity to infinity A delta of x dx and this is A into L and this integral is just 1. So, this just tells us this A is related to the area and this is telling us if we set A to 1 and if we set L to 1, this is equivalent to initializing a column of liquid which is infinitely tall and whose volume is unity. So, that is the physical interpretation of the initial condition.

Now, this initial condition has been chosen because this allows us to solve the integrals analytically. In particular these initial conditions are also very important. The delta function initial conditions if you recall from your course on ordinary differential equations recall that the solution to a equation with a delta function on the right hand side is the fundamental solution to the equation.

So, if you know the solution to that equation you can solve for more complicated right hand sides. So, this is the basic reason why we are doing it for delta function. So, with that let us proceed. So, we will require the Fourier transform of eta 0. So, the Fourier transform of eta 0 is just A 0 by 2 pi. This is essentially saying that the Fourier transform of a delta function is a constant or in other words if you have your interface being excited like a delta function at time t equal to 0.

In Fourier space we are exciting every possible wave number. So, that is why the Fourier transform is a constant. It basically tells us that every wave number in the spectrum is getting excited. Now, this is what we had solved the Cauchy-Poisson problem for. We wanted to solve for a spectrum of waves not just one wave. So, this delta function is exciting a spectrum of waves.

Now, because this is a linear problem, so, each wave will evolve independently in time and what we are going to see at any later time is going to be the linear superposition. So, the interface at later time is just a linear superposition of those waves each travelling with its own phase speed. This is a dispersive medium. So, each wave has its own phase speed given by square root g by k.

Now, let us solve this problem analytically and let us try to understand, what is the physical meaning of this solution. So, we will have eta of x comma t is A 0 by 2 pi. Similarly, there is a solution for phi of x comma z comma t that will also have only one term because the initial condition of impulse is 0. So, the phi solution is just minus A 0 by 2 pi. We have already written these solutions earlier. All I am doing is going back to the solutions and substituting the initial conditions.

Now, in this case it turns out that it is easier to solve the phi integral and from there infer the value of eta using the boundary conditions. We will do that. That has got to do with the fact that there is an exponential factor here. Recall that z is negative in this integral because z is going from 0 to minus infinity. So, the convergence of these integrals is easier to establish for the second integral.

So, we will start with the second integral and we will work out how does the velocity potential evolve in time and from there once we know phi we can infer eta using one of the boundary conditions.

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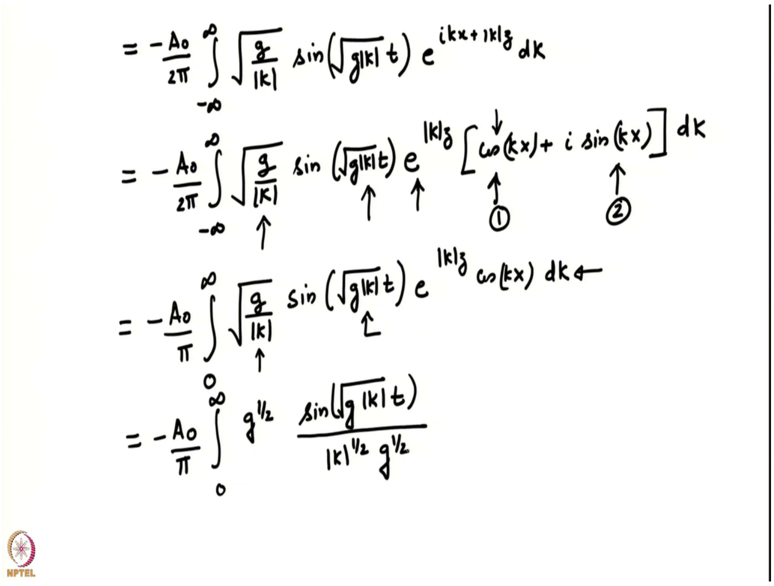

So, I am just going to rewrite the integrals. We had minus A 0 by 2 pi minus infinity to infinity. This is our expression for phi. Now, see observe that in the region between minus infinity to 0, k is negative, the limits of integral have a negative side also. Now, in that side you can clearly see that this part of the function you can clearly see that I can write this as sin of the same thing into e to the power mod k z into cos k x plus i sin k x into d k. Now, this sin has a g with a mod k.

Now, you can clearly see that I can split this into two integrals; one with the cos k x and another with the sin k x. Notice that all the terms other than these cos k x and sin k x are even or in other words if I replace k by minus k they give the same answer, the functions remain the same. This does not change because this has a mod k inside. So, although sin is an odd function, what is inside the argument is modulus of k.

So, making k to minus k does not change this. This anyway also has a mod k and there is a mod k here. So, you can see immediately that this part the first integral is a even integral and the second integral is an odd integral. So, over the limit it just goes to 0. So, I can write this as just the cos part the even part and I will make this twice the even part. So, I will get rid of a pi here.

We are basically saying that the first integral and the second integral, the second integral the integrand is odd. So, because of that the second integral goes to 0. The first integral is even that is because this is the only one function which does not have a mod inside it and cos of minus x is equal to cos of x.

So, I can just do the integration from 0 to infinity and make it 2 times to account for the minus infinity to 0 part and the second integral is anyway 0. So, this is how we obtain this dependence. Now, once we have obtained this we can write this as. So, what I want to do is I want to express this quantity as sin x by x. Why? Because sin x by x I can write it as a Taylor expansion. There is already a factor of k here and sin also has an argument in k and the integration is in k.

So, it is good to write it as sin x by x and then do a Taylor series expansion about z is about x equal to 0 and then integrate term by term, this is what we are going to do. So, we will have a g to the power half sin g mod k square root t divided by I already have a mod k to the power half. I would like to add a I would like to multiply with the g to the power half that will multiply with g at the top. So, it will just make it g at the top.

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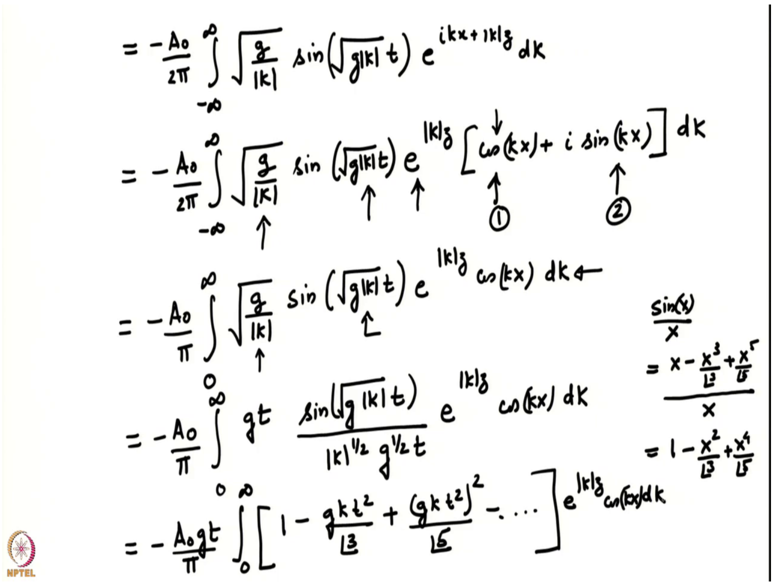

And then I will put a factor of t here and that will bring in a factor of t there. So, this stays the same. So, now, we will have to do a Taylor series expansion. I can pull the g and t out because the integration is in k. So, sin x by x if we just expand sin x has x minus x cube by factorial 3 plus x five by factorial 5 and so on and then there is an x. So, sin x by x starts from 1 and then there is x square by factorial 3 and x 4 by factorial 5 and so on. I will do the same here and we will obtain 1 minus x square.

So, we will obtain g k t square by factorial 3 plus x 4. So, g k t square whole square divided by factorial 5 minus dot dot dot into e to the power mod k z cos k x d k. We now need to integrate this term by term.