Introduction to interfacial waves
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Lecture - 36
Cauchy-Poisson problem in cylindrical coordinates (contd..)

We were looking at solving the problem of deep-water surface gravity waves in a cylindrical axisymmetric geometry. Our geometry was unbounded radially as well as depth wise and so, we were looking at the propagation of circular waves which would propagate outwards.

Firstly, we solve for simple initial conditions where our initial condition was the surface was perturbed in the form of a Bessel function. We found that the Bessel function plays the role of the Fourier mode cos k x and sin k x in Cartesian geometry and we solved it for the simple initial conditions we found the following solution. This solution was very analogous to the solution that we had found earlier in Cartesian geometry.

Now, let us build the analogous solution to the Cauchy-Poisson problem. Recall that in the Cauchy-Poisson problem in Cartesian geometry, we had solved the problem for arbitrary initial conditions arbitrary Fourier transformable initial conditions and we had expressed the answer as an integral a Fourier integral over those initial conditions.

We will do the same thing here. For that we require the equivalent of Fourier transform in a cylindrical geometry that transform turns out to be the Hankel transform. I had introduce the transform in the last video.

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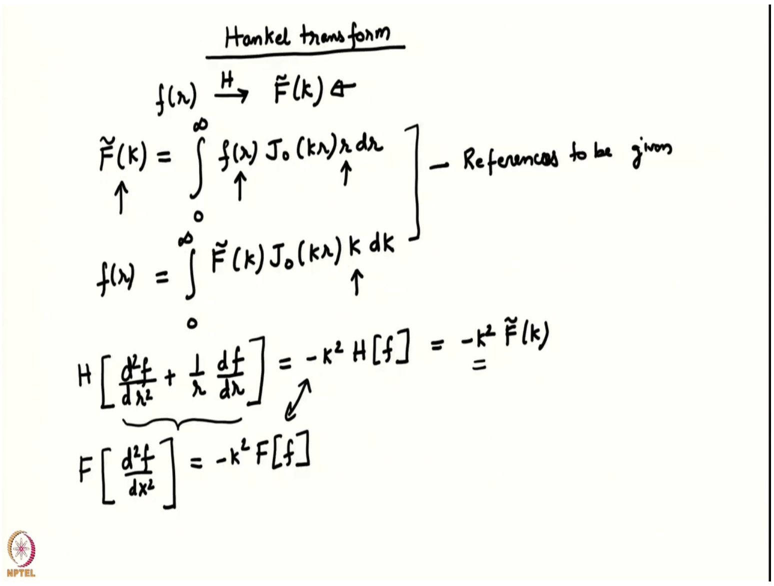

This is the structure of the transform and now, let us understand what are the properties of the transform. So, the most useful property of the transform is the following. So, if H represents the Hankel transform. So, H of d square by d square f by d r square plus 1 by r df by dr has the property that this is equal to minus k square H of the function f, this is equal to minus k square.

If we call the Hankel transform of F as f tilde of k the way we had indicated here, then this is telling us that the Hankel transform of the part of the; this is you can notice that where did I get this operator from. This is the part of the Laplacian operator which has derivatives with respect to r only.

So, the Hankel transform operates on this part of the Laplacian operator and produces minus k square into F tilde of k. You can notice the analogy with the Fourier transform. The Fourier transform in Cartesian coordinates would have taken the Laplacian operator there the horizontal direction was x.

So, the corresponding operator would be d square by d x square and this would have given us an i k whole square that would be minus k square into the Fourier transform of f. Notice the analogy between these two formulas. So, you can see that the Hankel transform will do in a cylindrical axisymmetric geometry what a Fourier transform will do in a Cartesian geometry.

So, this is the reason why we are doing the Hankel transform on this geometry and we will recover very very identical results compared to what we recovered earlier. The only difference is that, that right now the solution to the Cauchy-Poisson problem inside the will still be represented as an integral. The quantity inside the integral in the earlier case was e to the power i k x, here it will be J 0 of k r; it just represents the difference in basis functions ok.

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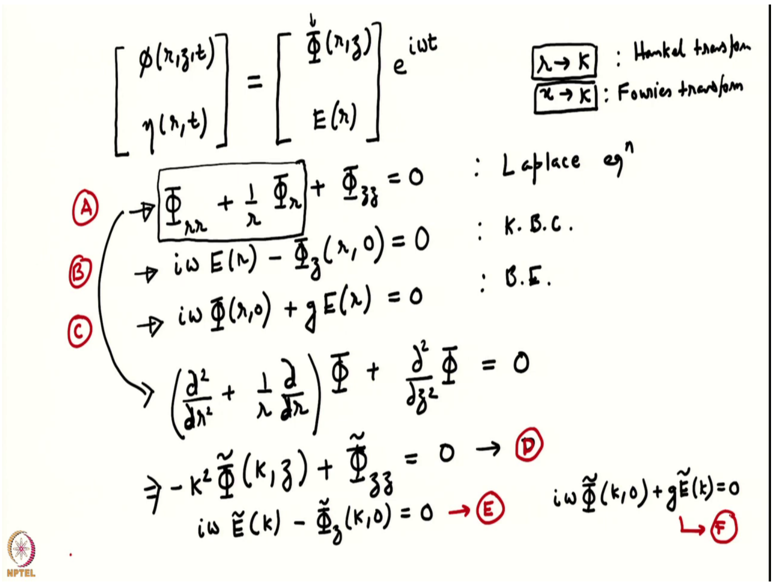

So, with that background let us proceed further. We will write like before, we will do a normal mode analysis combined with the Hankel transform. So, phi of r, z, t it is axisymmetric. So, there is no theta dependence and eta of r and t, eta by definition does not depend on the vertical coordinate is some eigen function capital phi which is a function only of the space variables eta.

So, I will call this E like before, but now E is a function only of r into the normal mode part, we are looking at oscillatory solutions about the quiescent base state. Once again remember that the base state is exactly the same as before, fluid is quiescent and pressure is hydrostatic.

We can derive the expressions for the pressure later in a very analogous manner to what we have done earlier. Now, with this approximation or with this assumption let us plug it into the equations. Our governing equation is the Laplace equation written in cylindrical axisymmetric coordinates.

If we do that, then this E to the power i omega t does not affect the Laplace equation, because there is no time derivative in the Laplace equation. So, it just converts it into an equation for this capital phi. I am going to indicate derivatives with subscript. So, I will get phi rr plus 1 by r phi r plus phi zz is equal to 0, this is my Laplace equation for the eigen function. Now, the boundary conditions; the boundary conditions retain the same structure as before.

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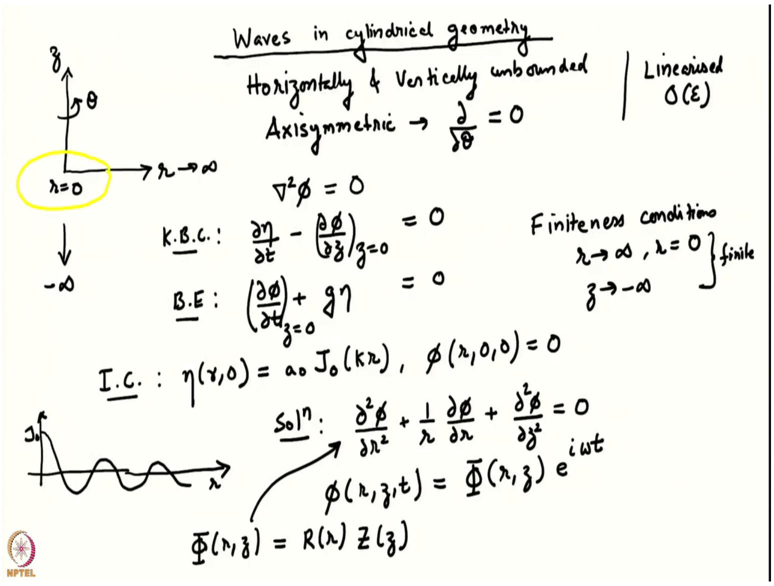

So, we just; if we plug these into the boundary conditions which we have written already earlier the boundary conditions that are indicated here as K B C and Bernoulli equation. So, we have a form for eta we have a form for phi if we just plug it, we obtain the following equations.

This is evaluated at the undisturbed location. The undisturbed interface is at z is equal to 0. We have placed our origin there is equal to 0, this is the kinematic boundary condition and then we have i omega phi once again at z is equal to 0 plus g of E, E is a function of r is equal to 0 and this is the Bernal equation both are linearized and hence apply at the undisturbed interface.

So, now this is our governing equation and those two are our boundary conditions. Like before we will be solving the Laplace equation using the Hankel transform. You can see that the Hankel transform is really going to simplify this part of the operator, why? Because as I have already told you the Hankel transform operates on the operator d square; d square by d r square plus 1 by r d by d r and it just converts it into minus k square times the Hankel transform of the function f.

So, you can see that this form of the operator appears here. So, we have see, if I this is d square by dr square plus 1 by r d by dr operating on capital phi plus del square by del z square these are also del, but I am just writing them as if they are d. So, you can convert this into del if you want and if I do a Hankel transform on this then this part the first part of the operator really simplifies. So, it just converts it into minus k square into the Hankel transform.

So, this also operates on phi. So, into the Hankel transform of phi which I will call phi tilde and so, this is equal to 0. So, this is a function now of k and z. The r has got replaced by k, just as in the Fourier transform x had got replaced by k. In the Fourier transform x had got replaced by k, here in the Hankel transform r is getting replaced by k, r plays the role of x, x was horizontal distance in the Cartesian case, r is the horizontal coordinate in the axisymmetric cylindrical case.

So, with this and then here for the second term we do the same trick that, because the Hankel transform involves an integral over r and r and z are independent. So, I can do the differentiation later and the integral first. So, I am exchanging the order of integration and differentiation and with that I can write this as phi tilde of zz is equal to 0.

The second derivative of the Hankel transform this is what I obtained. We had obtained a similar equation in the Cartesian case also. So, this is our equation, this is what our Laplace equation becomes. So, I will call like before I am going to call this A, B and C these are equation numbers. So, this I will call it equation D this is the Laplace equation written in the k space.

Similarly, we can take equations B and C which are basically boundary conditions, apply the Hankel transform on them just as we had done earlier and convert these into equations written in the k space. These are I will just write them here; i omega E tilde of k minus phi tilde of z at k comma 0 is equal to 0.

This is equation E and I can get a similar equation F from equation C. So, let me write it here. It is i omega phi tilde of k comma 0 plus g E tilde of k is equal to 0 and this is equation F. So, what I have done is I have taken equation A, B and C. The Hankel transform of A leads me to equation D, the Hankel transform of B leads to equation E and the Hankel transform of C leads to equation F.

So, I have three equations still. Now, written in k space, we will solve them in exactly the same manner. So, you can see that equation D can be treated as if it is an ordinary differential equation in z while writing the solution this the coefficients will not be constants, but they will be functions of the variable which is there in the form of k or in other words, it will be a function of k.

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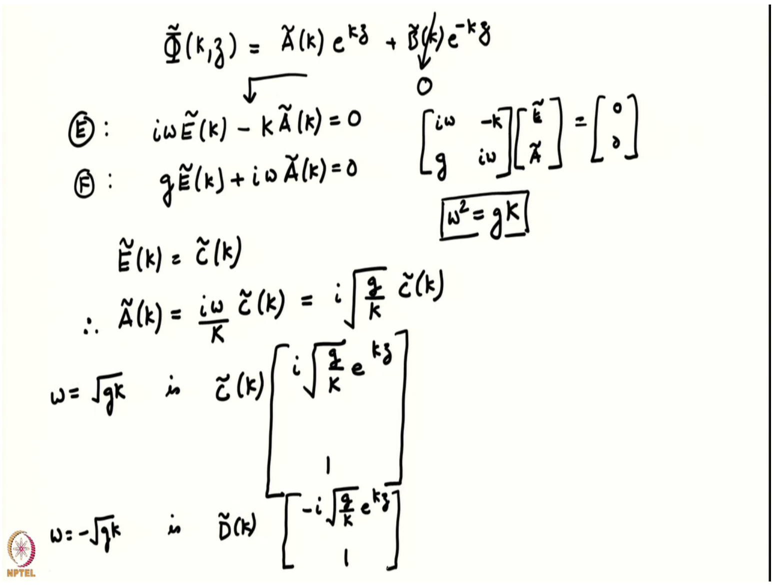

So, like before we will have phi tilde of k comma z is some A tilde of k e to the power k z plus B tilde of k e to the power minus k z. Now here, we really do not have to put a mod because recall that the Hankel integral in the Hankel transform goes from 0 to infinity. So, there are no negative values of k here. So, we do not have to worry about negative values of k. Unlike, we had to do when in the Fourier transform the range of integration is from minus infinity to plus infinity.

So, we had to put explicitly a mod there. So, here we do not have to put a modulus, k is always greater than 0. So, you can see that this is going to e to the power minus k z is going to diverge. So, I am going to set this to 0 and so, I am I have got this solution to the Laplace equation in k space. Now, let us write down the boundary conditions. So, boundary condition e leads to i omega E tilde of k minus phi z.

So, this will just give me k A tilde of k is equal to 0, this is from equation E. We have written equation E in the previous slide. Similarly, we can take equation F, substitute the form for phi tilde and get another equation. F gives us g E tilde of k plus i omega A tilde of k is equal to 0. Once again two equations in two unknowns, homogeneous, the determinant has to be 0. If you set the coefficient matrix you will find it to be i omega minus k g and i omega and this is E tilde and A tilde is equal to 0 0.

So, this tells you. So, that we again recover the old dispersion relation in deep water omega square is equal to g k. So, this is our dispersion relation like before. Now, we have to do exactly the same process. So, once again I am going to set E k E tilde of k is equal to some constant C tilde of k ok. Therefore, using this equation I can express A tilde of k as i omega by k C tilde of k and omega itself is square root g k. So, this becomes it is actually plus minus square root g k.

So, this becomes g by k C tilde of k. Once again we have two eigen modes. for omega is equal to square root g k, the eigen mode is C tilde of k i square root g by k e to the power kz and 1. and for omega is equal to minus square root g k, it is some other constant or other function and it is just minus of the above or the first term is minus the second term is still 1, we have done this before. So, I am not going into the details.

If you have any confusion, please look into how we did it in the Cartesian case. It is completely analogous to that process. And so, the final answer has to be once again written as a linear superposition of over the eigen functions, we can write it first in k space and then invert. When we invert there will be a in inversion of the Hankel integral ok. So, there will be an integral which will appear. I am going to write down the answer.

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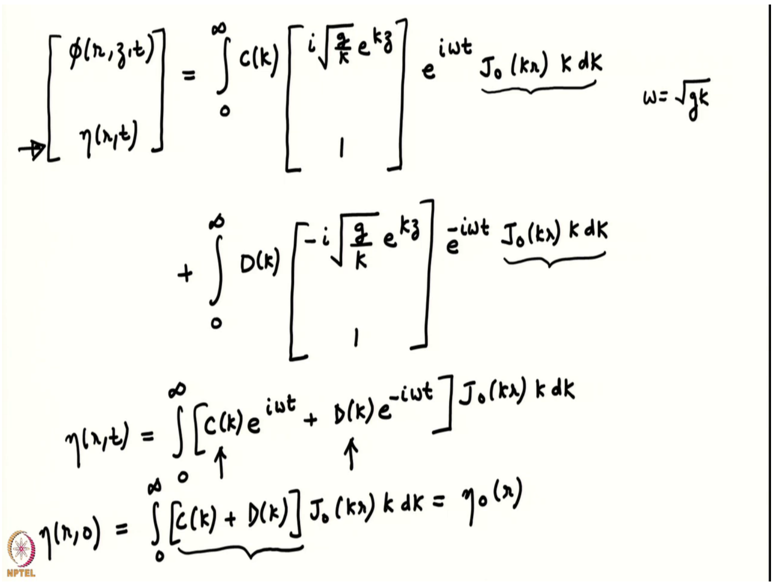

So, we will have in real space in r space rather is equal to like before two integrals one over C of k into the first eigen mode i e to the power i omega t J 0 of k r k d k plus integration 0 to infinity D of k it is just minus of this minus i omega t because the second part is minus square root g k, so minus omega J 0 of k r into k dk.

So, you can see that like before we had earlier in Cartesian we had interpreted one of them as a left traveling wave and the other one as a right traveling wave ok. Here also we have a similar structure and omega is square root gk. So, omega stays inside the integral. So, essentially this is the only part which is different in the two cases.

There it was e to the power i k x d k and here, it is J 0 of kr into k into dk. Now, like before we can take these expressions. So, if I take the second row of this expression then we obtain eta of r comma t. So, I am taking the second row of this matrix, this is a equation written in matrix form I am taking the second row.

So, I have to take the second row on the right and second row on the left is equal to integration 0 to infinity C of k e to the power i omega t plus D of k e to the power minus i omega t into J 0 of kr into k into dk. Let us relate. So, like before we will have initial conditions. So, I expect C k and D k to be related to the Hankel transform of the initial conditions.

So, let us do that. So, eta of r comma 0 is basically just infinity C of k plus D of k into J 0 of k r k d k is equal to and eta of r comma 0 we had called it some eta 0 of r let us say it is clear from this structure that this is the Hankel transform of eta 0 of r we had used the same argument in the earlier case also ok.

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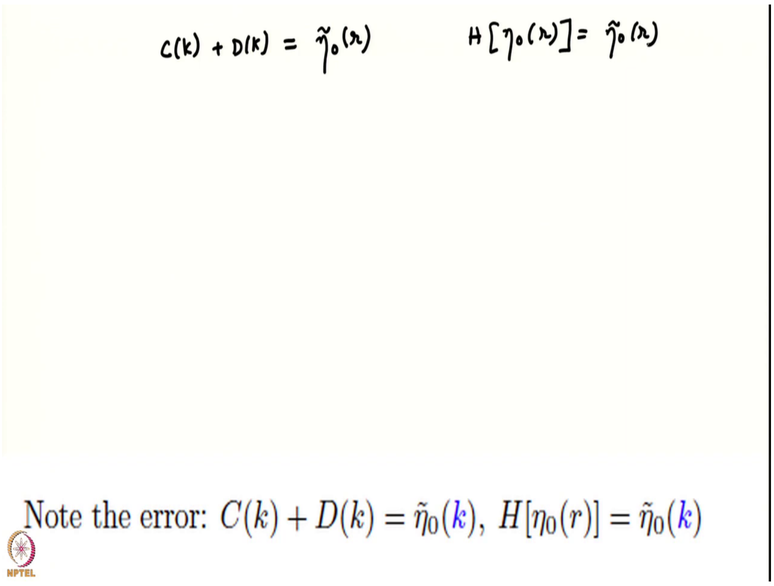

So, we can conclude from here that C k plus D k is basically just the Hankel transform of eta 0 of r we will call it eta 0 tilde of r. So, eta 0 tilde of r is Hankel transform of eta 0 of r. Similarly, you can write the expressions for phi from the first row in the matrix put time t equal to 0.

And you will find that some C k minus D k into some factor is the related to the Hankel transform for the initial conditions for phi at z is equal to 0 at time t equal to 0 ok. We since we have done it already once before. I am going to write down the answer straight away.

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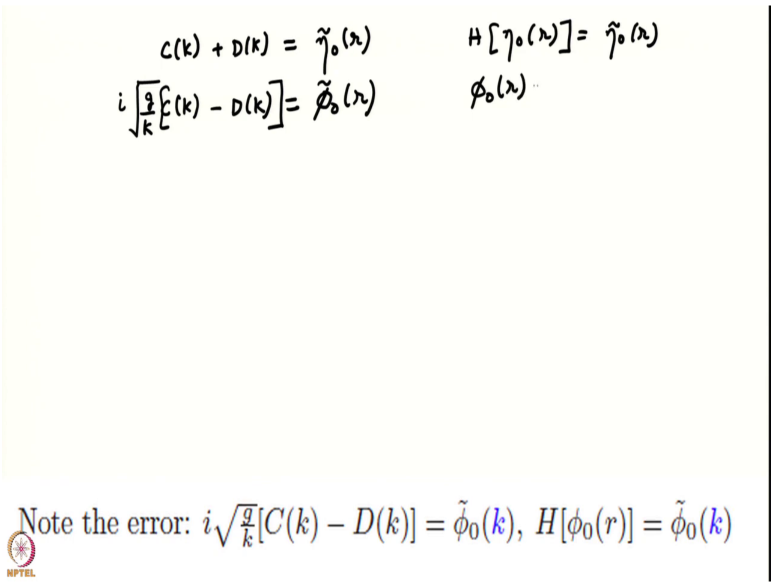

So, we will find that C k minus D k yeah. So, I have missed a tilde here, but I think it is clear. So, I am not putting an explicitly a tilde everywhere, but if you see there was a tilde in my original expressions, but it is understood that these are in the k space ok. So, I am not writing all the tildes. So, its C and D is equal to. So, there is a pre-factor square root g by k to this is equal to the corresponding Hankel transform on the phi initial condition. So, what is phi naught phi naught is a function of r.

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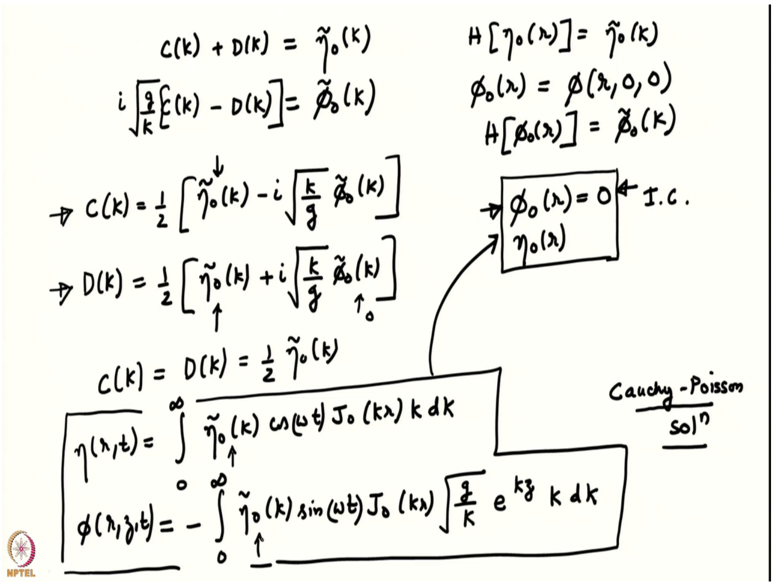

And this is basically the value of phi at r at z is equal to 0 at time t equal to 0 and the Hankel transform of phi naught of r is defined as phi naught tilde of and this is not r this is a mistake. So, these are all functions of k ok. So, once again we can eliminate C and D in favor of eta 0 tilde and phi 0 tilde ok and if we write the answer, then we will find that C of k is half eta 0 tilde of k minus i square root k by g phi 0 tilde of k.

You can note that the expressions are completely the same as what we had got in Cartesian geometry and that is because we have found the equivalent of the Fourier transform here. The Hankel transform completely converts it as if it is a Cartesian geometry problem ok. So, D of k is half.

Now, we are we can go back and in our final expressions. So, we have these expressions for phi and eta. So, the first row gives us the expression for phi and the second row is expression for eta. In these expressions we can replace the C's and D's with the expressions that we have just obtained in terms of eta 0 and phi 0 tilde and get analogous expressions like what we got earlier we are going to do a simpler exercise we are going to just take initial conditions such that phi 0 of r is 0 and eta 0 of r is something.

So, it is some localized initial condition some perturbation of the surface. So, we are going to solve it for these initial conditions and write the answer for these initial conditions. These are simple, because this quantity being 0 eliminates one of the integrals ok. So, we can readily see that if we substitute it in the expressions for c of k and d of k because phi 0 of r is 0 the corresponding Hankel transform is also 0.

So, for these initial conditions C of k and D of k become equal to each other and they just become half eta 0 tilde of k it is just the first term in the respective expressions because the second term is 0 because this is 0. So, the Hankel transform is also 0 if we do that and if we replace the expressions that we wrote earlier in matrix form into as an equation for phi and as an expression for eta then we will obtain eta of r comma t is equal to 0 to infinity eta 0 tilde of k cos omega t J 0 of kr k dk.

Similarly, phi of r comma z comma t is minus eta 0 tilde of k sin omega t J 0 of kr square root g by k e to the power kz k dk. So, these are the solutions to the Cauchy-Poisson problem for those kind of initial conditions. You can see that we have chosen simple initial conditions where only the surface is perturbed the surface perturbation is kept arbitrary.

But I am not imposing a corresponding phi perturbation one could have in principle of course, written down the taken a nonzero phi 0 of r and then we would have another term in the integral ok; however, let us just look at this form and let us recover the results that we have already recovered earlier.

So, in particular before we solve the Cauchy-Poisson problem using Hankel transform we had recovered this result that if we just put a single Bessel mode with 0 velocity potential perturbation then it leads to this solution. Now, we have the solution to the Cauchy-Poisson problem which is for arbitrary surface perturbations. So, in particular we could choose eta 0 of r to be a naught J 0 of some k 0 of r.

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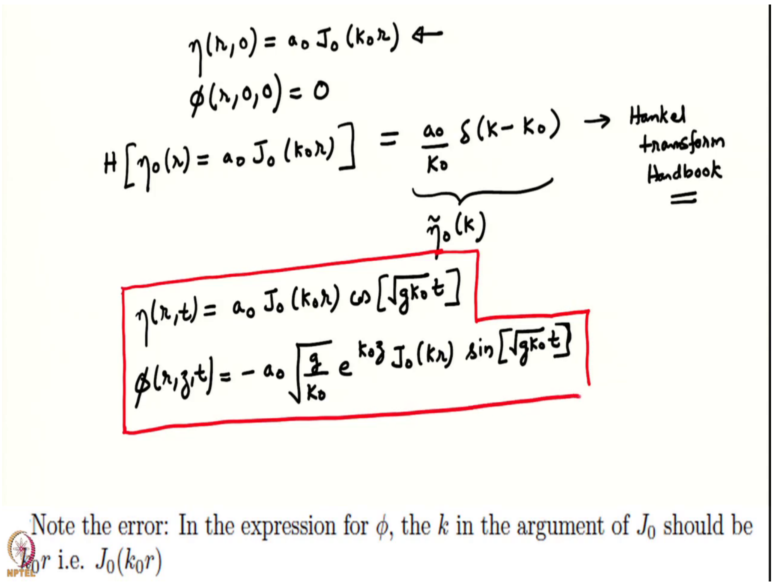

So, let us do that. So, we could choose initial conditions eta of r comma 0 to be some a naught J 0 and I am putting k naught here now, because k is one of the k is the variable of integration in Hankel transform. So, we do not want to use the same variable and the velocity potential perturbation is 0. So, at z equal to 0 at 0 is 0.

Now, this is one initial condition and I can plug this into the Cauchy-Poisson solution. So, this is the Cauchy-Poison solution written as an integral and we are going to what these integrals need is just the Hankel transform of the initial conditions once we plug in the Hankel transform then we have to think how to do these integrals, ok.

So, the input to these integrals is the Hankel transform of the initial conditions how do we take the Hankel transform of this object. Once again like before when we had put a single cosine mode of k 0 into x and it turned out to be a delta function. Here, also the Hankel transform of the Hankel transform of eta 0 of r which is equal to in this case J 0 k 0 of r is given by a 0 by k 0 a 0 is just a constant. So, it comes out of the thing and then there is a 1 by k 0 into a delta function k minus k naught.

There is only one delta function this you can look it up in any Hankel transform handbook. Once we know this what do we do this is basically what did we find we just found what is eta 0 tilde of k this is our form for eta 0 tilde of k you can go back and substitute it into the two expressions that I have put here.

So, you can substitute it here in this eta 0 tilde of k and here ok we are essentially asking the question that if we put if we deform the interface in the shape of a Bessel function with 0 velocity everywhere how is the interface going to move. We already know the answer the interface is going to move like a standing wave wherever the Bessel function was 0 they will remain 0 and the interface is going to go up and down like a standing wave with frequency square root g k t.

And the velocity field will be determined by the corresponding velocity potential if you plug in this expression for eta 0 tilde of k into those integrals and use the shifting property of the delta function. Essentially, it will just evaluate the integrants at k is equal to k naught it is very easy to show that you will recover the same expressions that we obtained earlier.

So, you will recover eta of r comma t would be just a 0 J 0 of k 0 into r cos square root and phi. Similarly, you have to go back to the expression for phi and plug in the delta function for eta 0 tilde if you do that then you will recover r z t is a minus as expected square root g by k naught e to the power k naught z J 0 of k r and a sin, because this was 0 at time t equal to 0.

So, there comes our standing wave solution. So, once again we have verified that the Cauchy-Poisson problem recovers results which we would have expected otherwise. We now have to think about how to solve these integrals for more complicated initial conditions, we will look at that shortly.

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