Introduction to interfacial waves

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Lecture - 33
Cauchy – Poisson initial value problem for surface – gravity waves in deep water

We were looking at the Cauchy Poisson problem, which is basically the initial value problem, corresponding to arbitrary Fourier transformable initial conditions, not just a single mode as we have done until now. So, we expect the answer to be representable as a superposition over every possible wave number, which the system allows ok.

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Cauchy- Poisson problem (Dimensional)

$\underline{\underline{O(\epsilon)}}$: Linear

$$\phi_{xx} + \phi_{zz} = 0$$

B.C. $\quad \eta_t - \phi_z = 0 \quad$ at $z = 0$

$\quad \phi_t + g\eta = 0 \quad$ at $z = 0$

Finiteness conditions at $x \to \pm\infty$ & $z \to -\infty$

I.C. : $\boxed{\begin{aligned} \eta(x,0) &= \eta_0(x) \\ \phi(x,0,0) &= \phi_0(x) \end{aligned}}$ arbitrary f^{ns} (Fourier transformable)

$$\begin{bmatrix} \phi(x,z,t) \\ \eta(x,t) \end{bmatrix} = \begin{bmatrix} \Phi(x,z) \\ E(x) \end{bmatrix} e^{i\omega t} \quad : \text{Normal mode}$$

And, so, we are going to use the technique of Fourier transforms extensively, while solving this problem. So, as I said earlier we have we are looking at the order epsilon problem. So, it is a linear problem. Looking at waves in deep water and we are asking the question that if we put an arbitrary interface perturbation of the form eta 0 of x.

And, some velocity potential impulse at the surface at z is equal to 0 and t equal to 0 phi 0 of x. Then, what waves are created as a result at later time t and what is the velocity potential as a result in the body of the fluid.

So, let us employ Fourier transforms for this. So, like before we will first write it in the normal mode form. So, we expect so, x, z, t, I mean eta of x, t let me write it. So, these are my two degrees of freedom, each of them itself is a function is some eigen function phi which is a function only of space into.

So, the eigen function for velocity potential is this capital phi and the eigen function for eta is this capital E. E is just a function of x capital phi is a function of x and z, into e to the power i omega t. This would be my normal mode, my normal mode, approximation. Now, we have to just go and plug this in into our governing equations.

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$$\Phi_{xx} + \Phi_{zz} = 0 \rightarrow (A) \quad \leftarrow \text{Laplace eq}^n$$

$$\left.\begin{aligned} & i\omega E(x) - \Phi_z(x,0) = 0 \rightarrow (B) \\ & i\omega \Phi(x,0) + gE = 0 \rightarrow (C) \end{aligned}\right\} \text{B.C.}$$

Solve (A), (B) & (C) by Fourier transform

Recap: $f(x)$ & its fourier transform $\tilde{f}(k)$

$$\underset{\text{output}}{\tilde{f}(k)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \underset{\text{input}}{f(x)}\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{+ikx} \tilde{f}(k)\, dk$$

$$F[f(x)] = \tilde{f}(k)$$

$$F[f^{(n)}(x)] = (ik)^n \tilde{f}(k)$$

So, we obtain the Laplace equation just becomes a equation for capital phi. Because, the I to the e to the power I omega t does not do anything, I will call this equation A. Then, in the boundary conditions I have i omega E of x minus the derivative phi, which is a function of x evaluated at z is equal to 0 is 0, this I will call it B. Then, I will have the linearized Bernoulli equation it is just del phi by del t. So, that brings i omega and capital phi is evaluated at 0 plus g times E is equal to 0.

So, this is my Laplace equation and those two are my boundary conditions, you can see that in both the equation B and C, z is always 0. So, it is just a function of x. Now, we are going to solve A, B and C, by solve A, B and C by Fourier transform. You can see that this has two advantages, first it will I will convert this into what will look like a ordinary differential equation. Secondly, it will allow me to represent arbitrary initial conditions easily.

So, define so, if I so, this is just for recap. So, please brush up your whatever you have learned in Fourier transforms in your math methods course. Recap, that if you have a function f of x and it is Fourier transform f tilde of k. So, we are going from x to k k is the wave number space.

So, the Fourier transform of the function f tilde of k is defined as this is the definition I will use and the reverse transform. So, this tells us that, if you give an input f x it throws an output f tilde of k, which is the Fourier transform of f of x. The reverse is if I know the Fourier transform of the function, the way to get back the original function is to do e to the power i k x into f tilde of k into d k. Because, this is integration with respect to x with respect to k so, I will get a function of x.

So, here this is the input and that is the output, here this is the input and that is the output. So, together they form the form a pair. So, I will indicate the Fourier transform operation as this. So, F of f of x would be f tilde of k. Now, the operation which makes it useful or the property rather which makes it useful is that that Fourier transform of the derivative.

So, I am looking to look at the nth derivative of F of x is just i k to the power n the Fourier transform of the function f itself. This is the property which makes this transform very useful and we will convert these equations A, B and C into ordinary differential equations, which will allow us to solve them as if they are constant coefficient ordinary differential equations. So, here we are interested in the Fourier transform with respect to the x variable, we are not transforming with respect to z.

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$$F[\Phi(x,z)] = \tilde{\Phi}(k,z) \leftarrow \quad x \to k$$

$$F[E(x)] = \tilde{E}(k)$$

$$\Phi_{xx} + \Phi_{zz} = 0$$

$$\Rightarrow (ik)^2 \tilde{\Phi}(k,z) + F[\Phi_{zz}] = 0$$

$$\Rightarrow -k^2 \tilde{\Phi} + \frac{\partial^2}{\partial z^2}\tilde{\Phi}$$

So, we will define Fourier transform and we have to remember that the Fourier transform is with respect to x, capital phi is a function of x and z, but I am Fourier transforming with respect to x. So, the x variable will go over to the k variable. So, this will become some function phi tilde and the x will get replaced by k and the z will remain z, because we have Fourier transform only the x variable. So, this is the Fourier transform version of phi ok.

Similarly, the Fourier transform of E of x, I am just writing the eigen functions in the normal mode analysis. So, every Fourier transform variable will have a tilde on top this is the convention that I am following and because E is just a function of x. So, it will become E tilde and x will get replaced by k.

So, the Fourier transform replaces x by k and it gives you a new function. So, if I do the Fourier transform. So, let me write down the Laplace equation; Laplace equation in x z space was just this. So, if I do the Fourier transform on this then I get i k square, because this is the second derivative.

Recall that we have written that the Fourier transform of the nth derivative is i k to the power n. So, the Fourier transform of the second derivative is i k to the power 2, into the Fourier transform of the original object which is phi and then the Fourier transform.

So, this becomes phi tilde and this is a function of k, z as we defined here plus. Here I am assuming, in the second term I am assuming that because the Fourier transform is with respect to the x variable. I can this is actually Fourier transform of phi z z equal to 0. Now, because the Fourier transform touches only the x variable, it does not do anything to the z variable. So, we are saying that the z derivatives can be taken after that Fourier transform.

So, I can swap the Fourier transform operation and do the two derivative with respect to z later. If, we do that then you can see that, I can pull out the 2 second derivatives with respect to z outside, and then this just tells me that it is the second derivative with respect to z of the Fourier transform of phi itself. So, I get minus k square phi tilde plus del square by del z square I pulled out this into the Fourier transform of phi, which is just phi tilde.

And, because I am always writing the derivatives as a subscript, I will not write it like this, I will just write it like this, I hope this step is clear.

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$$F[\Phi(x,z)] = \tilde{\Phi}(k,z) \leftarrow \quad x \to k$$

$$F[E(x)] = \tilde{E}(k)$$

$$\Phi_{xx} + \Phi_{zz} = 0$$

$$\Rightarrow (ik)^2 \tilde{\Phi}(k,z) + F[\Phi_{zz}] = 0$$

$$\Rightarrow -k^2 \tilde{\Phi} + \tilde{\Phi}_{zz} = 0$$

$$\Rightarrow \boxed{\tilde{\Phi}_{zz} - k^2 \tilde{\Phi} = 0} \rightarrow \text{D}$$

$$\text{B} \quad i\omega E(x) - \Phi_z(x,0) = 0 \leftarrow$$

$$\Rightarrow \boxed{i\omega \tilde{E}(k) - \tilde{\Phi}_z(k,0) = 0} \rightarrow \text{E}$$

$$\text{C}: \quad i\omega \Phi(x,0) + g E(x) = 0$$

$$\Rightarrow \boxed{i\omega \tilde{\Phi}(k,0) + g \tilde{E}(k) = 0} \rightarrow \text{F}$$

Sol^n to D

$$\tilde{\Phi}(k,z) = \tilde{A}(k) e^{|k|z} + \underbrace{\tilde{B}(k)}_{0} e^{-|k|z}$$

So, I will write the derivative term first minus k square phi tilde is equal to 0, this I will call equation D. We had A, B, C and now we have equation D, our equation B was i omega E of x minus phi z of x comma 0 is equal to 0.

If, I take the Fourier transform of this; this is really a boundary condition, but x is a independent variable here. If, I take the Fourier transform once again with respect to x then this becomes E tilde of k minus, we do the same trick that we did here, we do the differentiation with respect to z later. And, this just implies that this is the z derivative of the Fourier transform of phi. So, this is phi z and because this is a function of z, and because z was 0 in the original equation, so, this will be evaluated at z is equal to 0. So, this is equation E.

I can also write down a similar equation for equation C. So, this is equation B, which leads to this, I have already written equation B in the previous slide, this is equation B. Now I am going to take the Fourier transform of equation C. So, C I will write the equation again C is i omega phi of x 0 plus g E of x is equal to 0, if you take the Fourier transform of this, then with respect to x, then this is i omega phi tilde k comma 0 plus g E tilde of k is equal to 0.

So, now what we are doing is we have written down the Laplace equation in the Fourier domain, we have written down the two boundary conditions also in the Fourier domain. And, these are actually equations governing the eigen functions. We have eliminated the time dependence by saying that it is all simple harmonic time. So, it is everything is proportional to e to the power i omega t. So, these are all equations governing the eigen modes written in the Fourier domain.

So, now, we have to work on so, I will call this equation F, let me put these in green boxes, because these are equations that we will need to work on, so, this, this and this. My equations A, B, C written in the Fourier domain D E F; so, now, look at equation D. You can see that this is Laplace equation written in the Fourier domain, but now this is like an ordinary differential equation, because the derivative with respect to x has been eliminated, it appears as k square.

And, because the coefficient of phi tilde is in the first term the coefficient is just 1 and the second term the coefficient is just k square, it does not depend on z. I can solve this as if it is a ordinary differential equation. However, when we write down the constants of integration we have to remember that they are not really constants, they are functions of the other unknown which is k.

So, in one when we transform back to real domain the k will go back to x, we have seen such kind of things when we did the method of multiple scales. So, the solution to this solution to D, in general will be written as phi tilde of k z. This is just like an ordinary differential equation with constant coefficients in z. So, exponential of k z and exponential of minus k z. A linear combination of the 2, the prefactors will not be constants, but functions of k.

So, I will write this as some constants or rather some unknown functions of k into E to the power, I will put a mod around k, I will explain shortly why I am doing that, plus B tilde of k e to the power minus k z. Why are we putting a mod of k? K in until now was just a positive real number 0 to infinity it was related to the wavelength of our system.

However, because Fourier transform involves complex exponential notation, note that when we do the inverse Fourier transform, the limit of k actually goes from minus infinity to plus infinity. So, negative values of k are actually allowed when we are doing the reverse transform ok.

So, we have to be careful when we write down our expressions k now can be negative. Especially, when we write down the final answer in the form of an inverse Fourier integral, we will see that the limit of integration is from minus infinity to plus infinity, so, k negative values are allowed in the integration. So, we are going to have to get rid of one of these two exponentials to keep things finite as z goes to minus infinity.

So, unless I say that I am going to take the mod of k it is not clear, that whether k is positive or negative. So, to keep it clear I am going to say that this is mod of k. And, so, mod k is always positive whether k is negative or positive.

And, so, this allows me to say that this term the second term here always diverges. Independent of whether k is positive or negative ok. So, I am going to set this function to 0. This is similar to what we did earlier except that now we have negative k also in our math in our maths. So, we phi tilde of k z is just a of k e to the power k z, let us proceed further.

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$$\tilde{\Phi}(k,z) = \tilde{A}(k)\, e^{|k|z}$$

$$\left.\begin{aligned} \therefore\ \tilde{\Phi}_z(k,0) &= |k|\,\tilde{A}(k) \\ \tilde{\Phi}(k,0) &= \tilde{A}(k) \end{aligned}\right\}$$

Plugging the above in (E) & (F)

$$i\omega\,\tilde{E}(k) - |k|\,\tilde{A}(k) = 0$$

$$g\,\tilde{E}(k) + i\omega\,\tilde{A}(k) = 0$$

$$\Rightarrow \begin{bmatrix} i\omega & -|k| \\ g & i\omega \end{bmatrix} \begin{bmatrix} \tilde{E}(k) \\ \tilde{A}(k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For non-trivial

$$-\omega^2 + g|k| = 0$$

$$\Rightarrow \boxed{\omega^2 = g|k|}$$

$$\omega_1 = +g|k|$$

$$\omega_2 = -g|k|$$

So, we have phi tilde of k z is equal to A of k or rather A tilde of k into e to the power mod k z. Therefore, we will need this derivative phi tilde of z; the derivative of phi tilde with respect to z at z is equal to 0. And, you can see that this is mod k A tilde of k. And, we are also going to need the expression for phi tilde of k at 0, this is in the boundary conditions so, a tilde of k.

So, now if we go back and plug these in into our equations E and F, we are going to get, so, plugging the above in E and F. So, these two in E and F, we have written E and F earlier. So, E and F you can see that there is a derivative with respect to z in e and phi tilde of k comma 0 in F. So, we are going to use these two in E and F. And, once we do this two we will get the following equations i omega E tilde of k minus mod k, A tilde of k is equal to 0.

And, then we have g E tilde of k plus i omega A tilde of k is equal to 0. Very similar to what we did earlier, except that this E A are not constants, but unknown functions of k. Once again we do not want trivial answers. So, you can write this as a matrix, these are linear in E and k E k E A need not be linear functions of small k, but these equations are linear in E tilde and A tilde.

So, we can write this as i omega minus mod k g and i omega this into E tilde of k A tilde of k this is equal to 0 0. Once again for non trivial solutions, we have to set the determinant to be 0, you can readily work it out minus omega square plus g k g mod k is equal to 0. So, omega square is equal to g mod k. The mod will appear now onwards in the dispersion relation, because when we finally, write the answer we will have cos omega t. And, this omega will be g mod k because the integral limits will have negative values of k also ok.

So, this is the dispersion relation and we have recovered the same dispersion relation as we did earlier ok. Now, let us proceed further. So, there are two eigen values omega 1, you can see that this has the structure of an eigen value problem. So, omega 1 is equal to plus g mod k and omega 2 is equal to minus g mod k. Let us write down the eigen functions corresponding to omega 1 and omega 2.

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Eigen f^n for ω_1 is

Eigenf^{ns} (real s

$$i\sqrt{g|k|}\,\tilde{E}(k) - |k|\,\tilde{A}(k) = 0$$

Choose $\tilde{E}(k) = \tilde{C}(k)$

$$\tilde{A}(k) = \frac{i\sqrt{g|k|}}{|k|}\,\tilde{C}(k)$$

$$= \tilde{C}(k)\, i\sqrt{\frac{g}{|k|}}$$

For $\omega_1 = \sqrt{g|k_1|}$,

Note the error: It is $\omega_1 = \sqrt{gk}$

So, eigen function for omega 1 is we just go and substitute it in into one of the two equations whose matrix, whose determinant gave us the dispersion relation. So, for example, we can just substitute it into the first equation and that tells us 1 by square root g mod k, E tilde of k minus mod k A tilde of k is equal to 0.

Let us choose E tilde of k and represent it as sum function, C tilde of k, this is just for convenience. If this is true then this equation just becomes A tilde of k is i times square root g mod k divided by mod k into C tilde of k.

So, I can write this as I will write the C tilde of k first and then i times square root g by mod k. So, this is the eigen mode. So, for omega 1 is equal to square root g mod k 1 let us write down the eigen functions in Fourier space.

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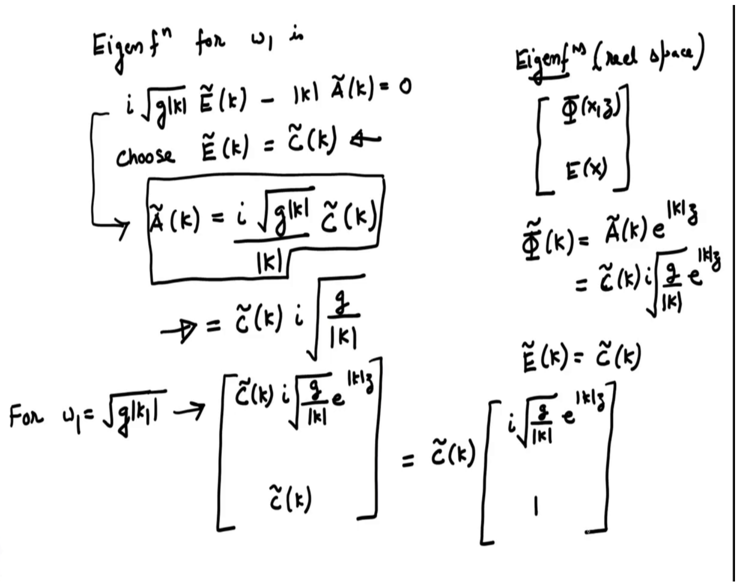

Recall that the eigen functions in real space, in real space by real space, i mean in x z space they were phi of x, z and E of x, we had multiplied this by e to the power i omega t, I am going to write down these things in the Fourier space. So, in the Fourier space we have found that phi tilde the equivalent of the phi in the Fourier space is just A tilde of k e to the power mod k z. And, we just found what is A tilde of k? So, this just becomes C, so, I am just going to use this expression for A tilde of k. So, C tilde of k into i times square root g by mod k e to the power mod k z.

What about E tilde of k? E tilde of k is just C tilde of k by this. So, my eigen functions corresponding to this omega 1 is just this C of k or C tilde of k i square root g by mod k e to the power mod k into z and C tilde of k. Once again, because C tilde of k is arbitrary I can pull it out and my eigen function is i square root g by mod k e to the power mod k z and 1.

Similarly, for omega 2; omega 2 is just minus of omega 1, you can find the corresponding eigen functions written in Fourier space.

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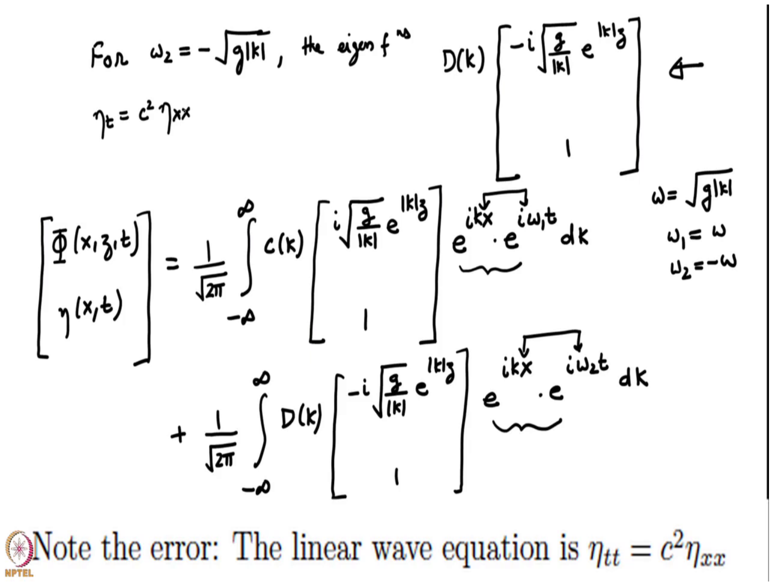

So, for omega 2 which is minus g mod k we have the eigen functions are; now we will put some arbitrary function D of k. And, you can show easily using the same thing that I showed in the last slide that it is just minus of this the first quantity just gets minus. What do we do with the these eigen modes? We know that the most general solution is written as a linear combination of the eigen modes.

In this case the linear combination actually allows k to go from all the way from minus infinity to plus infinity. So, we are going to do a Fourier integral. So, the most general solution is just the inverse Fourier transform of all of this. Use the definition of the inverse

Fourier transform that I had provided earlier and you can show that, the most general solution is just an integral C k.

The first eigen mode into e to the power i k x this comes from the inverse Fourier transform multiplied by e to the power i omega 1 t, this is the normal mode into d k this is an inverse Fourier integral, plus 1 by square root 2 pi minus infinity to infinity the second eigen mode. D k we have written it here write down the eigen vector, the eigen function here just the negative of that.

The first term is just the negative the second term remains 1. And, we have e to the power i k x that comes from the inverse Fourier transform and into e to the power i omega 2 t d k. You can notice something very interesting I can combine these two and I can combine those 2. If I write omega is square root g mod k, then you can immediately see that omega 1 is omega and omega 2 is minus omega.

So, you can see that the first exponentials in the first term will give you e to the power i k x plus omega t. And, these two will combine to give you e to the power i k x minus omega t. So, what is happening? It is telling us, that we are going to get travelling wave solutions. One wave is going from left to right; the other wave is going from right to left.

And, the solution will split up into two parts. Recall the wave equation the linear wave equation that we had seen, we had seen early in this course that the linear wave equation was eta t is equal to c square, eta x x. The linear wave equation if you do a normal mod analysis turns out to be a non dispersive equation.

So, the wave speed here is c independent of whichever Fourier mode you put in the system. So, you can try putting whatever Fourier modes you want in the system, whatever be the wavelength of the Fourier mode. There is a single phase speed c with which all waves propagate. These waves in deep water do not have the this property they are dispersive waves, each wave moves with it is own speed.

So, we will find that the solution to this problem. The solution for this eta will actually look far more complicated than the solution to the wave equation, later when we do shallow water approximation, we will find that in shallow water eta is governed by the wave equation.

But in deep water eta is not governed by the wave equation the expression for eta has to be written as an inverse Fourier integral. And, the physical content of that integral still contains left and right travelling waves, but those left and right travelling waves in general also change shape as they go along.

This is a property which is not present here; here the left and right travelling waves do not change shape. We will evaluate this solution in particular we will apply to the initial condition the very simple initial condition, that we had worked out where the interface was deformed as a single Fourier mode. And, we will apply this initial condition and we will work out what is the value of c of k and d of k for those initial conditions. And, we will find that the time evolution of phi and eta exactly gives us the same answer that we had found earlier.

So, this integral formulation actually generalizes what we know so far. It will allow us to solve for any initial condition, for which we can calculate the Fourier transform. If, we cannot calculate it analytically, we can at least do these integrals numerically.