Introduction to interfacial waves
Prof. Ratul Dasgupta
Department of Chemical Engineering
Indian Institute of Technology, Bombay

Lecture - 34
Cauchy-Poisson problem (contd..)

We were looking at the Cauchy-Poisson initial value problem for surfaces gravity waves in deep water.

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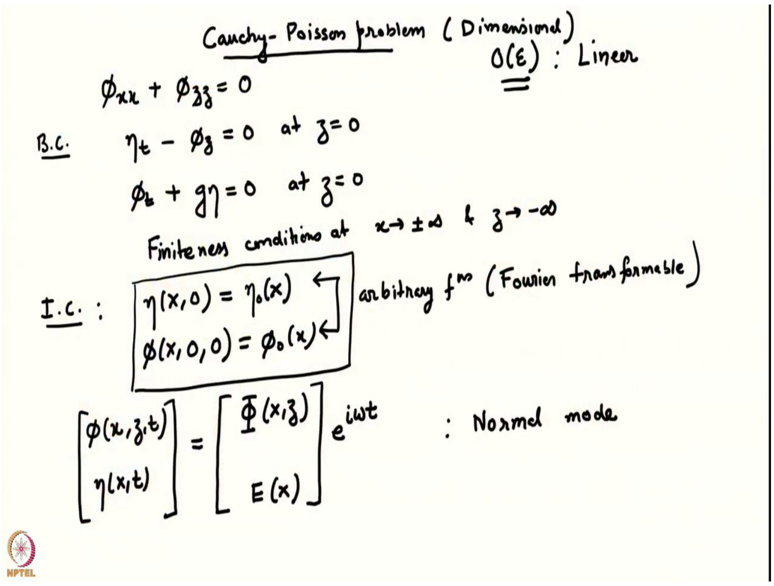

In particular, we were trying to solve the linearized initial value problem, where we had an initial arbitrary surface perturbation and an impulse in the form of a velocity potential at the surface. And we wanted to express the answer in terms of these initial conditions.

We used Fourier transforms in order to be able to make progress. So, we wrote the two-degrees of freedom as some eigenfunction into e to the power i omega t. And then, we wrote down the eigenfunctions in Fourier space by taking a Fourier transform of the equation.

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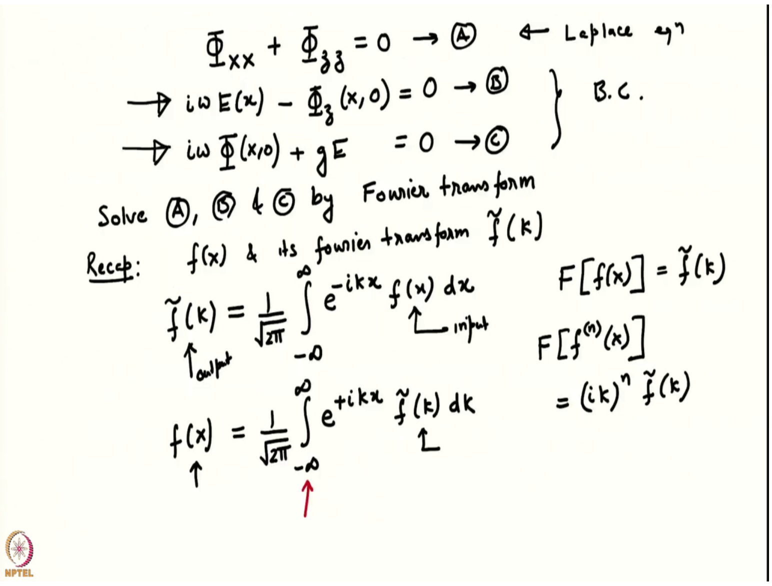

We discovered that by doing the Fourier transform, we can actually solve the Laplace equation and solve it like an ordinary differential equation.

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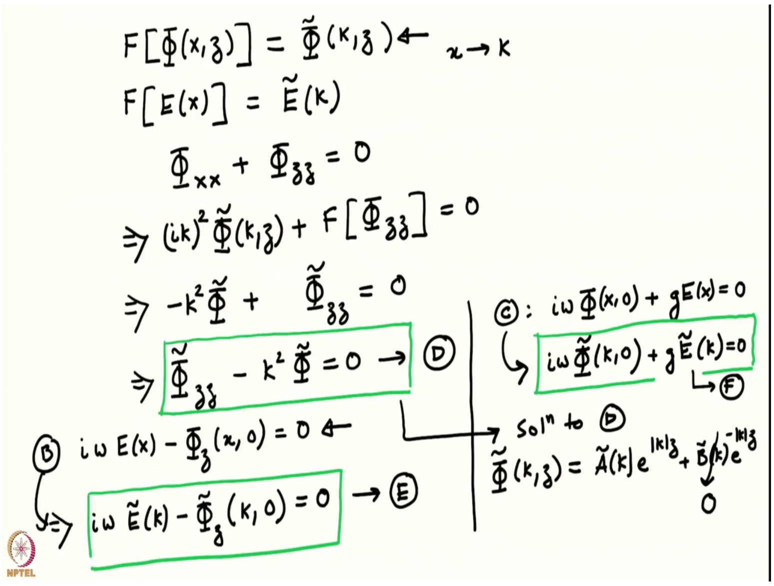

It turned out to be of this form.

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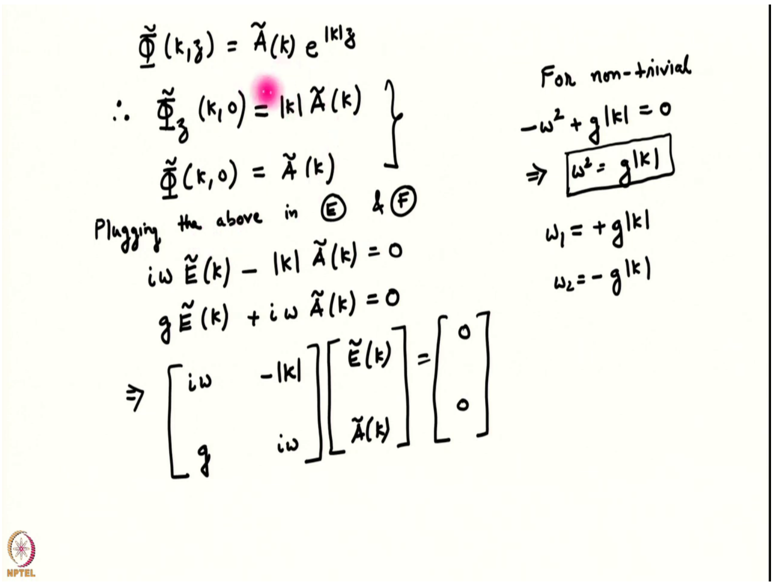

And then, we could write down its solution as an exponential, we eliminated the second exponential because of finiteness conditions at minus infinity. And then, we use the two boundary conditions respectively, the linearized kinematic boundary condition and the linearized Bernoulli equation to get two equations in two algebraic equations in two unknowns.

The determinant of the coefficient matrix gave us the dispersion relation that we had already found earlier. There are two branches to the dispersion relation.

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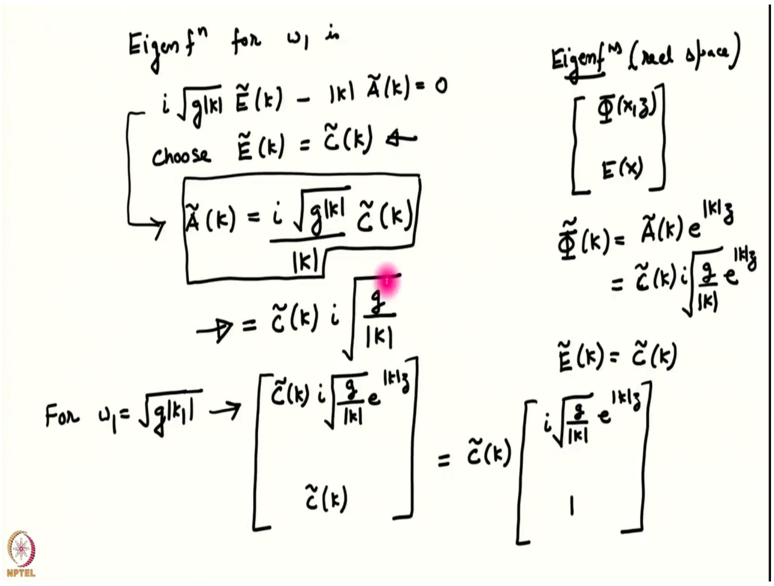

We then found for each of the frequencies, we found eigenfunction which was written as a column vector. And so, there were two such Eigen functions.

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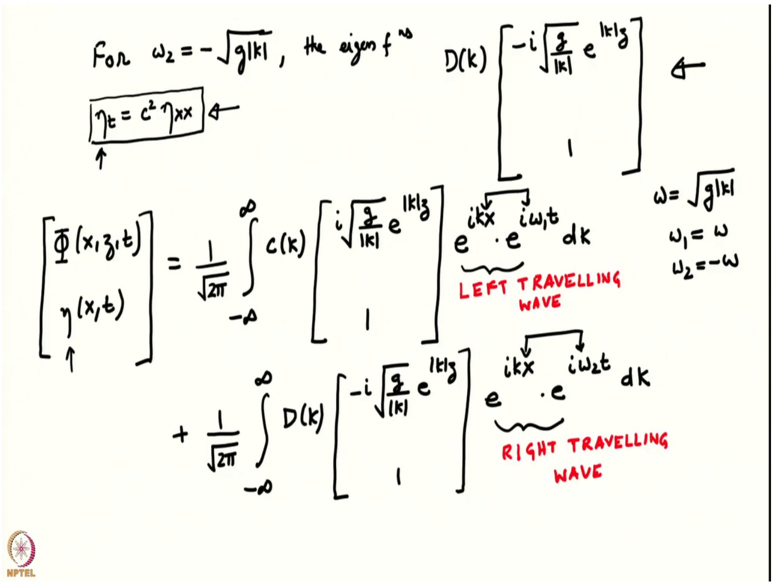

And so, the general solution is written as a linear combination of the eigenfunctions. Here, the prefactors are themselves functions of the wave number k.

In particular, we also saw that the solution has the form of a left travelling wave and a right travelling wave because this is of the form e to the power ikx plus omega t and this is of the form e to the power ikx minus omega t where omega is square root g k. So, let us proceed further.

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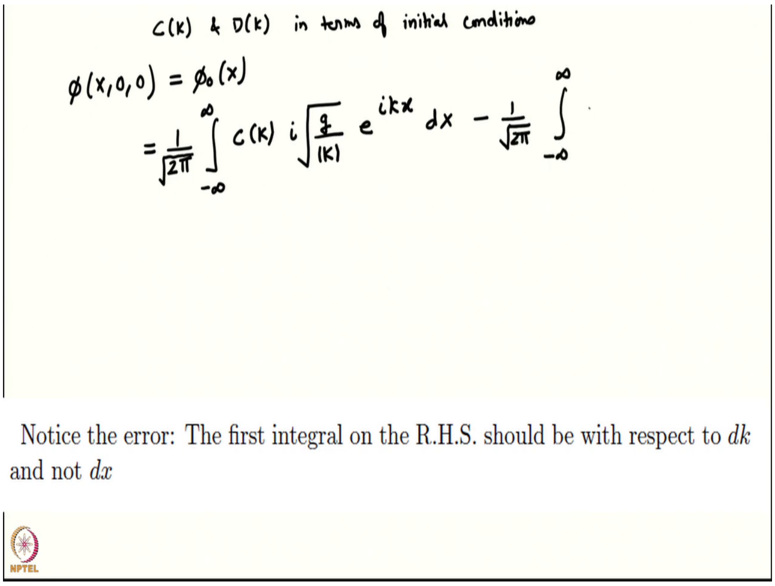

So, our task is now to determine C k and D k in terms of initial conditions. Intuitively, it is clear that C k and D k are going to be related to the Fourier transform of eta 0 of x and phi 0 of x. Let us find out the exact formula which gives us C k and D k in terms of the Fourier transform of those functions.

So, we have seen that phi 0 of or rather so if we just take the expressions in the previous page and we substitute time equal to 0, then we have the initial conditions. On the left-hand side we have the initial conditions, on the right-hand side, we have an expression. So, let us write that.

So, we have phi of x comma 0 comma 0, it is just a surface impulse and we had called this as some phi 0 of x. And this by the previous what we have written in the previous page is just the right-hand side with time substituted to be 0. This will eliminate all the e to the power i omega t because those will become unity. And so, we will have the following expression.

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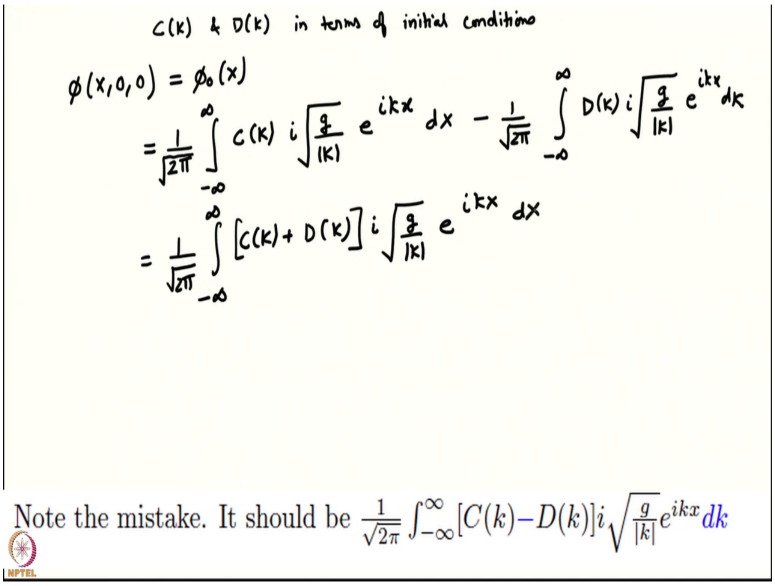

And I can of course, combine this and write it in a slightly more compact form.

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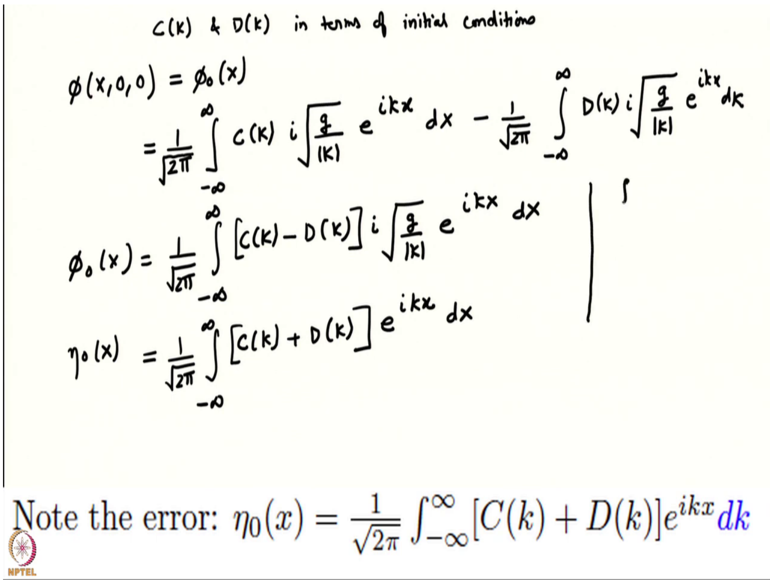

And this is equal to phi 0 of x.

Similarly, we can go back to the previous expression and put in the second row, we can put time equal to 0; this will give us an expression for eta of x at time t equal to 0. If we do the same, we will obtain this. So, eta 0 of x that is the left-hand side and then, I am straight away going to write it in a compact way so, this is minus. So, these are our formulas at time t equal to 0.

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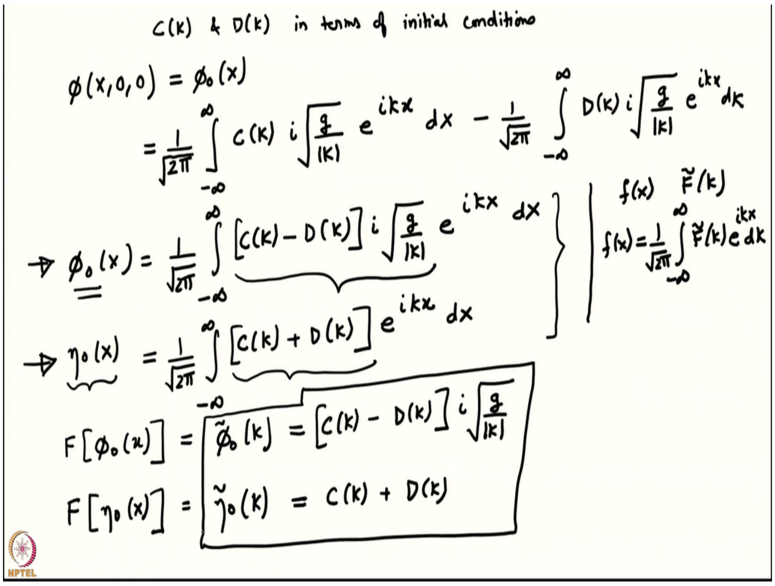

Now, notice the structure of these formulas, recall that I had said that if f x is a function and f tilde of k is its Fourier transform, then f x can be obtained through the inverse transform from f tilde of k using the following formula.

So now, you can see that this, this is precisely the structure of these two integrals, it has the structure of a inverse Fourier transform. I have phi 0 of x here and I have something some function of k into e to the power ikx dx. This implies that whatever I have which I have put in curly braces must be the Fourier transform of phi 0 of x.

Similarly, I have eta 0 of x here and I have C k plus D k here. So, this implies that C k plus D k must represent the Fourier transform of eta 0 of x. So we know, so we conclude that the

Fourier transform of phi 0 of x which is basically phi 0 tilde of k is from the first formula whatever I have put in curly braces ok, so this is C k minus D k into i square root g by mod k.

Similarly, we also find from the second formula that Fourier transform of eta 0 of x which we can represent as eta 0 tilde of k is C k plus D k. So as expected, these are the formulas which connect, these are the equations which connect C k and D k to the Fourier transform of the initial conditions.

What we have to do is we just have to solve these equations, these are two simple linear equations in C and D and if we solve them, we can express C and D in terms of the Fourier transform of phi 0 and eta 0. Let us do that.

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Note the error: The last term should be $= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{2}i\tilde{\phi}_0(k)\sqrt{\frac{|k|}{g}}\,i\sqrt{\frac{g}{|k|}}e^{i(kx-\omega t)}dk$

$$D(k) = \frac{1}{2}\left[\tilde{\eta}_0(k) + i\sqrt{\frac{|k|}{g}}\,\tilde{\phi}_0(k)\right]$$

$$\phi(x,z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{2}\tilde{\eta}_0(k)\,i\sqrt{\frac{g}{|k|}}\,e^{|k|z}e^{i(kx+\omega t)}dk$$

$$-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{2}i\,\tilde{\phi}_0(k)\sqrt{\frac{|k|}{g}}\,i\sqrt{\frac{g}{|k|}}\,e^{|k|z}e^{i(kx+\omega t)}dk$$

$$-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{2}\tilde{\eta}_0(k)\,i\sqrt{\frac{g}{|k|}}\,e^{|k|z}e^{i(kx-\omega t)}dk$$

$$-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{2}i\,\tilde{\phi}_0(k)\sqrt{\frac{|k|}{g}}\,i\sqrt{\frac{|k|}{g}}$$

So, if you just add and subtract these equations appropriately with the appropriate multiplying factors, you can very easily show that C of k is equal to half eta 0 tilde of k minus i times square root mod k by g phi 0 tilde of k. Similarly, D of k is equal to half eta 0 tilde of k plus i square root mod k by g phi 0 tilde of k. So, there are our expressions for C k and D k.

And so now, we can go back to our original expressions which we had written here in terms of a left travelling wave and a right travelling wave. We can substitute the expressions that we have got for C k and D k in terms of the Fourier transform of the initial conditions and that gives us the solution to our problem. We will do a, we will work a little bit more on it.

So we find that phi of x, z, t. So, recall that the original expression had two parts to it, a left travelling wave and a right travelling wave. If I have to get an expression for phi from here, I have to take the first row of this expression, this is written in matrix form. So, I have to take the first row of this expression both on the left and the right and that will give me an equation which tells me; how does phi evolve in time.

Now, these the on the right-hand side, there will be two terms, there will be two integrals, one containing a C k and one containing a D k. In turn, we have seen that C and D each contain two terms each. So, there is there are two terms in C of k and there are two terms in D of k. So, total there are going to be four terms, two terms from C and two terms from D in the expression for phi. So, I am straight away going to write that those four terms.

Note that the e to the power i omega t has to be inside the integral, because omega is a function of k. We know that omega is equal to in this case square root g mod k. So, we cannot take the e to the power i omega t out of the integral because the integration is over k.

So, this is the first term, then I will have a second term. The second term is minus because this is minus. So, there are two terms coming from C. So, and then, we will have a half i phi 0 tilde of k square root mod k by g. I am writing the whole thing, there will be some cancellations you can do that later. So, these two terms come from C k.

And then, we have to take into account the dependence on D k. So, D also has two terms and then, D has a overall minus sign. So, D has a overall minus sign here. So, you can see that there will be an overall minus sign because D is a sum of two terms this and that. So, we will have a minus 1 by square root 2 pi half eta 0 tilde of k i times g by mod k e to the power k z. And now, we will have a kx minus omega t.

This is a right travelling wave, those two are left travelling waves ok d k. And then again a minus phi 0 tilde of k square root mod k by g i, you can see that the two square root terms will cancel each other out. And once again a right travelling wave. So, there are our four terms.

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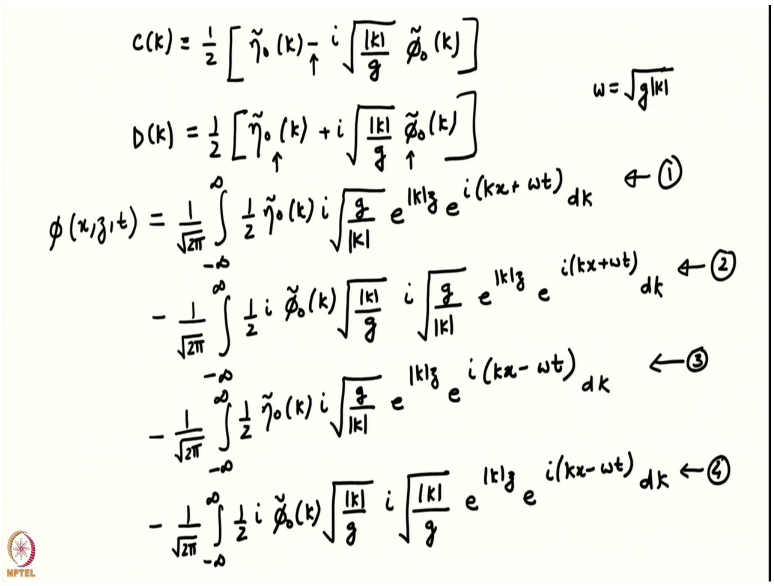

Now, I will call this term 1, term 2, term 3 and term 4. So now, I would like to convert all this to completely real notation without having any complex exponential notation. Of course, the

Fourier transform will still be there, and the Fourier transforms in general can be a complex quantity ok. But we will write it at least without any explicit i's in the expressions ok.

So, for that, you can see that I can combine the 1st and the 3rd term ok. If I do that because there is an i e to the power ikx and then, e to the power i omega t and e to the power minus i omega t will give me a cos t, ok. So, 1st and 3rd will be combined, 2nd and 4th will be combined. So, if you do that.

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$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2} \tilde{\eta}_0(k)\, i \sqrt{\frac{g}{|k|}}\, e^{|k|z} e^{ikx} \left(e^{i\omega t} - e^{-i\omega t}\right) dk$$

$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2} i^2 \tilde{\phi}_0(k)\, e^{|k|z} e^{ikx} \left(e^{i\omega t} + e^{-i\omega t}\right) dk$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\eta}_0(k) \sqrt{\frac{g}{|k|}}\, e^{|k|z + ikx} \sin(\omega t)\, dk$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\phi}_0(k)\, e^{|k|z + ikx} \cos(\omega t)\, dk$$

$$\phi(x,z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\tilde{\phi}_0(k) \cos(\omega t) - \tilde{\eta}_0(k) \sqrt{\frac{g}{k}} \sin(\omega t)\right] e^{ikx + |k|z}\, dk$$

So, combining the 1st and the 3rd terms and then of course, we are in the 2nd and the 4th terms, there are some terms which cancel out, there is a there are two square roots which cancel out and so, the 1st and the 3rd term combination gives me this.

You can see that this will eventually lead me to a sin omega t because this is e to the power i omega t minus e to the power minus i omega t, I need a two i, but there is a i here and there is a 2 there so, I can multiply top and bottom by i that will make it minus and then, minus 1 by 2i ok. So, that will introduce the minus sin omega t, ok.

And then, I will have a if I combine the 2nd and the 4th terms, then I obtain with a minus sign, the 2nd and the 4th terms both have a minus sign. And you can see that I can cancel out in the 2nd and the 4th terms each, there is an i square. So, if you look at the 2nd term, there is a i here and there is a i there, the two square roots cancel each other so and the same thing happens in the 4th term also, ok.

So, the i square is going to eventually make it a plus because there is overall a minus sign, but I will keep the i square right now and I will write the half here i square phi 0 tilde of k e to the power so, I should put a mod here, e the power mod k z e to the power ikx. And once again e to the power i omega t plus e to the power minus i omega t dk. And so, those can be combined.

And the 1st term I am going to have a minus sign because I wanted two i in the denominator, there is an i in the numerator so, minus. So, I will get minus 1 by square root 2 pi minus infinity-to-infinity eta 0 tilde of k; I have absorbed the half in writing cos omega t. So, I will get square root g by mod k e to the power k z plus ikx into a sin omega t dk.

Again, sin omega t cannot be taken out because omega is a function of k. So, sin omega t cannot be taken out of the integral. And the 2nd term becomes a plus because the i square and the minus makes it a plus. And then, we have phi 0 tilde of k e to the power mod k z plus ikx cos omega t dk.

And I can combine this entire thing and write it in a more compact notation. Combining both the terms so, I will have phi 0 tilde of k cos omega t minus eta 0 tilde of k square root g by k sin omega t this whole thing into e to the power ikx dk.

You can see that the whole thing has the structure of a inverse Fourier transform and what did we obtain? We obtain an expression for phi x. So we are missing, we are missing plus mod k z into dk. So, and let me put this in a box.

So, this is telling us that if we prescribe initial conditions, arbitrary Fourier transformable initial conditions, then in terms of those initial conditions, how is the velocity potential going to look like at all later times? The answer is expressed as a inverse Fourier transform ok. So, you can see that the whole thing, this bracket into e to the power mod k into z, if you call this some function of k into e to the power ikx dk. So, that is the structure of the solution.

You can in general see that if I have initial conditions where I perturb introduce a perturbation on eta, but I do not introduce a perturbation on phi at the surface, then it will still produce a velocity, we will see that shortly. You can see that readily from here that if I just have a perturbation on eta, then eta 0 tilde of k is going to be non-zero whereas, phi 0 tilde is going to be 0, but that is still going to produce a velocity field within the bulk of the fluid.

Similarly, if we have only a, if we start with a flat surface but give it a surface perturbation in the form of phi 0, but do not deform the surface initially, it is still going to produce waves. As we will see when we write down the corresponding expression for eta, ok.

So, this is telling us how the velocity potential and the interface move together in a coupled manner producing surface gravity waves, ok. The solution is more complicated than what we would have obtained from a wave equation. Remember that this is also linearized the wave equation is also linearized.

However, the wave equation that we have seen until now represents the equation governing waves in a non-dispersive medium, this is a dispersive medium and consequently, the shape of the wave packet will keep changing because every Fourier mode travels with its own speed. When we look at the numerical solutions of this, it will become apparent. Right now, let us write down the corresponding solution for eta of x comma t.

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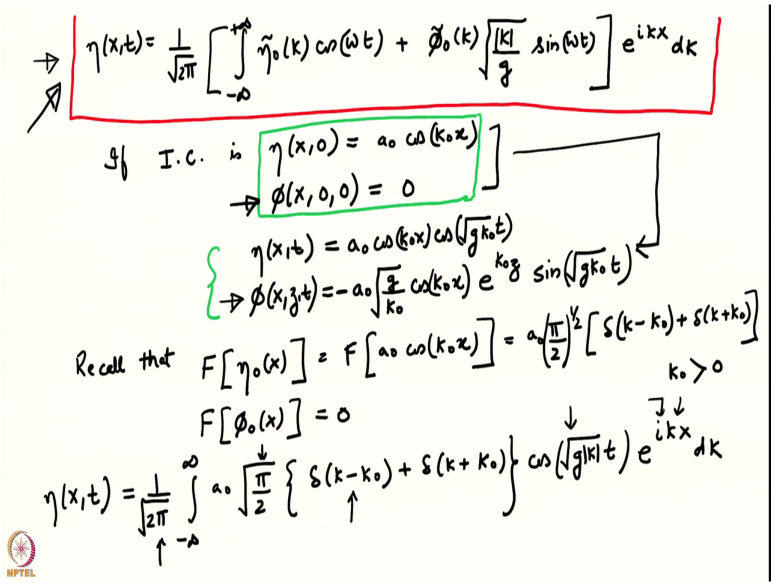

So, using the same argument using, the same set of algebra, very similar one can also show that eta of x comma t is just given by this expression. I leave the details to you, it is very easy, you can do it yourself. You can just look at the way I have done it for phi and do the whole thing yourself.

This is the solution, form of the solution which was obtained by Gaussian Poisson. And this was considered quite a landmark achievement at that time because many of these mathematical techniques were not well-known, and they had to invent many of these techniques in order to obtain the final answer as an integral superposition of all the modes. So, this basically completes the solution to our initial value problem formally, ok.

Now, a couple of things. As I told you sometimes you will see that eta 0 of k can be complex or in general because eta 0 of k is a is the Fourier transform of something, but as long as we are taking real functions as initial conditions, this expression is guaranteed to be real, ok. Let us first test this on a problem which we have already solved.

So earlier we saw; recall that we had solved this problem earlier. So, we had set our initial condition to be just a pure cosine mode and the initial surface impulse to be 0 and that had given us a solution like this. Let us write this solution in terms of dimensional variables. So, one can take this eta 1 and phi 1 and dimensionalize it and get the solution. You can do it using the scales that are already known to you; I am just going to write down the dimensional form of the solution.

So, we had seen earlier, if the initial condition is eta of x comma 0 is some a naught cos let us say k naught x ok. I do not want to use the same k because k is now reserved for the integration variable of the inverse Fourier transform. So, I am just using some k naught x, some wave number k naught.

And if phi of x at z is equal to 0 and time t equal to 0 was 0, then the solution to this initial value problem is this that eta of x comma t so, the surface at all later times looks like a standing wave which moves up and down harmonically in time with a frequency which is given by this.

The corresponding velocity field which is developed is represented by a velocity potential. The velocity potential has a minus sign for these initial conditions. So, this should be k naught and this is sin g k naught t, so this is the solution.

So, this is telling us that if we just deform the surface initially as a cosine wave with an amplitude a 0. It is going to end with 0 velocity everywhere in the fluid. It is going to go up and down as a standing wave with the frequency square roots g k 0 and there will be velocity fields created which will decay exponentially with depth. You can get the velocity fields by differentiating this with respect to x and with respect to z. We have solved this problem initially, ok.

Now, let us verify that the two solutions that we have just written in red boxes should also contain these as special cases ok. So, let us see whether we can use these initial conditions on the expressions in the red boxes and recover the same results that will be a test that our expressions are correct. So, let us try that.

So, our what we really want in evaluating the expressions in the red boxes are basically the Fourier transforms of the initial conditions. Everything else is already there in those expressions, the only input that these expressions require before the integration can be done is the Fourier transform of the initial conditions.

Our initial conditions, one of them is very simple phi is just 0 so, its Fourier transform is also 0. So, we only have to worry about the Fourier transform of eta that is a 0 cos k 0 of x. So, recall that the Fourier transform of eta 0 of x which in this case is a 0 cos k 0 of x is basically given by this formula, k 0 is of course, greater than 0. So, it is a sum of two delta functions.

Those of you who are not familiar with this, please brush up your Fourier transforms, look up what is the Fourier transform of cosine and what is the Fourier transform of sin. Intuitively, you are expressing the Fourier transform expresses, what is the sin and the cosine content of a function. If a function itself is just a Fourier mode with wave number k, then it tells you that there is only one wave number present and these delta functions just tell us that, there are two of them, one at the positive side and one at the negative side.

Now, let us plug in this result into our integrals that we have derived in the red boxes. So of course, we have, we also know that Fourier transform of phi 0 of x in this case is 0 because our initial condition just says that there is no disturbance at the surface as far as phi is concerned.

So, if we plug this in into our expression for eta so, I am just going to substitute in this integral and the second term is identically 0 because phi 0 tilde of k is 0. So, eta is just square root 2 pi into minus infinity to infinity and I am just going to substitute the value of whatever we have here.

So, this is a 0 square root pi by 2 so, some things will get cancelled out. And then, inside we will have the two delta functions, this into cos omega or let me write it as cos square root g mod k into t into e to the power ikx dk. So that is it, the second term is 0.

Now, from the shifting property of the delta function, whenever there is a delta function and the argument of the delta function goes to 0 within the limits of integration, then what the delta function does is it just evaluates whatever is there in the integrand at the point where the argument goes to 0.

So, what the first delta function is going to do is it is just going to evaluate the rest of the quantities at k is equal to k 0 and the other delta function will evaluate it at k is equal to minus k 0. You can see that because everywhere we have a mod k except e to the power ikx so, making k 0 to be minus k 0, will not make any difference other than this e to the power ikx term.

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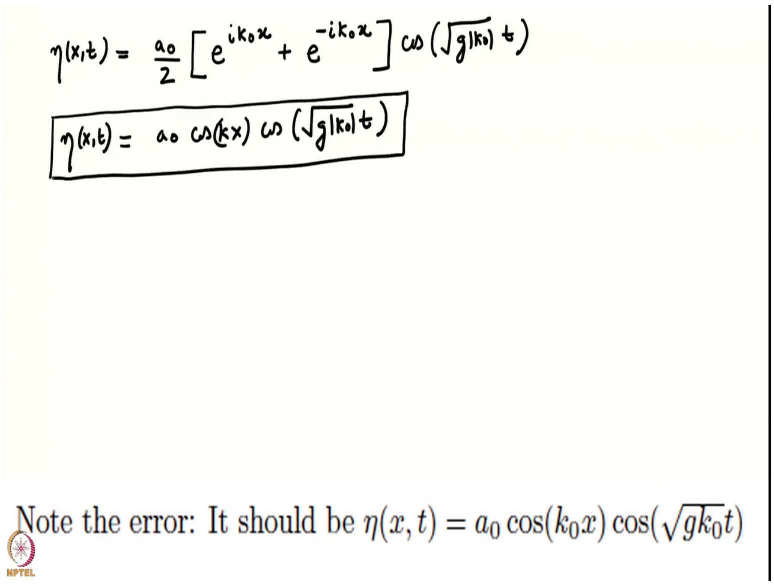

So, we can straight away write. Eta of x comma t is equal to a 0 by 2, I have cancel out the square root 2 pi and the square root pi here and then, I have cancelled out the pi here and the pi there and then, there is a square root 2, there are two square roots 2 that gives us a factor of half.

So, that is why I have a 0 by 2. So, a 0 by 2 into. The delta functions just evaluate the so, this delta function; this delta function just evaluates this and this term at k is equal to plus k naught.

So, I have e to the power i k naught x so, second delta function will evaluate it at minus k naught so, I will have e to the power minus k naught x. And both the delta functions will do the same thing to cos because cos has k inside the argument, inside the under the square root and that is a mod. So, whether it is k is equal to k naught or k is equal to minus k naught, I get the same expression. So, I will just write it as cos of square root g mod k naught into t.

Please make sure you have understood how did we get this. You can see that this is very simple, this just boils down to cos kx cos square root g k 0 into t eta of x, t. So, indeed we are recovering whatever we had obtained earlier, ok. We had this is exactly what I have written here. This is the same expression that I have written here.

You can check that even for phi, we get the same expression. You substitute, the go back to the expression for phi in this red box, substitute the expression for eta 0, phi 0 is 0 here, but there is a minus eta 0 in the second term. Eta 0 tilde of k is not 0, eta 0 tilde of k is the sum of two delta functions.

Once again we substitute it here and you will see that there is a minus sign overall, that is the minus sign which came in our earlier solution also. We have seen this minus sign earlier so, there is a minus sign here. So, we expect a minus sign from the from this expression as well. So, if you substitute it here and work out the two delta functions and allow the same shifting property, you will just get this.

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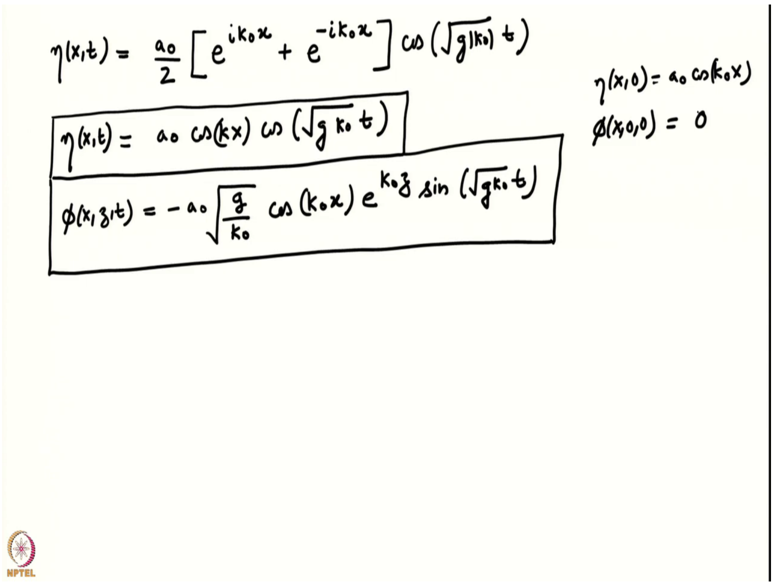

So, you will get that phi of x comma z comma t is equal to minus a 0 square root g by k 0. And because now k 0 is anyway a positive number so, I am; I will just skip writing the mod cos k 0 of x e to the power k 0 of z sin square root g k 0 into t. So, those are our solutions when the initial condition is a 0 cos k 0 x and phi is 0. You can see that if you put time equal to 0 and z is equal to 0 in these conditions, you will recover the initial conditions.

So, this is reassuring because this is telling us that our expressions are consistent with whatever we have done so far and it recovers the correct limit. We will look more at these solutions by looking at some examples of more complicated initial conditions, when and we will plot these initial conditions and we will track how does the interface vary.

We will see that this in particular if we just have just some perturbations in eta 0 so, I could take a eta 0 in the form of a Gaussian so, it is some kind of a elevation and we can ask what happens if you put an elevation on the interface. You will see that there is a wave packet. One wave packet we will travel from right to left, another we will travel from left to right.

And we will, if our initial perturbation is symmetric, the two wave packets will look exactly the same. And then, we can analyze what happens inside the wave packet. In the process, we will also find out something about group velocity when we look at those wave packets.

In the next video, I am going to do the same Cauchy-Poisson problem for a circular geometry. As I said before we have, we would like to understand when we throw stone into a pond, what perturbations it causes and how do the ripples spread out, ok. This is also relevant in many kind of flow situations where there are perturbations on the surface which can create travelling waves. We will look at that problem in a cylindrical axisymmetric geometry next.