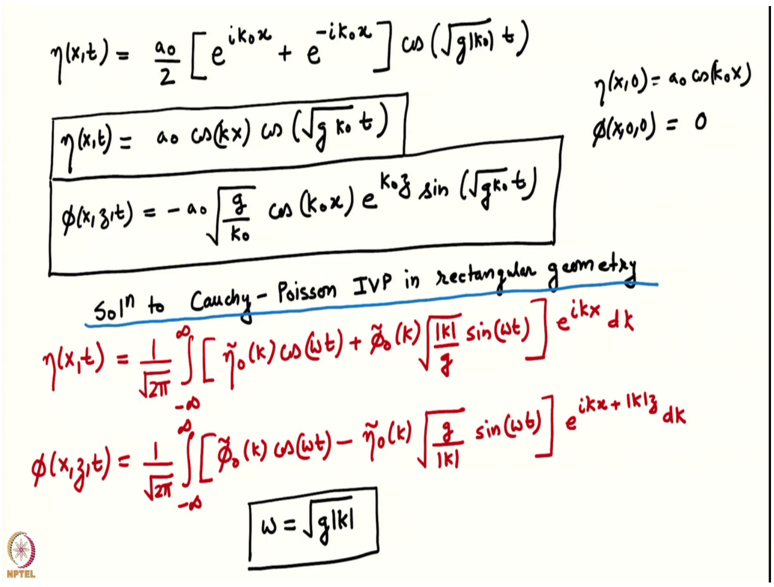
Introduction to interfacial waves
Prof. Ratul Dasgupta
Department of Chemical Engineering
Indian Institute of Technology, Bombay

Lecture - 35
Cauchy-Poisson problem in cylindrical coordinates

We had obtained the solution to the Cauchy-Poisson problem for deep water waves, surface gravity waves in rectangular geometry. The final answer was expressible as a Fourier integral. A natural question arises how does one evaluate these integrals particularly for more complicated initial conditions? We will look at some examples later on.

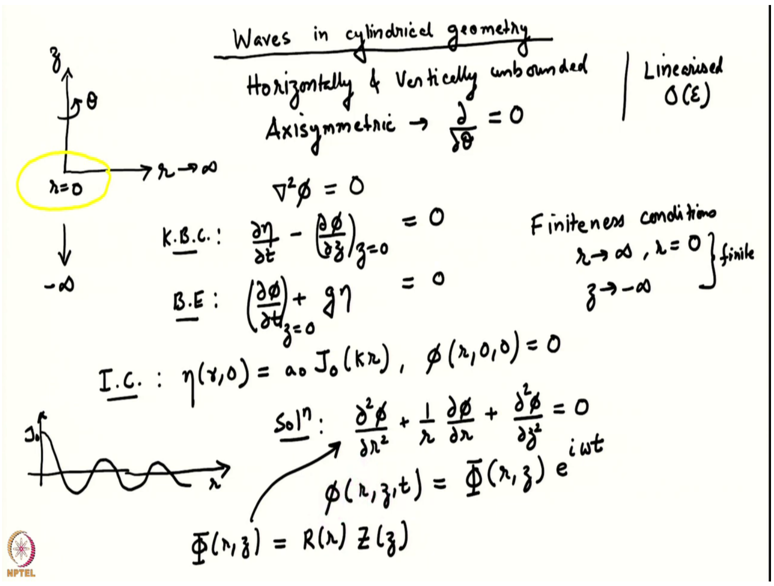
(Refer Slide Time: 00:35)



But let us now keep the answers right now in integral form. We will also see that one can also approximate these integrals in the limit of t going to infinity using a particular kind of a mathematical technique.

Now, before we do this let us extend the solution to the Cauchy-Poisson problem for waves in a circular geometry or rather waves in a cylindrical geometry. So, you can think of throwing a stone in a pool.

(Refer Slide Time: 00:57)



So, you can see that in when we throw a stone in a pool, waves come out in a circular manner. And so that problem is naturally defined in a cylindrical coordinate system where the coordinate r represents the radial coordinate. There is a center of the pool and we will assume that the pool is horizontally unbounded like before.

So, horizontally and vertically unbounded like before. This implies that in this r z coordinate system. So, we are also going to assume that the waves are axisymmetric which implies that all del by del theta is equal to 0.

So, this is my coordinate system as I have indicated on the left and the axis of symmetry is the vertical axis, the z axis and there is no variation if I go around the z axis. So, this becomes a planar problem, but we have to remember that we have to visualize it in three dimensions and we have to use a cylindrical coordinate system.

Once again my depth of my pool is infinite and in the radial direction, it goes up to infinity. So, we ignore the presence of any confining boundaries. So, you can think of a very large pond where the boundaries are very far away and we throw some kind of a stone or something and then it causes ripples. And we want to understand how these ripples spread out.

This problem is also known as the Cauchy-Poisson problem. Cauchy-Poisson has provided the solution to the problem, both in cylindrical as well as Cartesian geometry. We have already done the Cartesian geometry. Let us understand how we do the cylindrical geometry probably.

So, first we will do it for a simple case, where there is only one mode present in the system. Just as we had done it earlier. We will do it, first for the simple case and then we will make the initial conditions more complicated. So our governing equations like before and I am straight away going to do a linearized analysis. So, linearized, so everything is at order epsilon only. We know what are the equations at order epsilon; there is a linearized kinematic boundary condition, there is a linearized Bernoulli equation and there is a Laplace equation.

So, we just have to write the Laplace equation in cylindrical axisymmetric coordinates. Earlier we had written it in Cartesian coordinates and now we will write it in cylindrical axisymmetric coordinates. And the form of the kinematic boundary condition and the Bernoulli equation remains the same.

So, del eta by del t minus del phi by del z and z is equal to 0 is 0. This is my familiar kinematic boundary condition; linearized. Then the linearized Bernoulli equation is del phi by del t plus g eta is equal to 0 and this is evaluated at z is equal to 0. This is obtained once again by saying that the gas above which in this case could be air exerts negligible pressure. So, the pressure is 0. We put pressure equal to 0 and ignore the non-linear terms.

And then, of course we have finiteness conditions because our domain is unbounded. So, we have to make sure that as r goes to infinity, in this case we will also have to worry about r equal to 0. So, and z goes to minus infinity, everything is finite. So, we do not want things to diverge that will eliminate some of the constants of integration, ok.

So, now let us start with and we would like to also maybe specify some initial conditions. So, now let us specify the initial condition as a 0, because this is cylindrical geometry you will see that the Bessel function which we have encountered before when we studied vibrations small amplitude vibrations of a circular membrane clamped at the boundaries, we have made the Bessel function before. So, I am going to specify this as a Bessel function.

You will see that the Bessel function here plays the same role as cos and sin plays in a rectangular geometry, ok. So, I will keep the initial conditions very simple. Just deform the surface in the form of a Bessel function and we do not give any initial impulse at time t equal to 0.

This is the equivalent of what we had done earlier for Cartesian coordinates. There I had deformed the surface as a cos k 0 x and phi was 0 at the surface. This was the initial condition for which we solved. Here the eta is defined in the shape of a Bessel function j 0. It is j 0 because it is axisymmetric, ok.

So, if you want to recap, go back and look at your, the video where we discussed the Bessel function. The Bessel function will is an oscillatory function and it decays very slowly with distance, ok. So, j 0. And its argument is non-dimensional. So, k is like a wave number and it has the dimensions of inverse length.

So, now let us solve this problem first before we hit the Cauchy-Poisson problem where our initial condition will be arbitrary. So, we will have to specify some arbitrary function of radius, ok and that will represent the pebble in the pond problem. But right now let us take a slightly artificial initial condition as a single Bessel mode of a given wave number k.

So, how do we solve this? So, once again let us go back. So, solution so this is r. So, solution, so let us write the Laplace equation in cylindrical axisymmetric coordinates. Once again, we are going to do a normal mode analysis. So, r z and t and I will write it as some eigen mode like before which is a function of the spatial variables only into e to the power i omega t. I am looking for standing wave kind of solutions.

Now like the Cartesian coordinate system even in this coordinate system the Laplacian is variable separable. So, the Laplace equation can be solved by variable separation ok. So, I am going to say that this phi which depends on r and z in the Cartesian case, it was x, y or x, z here it is r, z. So, it will be some function of small r and some function of small z. They are separable. We go back and plug this form back into the Laplace equation. If we do that, you will find the following equation.

(Refer Slide Time: 07:59)

$$Z \frac{d^2R}{dr^2} + \frac{Z}{r}\frac{dR}{dr} + R\frac{d^2Z}{dz^2} = 0$$

$$\Rightarrow \frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{R}\frac{1}{r}\frac{dR}{dr} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}} = 0$$

$$\Rightarrow \underline{\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{R}}\frac{1}{r}\frac{dR}{dr} = -\frac{1}{Z}\frac{d^2Z}{dz^2} = -k^2$$

sign of separation const.

$$\Rightarrow \frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + k^2R = 0$$

$$\bar{r} = kr$$

$$\bar{r}^2\frac{d^2R}{d\bar{r}^2} + \bar{r}\frac{dR}{d\bar{r}} + \bar{r}^2 R = 0$$

$$R(\bar{r}) = C_1 J_0(\bar{r}) + \cancel{C_2}^{\,0}\, Y_0(\bar{r})$$

$$\frac{d^2Z}{dz^2} - k^2 Z = 0$$

$$\Rightarrow Z(z) = D_1 e^{kz} + \cancel{D_2}^{\,0}\, e^{-kz} = D_1 e^{kz}$$

$$\therefore R(\bar{r}) = C_1 J_0(kr)$$

So, z d square R by d r square plus z by r d R by d r plus R d square Z by d z square is equal to 0. I define both sides by capital R and capital Z, the product of capital R and capital Z. So, I will get a 1 by R d square R by d r square plus 1 by R 1 by z d square Z by d z square equal to 0. Now you can see that this part depends only on small z.

So, I do the same trick that we did before. We take everything which depends on small r on one side, everything which depends on small z on the other. So, we have 1 by R d square R by d r square plus 1 by R. And this I will put it equal to some separation constant k. This k is the same k that we want in the initial conditions, it will represent a wave number. Once again, I would encourage you to think why did we choose the negative sign of the separation constant. Think what happens if we choose a positive sign?

Now, with this we get two equations for capital R and capital Z. Let us write the capital R equation first. So, you can see that I can multiply both sides by r capital R and so, I will get d square R by d small r square plus 1 by R plus k square into R. So, I am using this equality and I am equating the left-hand side to minus k square and this is going to give me one equation for capital R.

We have done a similar procedure when we looked at vibrations of the membrane. When we did vibrations of a circular membrane, our membrane was clamped here. The liquid pool on which we are going to get waves, it is going from R equal to 0 to all the way to R equal to infinity. So, there is no confining boundaries. That is the only difference here and you will see that we will find standing wave kind of solutions here as well.

So, this is our equation for R. Similarly, we can get an equation for Z by equating those two and the equation for z just turns out to be d square Z by d z square minus k square Z is equal to 0. Now we have looked at this equation before. So, I am not going to repeat those things. Recall that this equation is related to the Bessel differential equation. It has two linearly independent solutions j 0 and y 0. Before we express it in that form, we need to non-dimensionalize r because the argument of Bessel function, both the Bessel functions are non-dimensional

So, let us say that some r bar is k into small r k has the dimensions of 1 by length. So, r bar is non-dimensional. If we do that and I multiply this equation throughout by small r square, then I can straight away write r bar square d square R by d r bar square plus r bar d R by d r bar plus r bar square into R is equal to 0.

You, if we have done two steps in going from here to here. Firstly, we have multiplied the equation at the top with small r square. So, that has introduced r square in the first term and r in the second term and a k square r square in the third term as its coefficients. Then we have converted from small r to small r bar. If you do that everywhere consistently, you are going to get that equation.

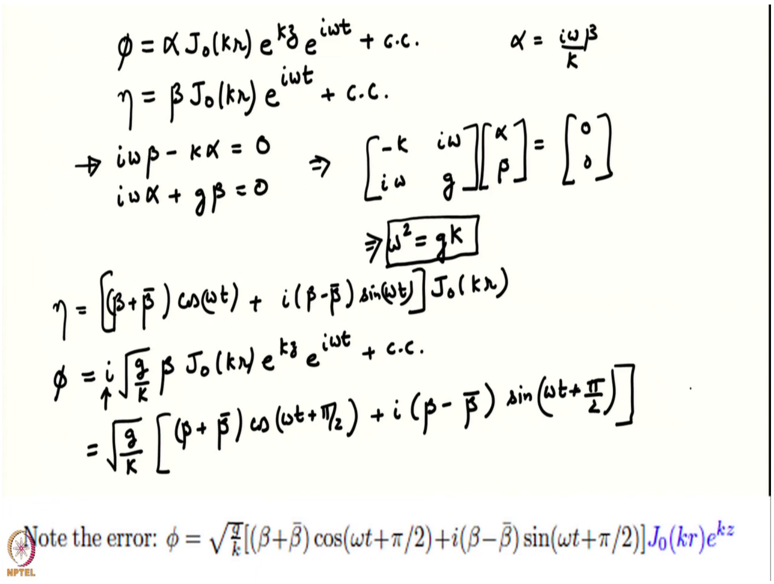
We know that this is the Bessel's equation and the order of the equation is 0. So, the solutions are; so R in this case is a function of r bar is some linear combination of J 0 r bar plus y 0 r bar. We have encountered both of these functions before. I have also told you that y 0 diverges at r equal to 0, ok.

So, in order to prevent that divergence as I said earlier in the last slide that we will have to worry about divergences both at r equal to 0 and r equal to infinity. So, the r equal to 0, divergence is there in y 0. So, we will have to get rid of C 2. So, this is 0. So, therefore, we obtain R of r bar is equal to sum C 1 J 0 of k. We can straight away write r bar is kr.

Similarly, this equation is also solvable easily. So, capital Z of small z is just again the same procedure e to the power k z plus some other constant e to the power minus k z because of divergence at z is equal to minus infinity we have to set this to 0, and so this just gives us D 1 e to the power k z.

You can see that our Eigen function capital phi for the velocity potential had capital R into capital Z. We have determined capital R into capital Z. Both of them contain one arbitrary constant of integration. The product of those two in this case, C 1 and D 1, we can represent it as some other constant, ok. So, generically I can write down the form for phi.

(Refer Slide Time: 14:19)



So, phi in this case is going to have the form some constant C 1 into D 1, I will call it alpha. Alpha could be complex in general; J 0 of k r the same exponential that we encountered earlier into e to the power i omega t, that is because we are doing normal modes. Again here the base state is again quiescent. The pressure variation is hydrostatic like before. It is only we are doing the problem in a cylindrical coordinate system, not in a Cartesian coordinate system.

Now, looking at the form of eta in the equations, recall that the Laplace equation satisfies governs phi and then, there are boundary conditions which couple phi to eta. If you look at the boundary conditions for the kinematic boundary condition and the Bernoulli equation, once again you find that phi and eta have the same radial dependence. This was the same argument that we used earlier where we found the horizontal dependence of phi. It was some linear combination of cos k x and sin k x, here it is just J 0 of kr, ok.

So, you can readily see that the horizontal dependence. In this case, the r dependence of eta is going to be the same as the r dependence of phi. If that is the case, then we can write some other constant I will put beta in general complex. Again, it is the same horizontal dependence as phi and then, eta is not a function of z. So, I do not have to worry about the e to the power k z and then, once again normal modes into plus c c. So, this is our anticipation for the form of phi and eta.

Now, we do the same procedure. We plug these forms into the two boundary conditions it will give us two linear equations in alpha and beta. These are homogeneous equations. The determinant of the matrix will tell us the allowable frequencies. The equations just turn out to be; the equations are i omega beta minus k alpha is equal to 0 and then, we have i omega alpha plus g beta is equal to 0. The first one is the kinematic boundary condition, the second one is the linearized Bernoulli equation, there is a gravity in it.

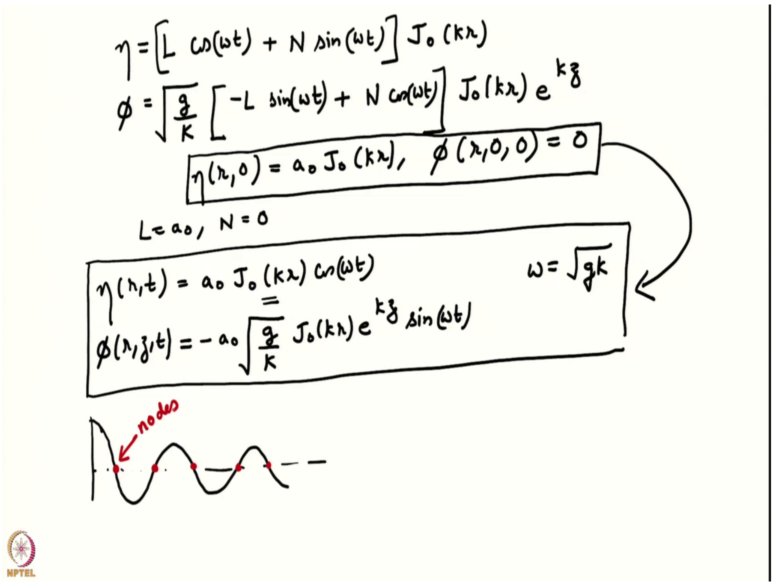
This we can write it as a matrix. So, minus k i omega; i omega g into alpha beta is equal to 0. This again gives us the same dispersion relation. The determinant of the matrix has to be 0 omega square is equal to g k. Interestingly we find that waves in deep water have the same dispersion relation independent of whichever geometry they are described in. At least in these two geometries, you can expect the same result from dimensional arguments.

Now let us anticipate the general solution. So, the general solution is the same as before. So, I will write it as beta times J 0 e to the power i omega t plus c c. So, it will be beta plus beta bar into cos omega t J 0 is any variable plus i times beta minus beta bar sin omega t, the whole thing multiplied by J 0 of kr.

And phi, you can see that we can express alpha from this equation. For example, we can express alpha as i omega beta by k and omega is square root g k. So, it is i times square root g by k beta. So, I only have one constant one complex constant in both the expressions; into J 0 of k r e to the power k z e to the power i omega t plus c c.

And now, I can absorb this i into e to the power i omega t. This is exactly the same procedure as before g by k and then, I can write the final expression as beta plus beta bar cos it will be omega t plus pi by 2 because there is a factor of i here which will be absorbed into e to the power i omega t, it will be e to the power i pi by 2; plus i beta minus beta bar sin omega t plus pi by 2. And once again this is real, this is real in both the expressions so we can replace it by some equivalent real expressions.

(Refer Slide Time: 19:20)



So, let us write it in terms of real expressions. Eta is equal to L cos omega t plus N sin omega t whole thing into J 0 of kr. Phi is equal to square root g by k minus l sin omega t plus N cos omega t into J 0 of k r e to the power k z. Those are the solutions, let us take the initial condition. The initial condition was eta of r, 0 is sum a 0 into J 0 of kr and we had not put any impulse at the surface. So, at z is equal to 0 at time equal to 0 is just 0. We have written this initial conditions earlier. You can see them here, ok.

So, I want to determine the constants, the unknown constants using these initial conditions just like we did it in Cartesian coordinates. So, using this if you just substitute and time equal to 0 in these and equate it to these initial conditions, you will find readily that L is equal to 0, n is equal to 0.

And thus, we have eta of r, t is equal to a 0 J 0 of k r cos omega t and phi of r coma z coma t is minus a 0 g by k, J 0 of kr e to the power k z sin omega t and we have already found that omega is equal to square root g k, ok. So, these are our solutions for those initial conditions. For these initial conditions.

Compare this to what we found earlier in Cartesian geometry. The only difference was instead of J 0, kr we had a cos k x, ok. That is because we had a cos k x in the initial condition, ok. So, J 0 k r behaves like cos k x cos k x is a periodic function of x. J 0 k r is not a periodic function of x; its oscillatory, but the distances between the roots are not exactly equal. I have told you what those roots are when we were looking at the vibrations of a circular membrane.

Note that this is also a standing wave solution. The space and the time part are separate. So, every place where the space part goes to 0, the displacement of the interface or the free surface remains 0 at those points at all times. So, the places where the Bessel function J 0 of k r depending on k, you will have intersections with the undisturbed location.

So the, so it will look like this. So, if this is the undisturbed location and this is your Bessel function initially, so it is a slowly decaying function as we go further and further. So, these are the points where it vanishes, where eta is 0 at time t equal to 0.

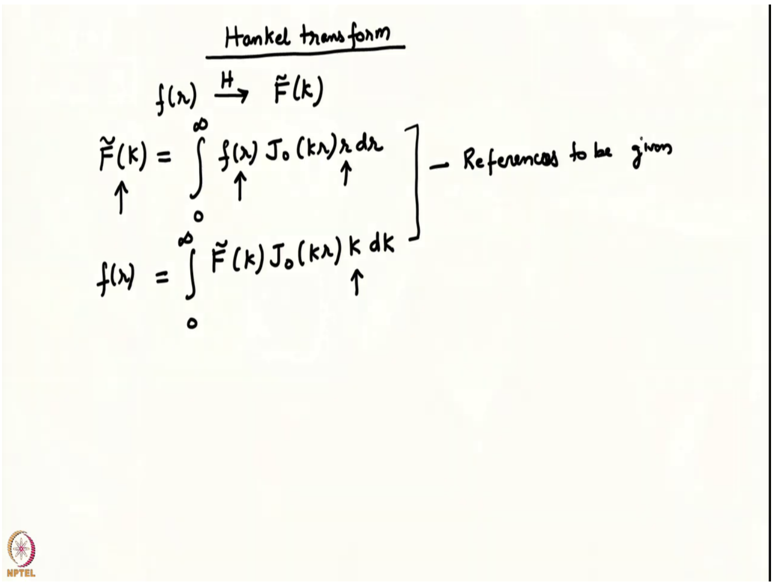
You can note from the expression that these at these places at these nodes, eta will always remain 0. That is the property of a standing wave. So, we have found standing wave like solutions even in a cylindrical axisymmetric geometry. You can go back and substitute this, you can do this exercise using the full problem where we do not say that del by del theta is 0.

So, we could put a perturbation along the theta direction as well. So, it could be, it would be have to be of the form cos m theta, where m has to be an integer. So, you can put that and you can see what is the effect on the dispersion relation.

So, now we are looking at waves in a cylindrical geometry deep water linearized surface gravity waves. Now, let us ask the question that we are now going to perturb the interface let us say via Bessel mode. Just as we had asked the same question earlier that our initial condition is not a cos k x mode. It is a more general initial condition. It could be some arbitrary function of x f of x. And in that case, we had found that the general answer can be written as a Fourier superposition over the Fourier transforms of the initial conditions.

Let us see what is the equivalent answer here. And this process of what whatever we are going to do now we will answer the pebble in the pond problem. So, let us first do.

(Refer Slide Time: 24:16)



So for this, we will introduce a transform which is called a Hankel transform. This is once again an integral transform. It is like a Fourier transform. I will give you references for reading up on the Hankel transform in case you have not encountered it before. One can derive the Hankel transform from the two-dimensional Fourier transform. It involves the transform involves the Bessel functions.

So, the Hankel transform of a function f of r. So, if you have a function f of r and its Hankel transform is indicated. Let us say by F tilde of once again k, but this is now a Hankel transform. So, I will call it H. So, the relation between, so given f of r how do we obtain F tilde of k? This is obtained by the following formula 0 to infinity f of r, J 0 of k r r d r and how do we go? So, this this gives us that if we get f of r and if you plug it into this integral, then if this integral converges, then it will give us the Hankel transform of the function f of r.

Similarly, if we want the function back from its Hankel transform, then we will get f of r and the inverse is represented as f of k or F tilde of k. Once again J 0 of k r and this is k d k. Notice the extra factor of r here and the extra factor of k there, ok. We I will give you some references where you can reference to be given where you can read a little bit more about this integral transformation.

What is the utility of these transforms? Recall that in the Cartesian case, the Hankel; the Fourier transform converted the del by del x square to minus k square. This is what allowed us to solve the Laplace equation as if it was an ordinary differential equation and we could write it down its solution along the z direction as exponentially in z with the prefactor which was k dependent.

We will see that the Hankel transform will do the same thing here. The Hankel transform here, the horizontal direction, the Laplacian operator has a more complicated structure. There is a del r square by del r square plus there is a 1 by r del by del r. The Hankel transform precisely has the property that when it applies on that part of the operator, it converts it into minus k square into the Hankel transform of the function.

So, we will recover very analogous results compared to what we did earlier.

(Refer Slide Time: 26:55)