Introduction to interfacial waves
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Lecture - 32
Standing and travelling waves in deep water

We were looking at linearized surface gravity waves in deep water. We found the pressure field corresponding to a deep water surface gravity wave and we found that it decays exponentially with respect to depth. The perturbation pressure field decays exponentially.

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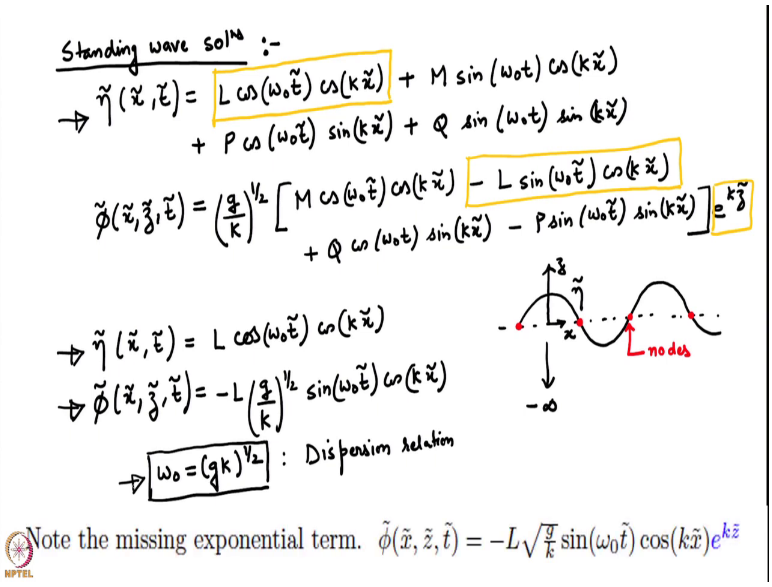

Let us look at the structure of the solutions that we have found so far. So, for I have just written it down here whatever we have found so far and I have written it as four separate parts. Recall that we had 4 constants of integration; L, M, P, Q and these constants also appeared in the expressions for phi and the perturbation pressure p. I have written everything in dimensional variables here.

Recall that we had also looked at once easy initial condition, where the interface was deformed just as a cosine wave; eta of x comma 0 was just cos k x tilde and we had found that the solution was very simple in time. The dimensional versions of the solution that we found is written here. So, eta is just a cos wave in space and it oscillates up and down in a harmonic manner in time with a frequency which is given by the dispersion relation. Similarly, phi is the velocity potential.

Recall that there was a negative sign if for these initial condition, there was a negative sign. Now, you can see that this initial condition which led to the solution is equivalent to just choosing these two terms which I have indicated in yellow boxes. All the other terms were 0 because M, P and Q turned out to be 0 for the initial condition.

Now, in the way I have written it for example, the expression for eta tilde, there are four terms. You can see that each of this term is what is known as a standing wave solution to the problem. What do we mean by a standing wave? So, if you imagine that eta of x comma 0 is given by this picture on the right, this picture on the right. So, I start with eta of x comma 0 being a cosine wave.

So, if you put time t equal to 0 in that formula, it is just a cosine wave with an amplitude L. L has the units of distance. So, it is just a cosine wave. Now, we found that in time, it will oscillate up and down with the frequency cos omega 0 t. It will go up and down harmonically in time and omega 0 is given by this relation.

So, what will with this wave do if I evolve it in time? It will just go up and down ok. In particular notice that there are these points which I have indicated in red, which are the places where cos k x vanishes at time t equal to 0. You will see that because time and space part are separate in this formulation, the places where cos k x vanishes initially, the wave will always continue to have a 0 displacement there at all times ok.

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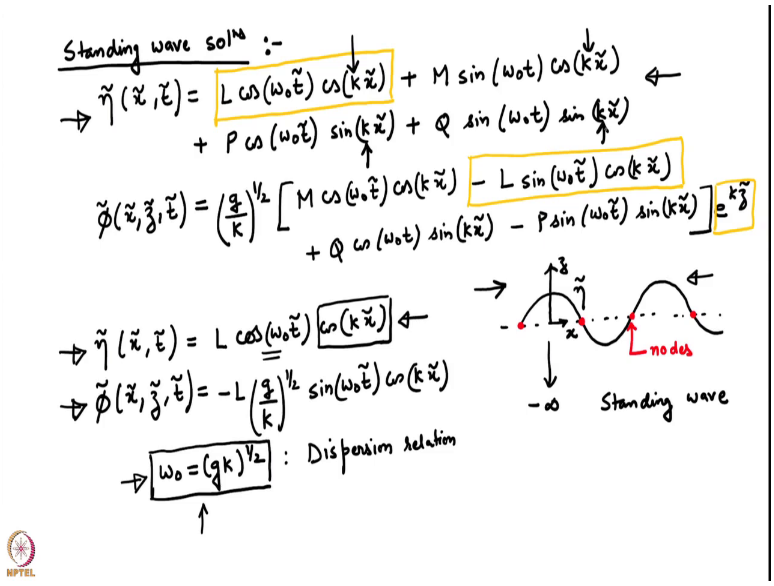

So, these points are called nodes. The resultant system behaves like what is called a standing wave. So, between two nodes, the wave will just go up and down harmonically in time with that frequency. This is the standing wave form. You can see that the solution that I have written eta of tilde at the top of the slide, each of the terms here, there are four terms, each term has a standing wave form the space and the time part is separate.

So, the places where the space part goes to 0, the expression will go to 0, the expression will be 0 at all times at those points. The sum of such, so each term here is a standing wave solution; the sum of them need not be a standing wave. So, we will study another kind of solutions to our equations. Until now, we have written down the general solution as a sum over standing waves. Let us now write what is known as a travelling wave solution.

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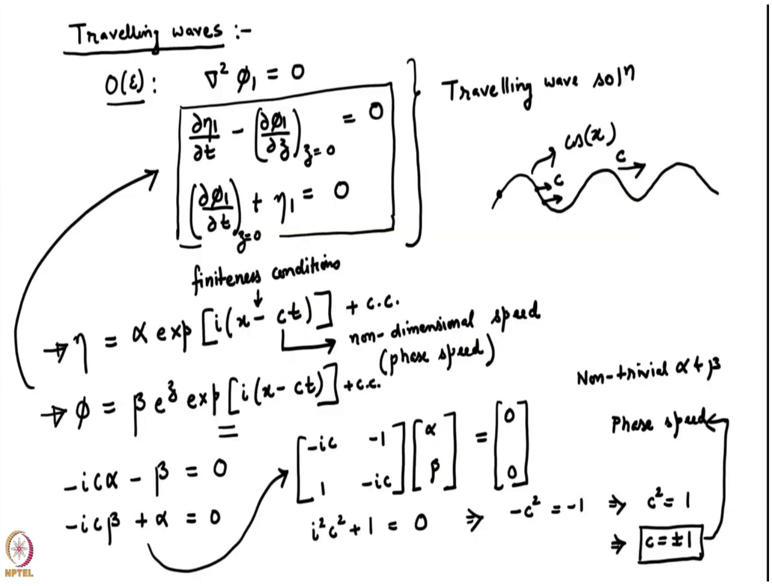

So, let us understand what is a travelling wave solution; travelling waves and we will see that our system also admits travelling waves. So, at order epsilon, our system was like before. I am just rewriting the equations that we had found. This is one of our boundary conditions, the linearized kinematic boundary condition at z is equal to 0 and then, in addition, we will also have finiteness conditions at x plus minus infinity and z minus infinity.

Now, we wish to look for a travelling wave solution to these set of equations. How do we look for it? So, we set eta is equal to some constant alpha. In general, it could be a complex constant into exponential of because it is this is all non-dimensional, I will write it as i x minus ct because t is non-dimensional, c is a non-dimensional speed.

We could have also chosen e to the power i x plus ct. Each of this such a solution will be a travelling wave solution. What does it mean? It means that if I draw the profile at time t equal to 0, so for example, if you take the real part of this expression at time t equal to 0, this is just e to the power i alpha x. If you take the real part, it is just cos i cos x. If at later time t, it will be cos of x minus ct.

So, it will be a wave which will travel to the right. If you take plus ct, the wave will travel to the left. Entire profile will travel without any change of shape. We will see the consequences of change of shape later on. Now, this is a non-dimensional speed. Notice that, there is no frequency which is appearing in the expression. When we look for standing wave solutions using method of normal mods, then we look for solutions which are proportional to e to the power i omega t.

So, omega was a non-dimensional frequency. When we dimensionalize it, we will found that it is equal to square root g k. Here, we are looking for a non-dimensional speed ok. We it will turn out to be what is known as the phase speed or the phase velocity of the system. Let us understand. So, this is the speed with which this profile will travel; it will just travel from left to right without any change of shape.

This is a cosine mode, cos x at time t equal to 0 ok. So, let us find now travelling wave solutions to our equation and in the process, we will find the equivalent of the dispersion relation. In the earlier example, we had found omega as a function of k. Here, we will find c as a function of k. In this case, once again c will turn out to be a constant because c is non-dimensional. Once we dimensionalize it, we will get the dimensional c as a function of k. Let us find that.

So, we will write phi is equal to some beta e to the power z into once again exponential of i x minus ct. The motivation for this comes like before phi has to be a solution to the Laplace equation. You can go back and substitute this form of phi and check that this solves the Laplace equation ok. You can also see that that e to the power z comes from the fact that from variable separation.

There would be e to the power z and e to the power minus z; we have to set the pre-factor of e to the power minus z to 0 in order to prevent divergence ok. Now, the essential difference between what was there earlier and what is here is that that here the x and the t both appear within the argument of the complex exponential.

Earlier, the argument of complex exponential contained only time, the space part was separate and was written as a real function either cos k x or sin k x and things like that. Here, the argument of the exponential contains both x and t ok. Now, let us find solutions and alpha and beta like usual are complex constants.

Let us now substitute into the these forms into the governing equations and the boundary conditions. I will leave the I will leave it to you as an exercise to convince yourself that for any value of beta and any value of c, this is a solution to the Laplace equation. So, I will not substitute it in the Laplace equation that is already satisfied, I will just have to satisfy the boundary conditions. So, let us do that.

So, the first boundary condition just tells minus ic alpha minus del phi by del z at beta. So, it is just beta; exponential of i x minus ct is just a cos or a sine, it is never 0 at all times, at all space and so, I eliminate that and this gives me one algebraic equation in alpha and beta.

If I take the second boundary condition, I will get the second algebraic equation. So, del phi 1 by del t that is minus i c beta plus alpha is equal to 0. So, you can think of this as two equations in two unknowns; two algebraic equations, once again the determinant has to be 0. So, this is minus i c, if I write it in matrix notation, minus i c the coefficient of alpha minus 1 1 minus i c into alpha beta is equal to 0.

Now, of course, I am not writing explicitly that you have to add a complex conjugate. You have to add a c c. ok. We will write that later and in order for non-trivial alpha beta, this is just like before, we do not want alpha beta both to be 0. If I substitute both to be 0, this is the solution to the equation.

So, I should write this as a matrix. This is just these two equations being written in matrix form. The determinant has to be equal to 0. You can readily see that the determinant is i square c square plus 1 is equal to 0. This just tells me minus c square is equal to minus 1 implies c square is equal to 1 or c is equal to plus minus 1. So, in non-dimensional variables, my phase speed; so, this is what is known as the phase speed ok. If I take any point on this wave, every point moves with the same speed which is c ok.

So, the profile as a whole travels as if it is a rigid profile and it just goes from left to right; it goes from left to right because we have used a minus here. I encourage you to try this analysis with this minus replaced with a plus and see whether you get a different dispersion relation. You will find that you get exactly the same dispersion relation. So, this equation admits left to right travelling waves; but the same set of equations also admit right to left travelling waves.

Now, these solutions that we are going to find are what are known as travelling wave solutions. Unlike the standing wave solutions, they do not have nodes. Nodes by definition are the point on the interface, if you are looking at eta for example, eta 1. Nodes by definition are the points, where eta 1 is always 0 at all times.

You can see that in this kind of a profile, at any time, there will be points on the interface, where the displacement is 0. But at the next instead of time the displacement at that point will not be 0 because the wave will be travelling and what is 0 now will not be 0 later on and some other point will become 0 at a later instant.

So, there will not be any nodes, when we have travelling wave solutions. In general, one can express because these are linear systems, one can express a sum of standing waves to obtain travelling waves and one can obtain use a sum of travelling waves to obtain standing waves.

Recall that in the previous example, we had looked at we had written eta as a sum of four parts, each of which is a standing wave solution. The sum could give us a travelling wave for example, ok. So, with that in mind, let us continue. So, we have found the dispersion relation. In this case, it tells us what is the phase speed because we are looking for a traveling wave solution.

This process is also equivalent to doing a normal mode analysis because we are still saying that the time dependence is exponential i to the power ct. In this case, it is e to the power minus i c t. But we are not instead of writing it as a frequency into time, we are writing it as a speed and that is because when something travels, then instead of coating its frequency, we would like to coat its speed.

For a standing wave, it makes more sense to talk about the frequency because the standing wave by definition, does not travel ok. These are travelling waves. So, it is common when we obtain dispersion relation and we are looking for travelling wave solutions, it is common to obtain c, the dimensional phase speed of a Fourier mode as a function of its wave number; l whereas, in the case of a standing wave solution, we would obtain omega as a function of k, they are interrelated to each other as we will see shortly.

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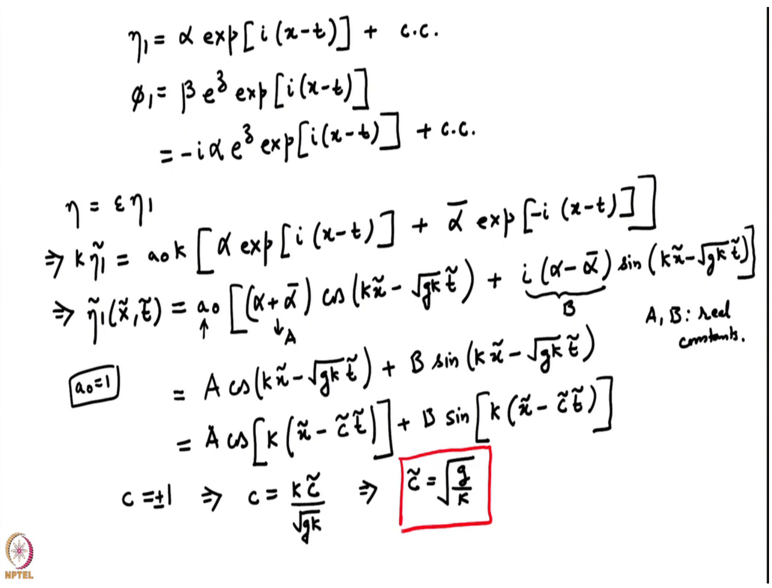

So, we have found the dispersion relation and so, eta 1 is alpha exponential of i x minus t; c is 1, we have just found it plus its complex conjugate phi 1, we have found that. So, phi 1 was beta e to the power z exponential of i x minus t and then, we have found that beta is equal to minus i c alpha.

So, beta is equal to minus i c alpha that is one of the equations that we have written here. So, you can see that if you use this equation to express beta in terms of alpha, then beta is minus i c alpha and c is 1. So, I can just write it as minus i alpha e to the power z exponential of i x minus t plus of course its complex conjugate.

Now, if we dimensionalize these expressions, then we get something physically little bit more meaningful and so, like before eta is k times eta 1 tilde. This is a naught into k into eta 1 which is alpha exponential of i of x minus t plus alpha bar exponential of the i has to be replaced by minus i x minus t and if you cancel out the in the k on both sides, then we obtain this is equal to a naught into.

So, we can do the same thing that we have done before, alpha plus alpha prime into cos x minus t. So, we will dimensionalize everything. So, alpha plus alpha prime cos its x is k x tilde minus square root g k t tilde ok plus i alpha minus alpha prime sin k x tilde minus square root g k t tilde.

Now, we can absorb this alpha bar is real and i times alpha minus alpha bar is also real. So, we can absorb this into 2 real constants and we can call them A and B and this a naught, small a naught like before can be set to be equal to 1 because I can absorb them into the constants capital A and capital B. So, this I am replacing it with capital A and this entire thing I am replacing it with capital B, where A and B are real. And this a naught, I am just setting it equal to 1 because I can observe it into capital A and capital B.

So, the answer looks like capital A cos of k x tilde minus root g k t tilde plus B sin k x tilde minus root g k t tilde. So, where A and B are real; A and B are real constants. Now, you can see that you can also write it as cos of I can pull the k out and I can write this as x tilde root g by k; root g by k, I am going to call it speed. You can convince yourself that square root g by k is a speed ok.

So, let us call it c tilde t tilde plus B sin k of x tilde minus c tilde t tilde; c t tilde has the dimensions of length. So, k into c into t tilde is dimension less and what is c tilde? We have seen that c is equal to 1 or c is equal to plus minus 1 ok. We are looking at the right travelling waves. So, plus minus 1, one will give you a left traveling wave, one will give you a right travelling wave. So, plus minus 1 and the relation would be this is a non-dimensional c.

So, we have to take the dimensional c and non-dimensionalize it. Velocity is length by time. The length scale is k, the time scale we have chosen to be root g k. So, this is the relation between c and c tilde c was 1 and so, this just tells us that c tilde is root g by k. So, this is the dispersion relation here, the dimensional dispersion relation and that is our expression for eta 1 tilde in terms of unknown constants A and B. Once again, A and B have to be determined from initial condition.

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$$\tilde{\phi} = \left(\frac{g}{k}\right)^{1/2} e^{k\tilde{z}}\left[A \sin\left\{k(\tilde{x} - \tilde{c}\tilde{t})\right\} + B \cos\left\{k(\tilde{x} - \tilde{c}\tilde{t})\right\}\right]$$

$$\tilde{\omega} = \sqrt{\frac{g}{k}} \rightarrow \text{Standing waves}$$

c

Notice the error: $\tilde{\omega} = \sqrt{gk}$

So, the corresponding solutions for phi 1 can also be shown to be equal to phi tilde or phi 1 tilde; phi tilde is equal to g by k to the power half e to the power k z into A sin k x tilde minus c tilde t tilde minus B cos again the same thing, k x tilde minus c t tilde.

Once again, you can work out the pressure field under a travelling wave ok. Here, we are writing the solution as a sum over two traveling waves. So, here if you look at the solution for eta, you can see that each piece is a traveling wave ok. So, now, this is what we have learnt so far.

Now, let us understand the connection between the c that we had found and the omega that we have found, we had found omega tilde to be equal to g by k and this was while doing standing waves. When we separated the space and the time part, then we got this.

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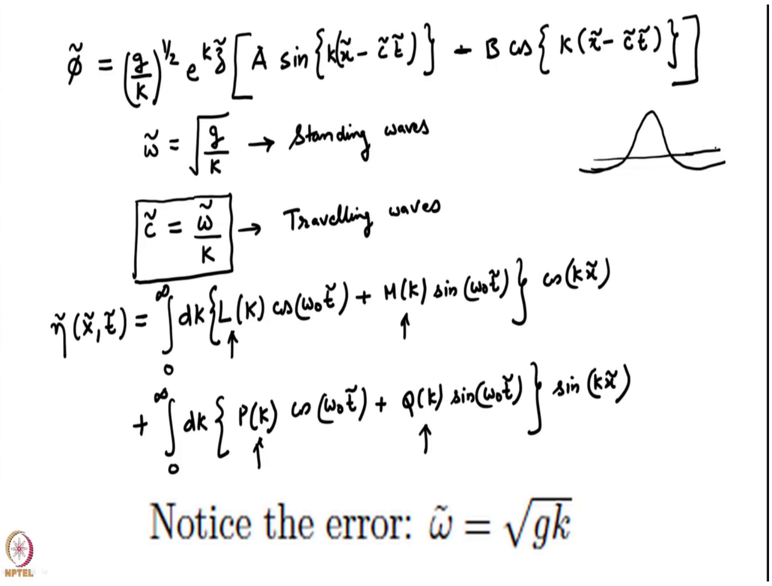

When we look for travelling wave solutions, then we also got the same thing omega tilde; but when we pulled out the k, then we got c tilde is equal to omega tilde by k and this we got for travelling waves. So, this is the relation c is equal to omega tilde by k; c is called the phase speed, omega tilde is called the frequency of the system and this is the relation between them.

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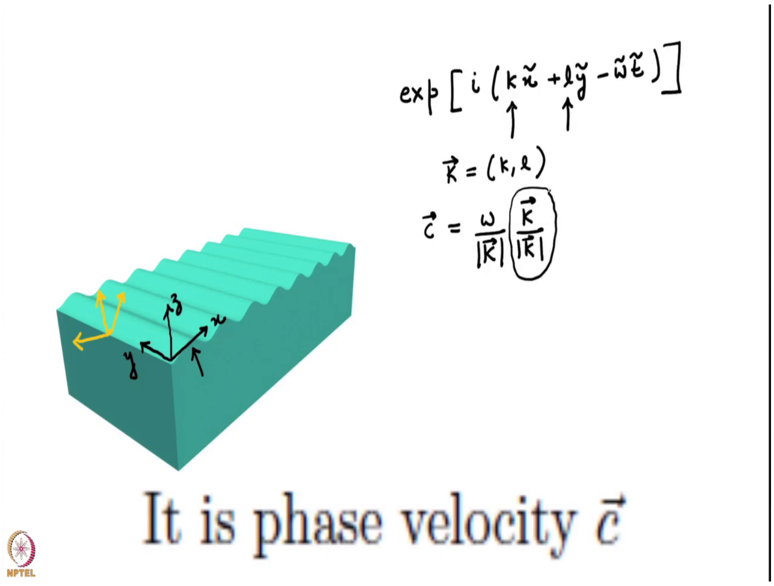

When we have when we are looking for waves in three dimensions. So, for example, until now, we have looked at waves which are two dimensional. So, we have perturbed our, we have chosen our coordinate system to be in this manner so that this was the x direction, this was the z direction and there was no variation in the y direction.

Suppose, I take the keeping the same perturbation the same, I choose another coordinate system. So, I will just shift it for better visualization. So, I am going to draw the coordinate system here. So, that is my origin; but instead of taking the x direction to be this, I choose the x direction to be that.

The y has to be orthogonal, so it would become something like this and the z would continue to remain this. You can immediately see that a eta would become a function of both x and y. In general, when we do normal mode analysis or when we look for travelling wave solutions, now our wave number will have two components; one along each direction.

Until now, our wave number had only one component and so, we wrote it as a scalar. Once our wave is not oriented along one of the coordinate axis, our wave number will have two directions. So, the wave number can be so our traveling wave solution can be in general written as exponential of i k x tilde; k is the wave number along the x direction plus l y tilde minus omega t tilde.

So, this k and this l; until now, l was 0 for us ok that is because our coordinate system was oriented like this. So, we can even three dimensional waves or we can have a superposition of many three dimensional surface waves. So, we have to define a wave number k, whose components are k and l and one can still extend the same notion of phase speed there.

And so, the phase speed becomes omega by k mod of the vector k into the vector k into mod of the vector. So, the phase speed is along a unit vector along the k vector. So, this is just a unit vector. The vector divided by its modulus just gives us a unit vector along that direction ok.

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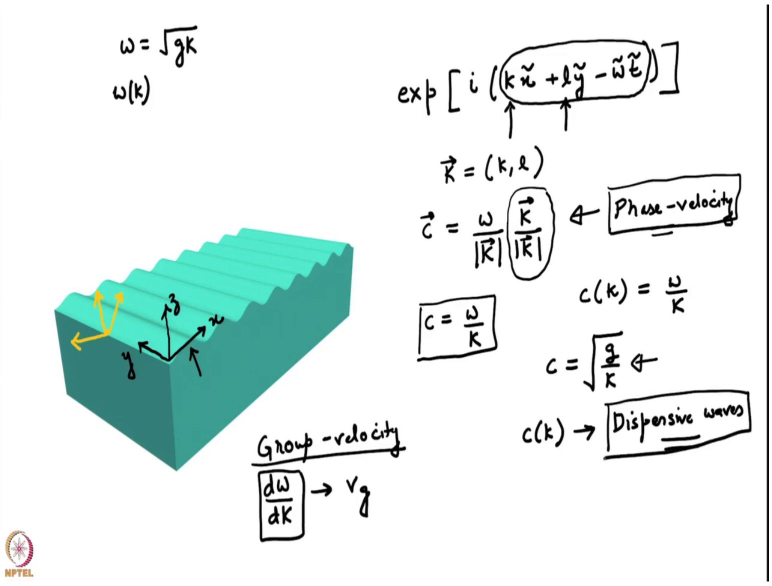

So, you can see that what we have been doing until now is just the one component. So, k, the vector k had was just small k for us. So, c was just omega by k. However, in more complicated problems you may have to look at a phase velocity which is a vector. So, that is why I am calling it a velocity.

We will not get into a two dimensional example right away, but this is just for your information. Now, is this. So, we have looked at a phase velocity, you can see why this is called a phase velocity because this is like the phase of the exponential and so, this is the speed with which this phase moves. You can think about drawing lines, you can think about on the in this three dimensional example on the x y plane, what would curves of constant phase look like?

You can find that they are straight lines; k x plus l y minus omega at any instant of time t naught would be the equation of equal to constant, would be the equation of a straight line ok. So, now, we have encountered this word called phase velocity. It tells us if you have a single Fourier mode in the system, how fast it is travelling. The speed of course c.

Let us go back to our two dimensional example, where eta is just a function of x and so, I do not have, so l is equal to 0. So, c is just a function of k and c is given by omega by k. Now, in general, we have seen or for our system c is given by g of square root g by k. So, c is a function of k, these kind of systems where the speed of the wave depends on the wavelength or the wave number are called dispersive waves.

They are called dispersive because waves, the wave speed depends on the wavelength. So, if you start with waves of many given wavelength at a given place at an instant of time and wait for some time, you will find that all of the waves because they have different speeds of propagation given by this formula. Then, after sometime, they will separate out.

The faster moving waves will be ahead, the slower moving waves will be behind. We will see an example of this this kind of behavior, when we do the Cauchy-Poisson problem. Having mentioned the phase velocity, another important thing that needs mentioning is the group velocity.

The group velocity is given by this is another kind of velocity which is meaningful for dispersive waves and that is given by d omega by d k. So, if you have the dispersion relation in this case omega is equal to square root g k. So, omega is a function of k, d omega by d k is what is given known as group velocity. Once we do the Cauchy-Poisson problem, I will give you a physical meaning of the group velocity.

This is in some sense, a more meaningful velocity than the phase velocity; especially, in linear dispersive systems. Systems which are where waves are dispersive that is all Fourier modes have their own distinct speeds and the speeds are different from each other, the speed depends on the wavelength and this group velocity is a more meaningful concept than phase velocity in linear dispersive systems.

So, with that, let us now go over to the next problem that is a very important problem. Until now, we have looked at one wave. So, we have looked at a single Fourier mode or things like that you know. So, for example, when we wrote down our expressions for example of the standing wave solutions, we had a single k in the system; this k, this k and of course, we wrote it as a super position of four different parts.

But in general, depending on initial conditions L, M, P and Q would be excited. It is clear that we need not have only one k present in our system initially. Now, because we are looking at a linear system, our system is still at order epsilon, it is still a linear system. If I have many k's, each of them will produce a resultant motion which will behave independently. I can solve using superposition because we are dealing with linear systems.

So, just by looking at the structure of this solution, I can already anticipate a more general solution to my equations of motion, where which will also take into account the fact that I can have many k's to start with in particular because of my boundary condition, because my domain is horizontally unbounded; I can have any k on the system.

So, I can instead of having a discrete number of k's initially or a countable infinite number of k's initially, I can have any number of k's in my system, I can have an uncountable infinite number of k's in my system initially and so, we will quickly see that this will lead us to the Fourier transform operation.

So, let us write down the intuitively, let us write down the general solution to our system. So, the general solution to our system is eta tilde, I am just going to write it for the variable eta tilde. You can write down an analogue solution for phi tilde and p tilde. The perturbation velocity potential and pressure, eta tilde is the perturbation of the surface.

So, in general, I will have four parts; L, M, P, K and those L, M, P, K are to be determined from initial conditions. But those initial conditions may involve a whole spectrum of Fourier modes which are present in the system. So, the constants which are there L, M, P, K in general can become a function k itself.

So, the because this is a linear system, I can write the general solution as a linear superposition because this is the system, where any value of k is allowed, it will not be a summation. Because summation is when I have a countable infinite number of k's. Here, it will be an integration.

So, I will integrate over all possible values of k from 0 to infinity dk. My L will become a function of k. So, instead of writing L as a constant, I am writing L as a function of k an unknown function as of now, cos omega 0 t tilde. I am just writing the answer as a linear superposition over all k's plus M k sin plus the second part.

So, there are two parts to in the first part already and then, the second part is also the same. We just have to take the constants of integration and consider them as unknown functions of k. So, I expect the structure of my most general solution to the problem to the to this order epsilon problem for eta to look like this. Here, it is clear that L of k, M of k, P of k and Q of k have to be determined once again from initial conditions.

You will soon see that these are related to the Fourier transform of the initial conditions ok. So, this is the more generic structure. This also brings us closer to real life situations, where I need not have a single Fourier mode in the system present initially.

I can have an infinite number of Fourier modes present in the system; in particular, I could have I could have a pool of deep water, where I give it a perturbation which is like this. It is some kind of a hump and I ask what happens next, if I deform the surface like this at time t equal to 0?

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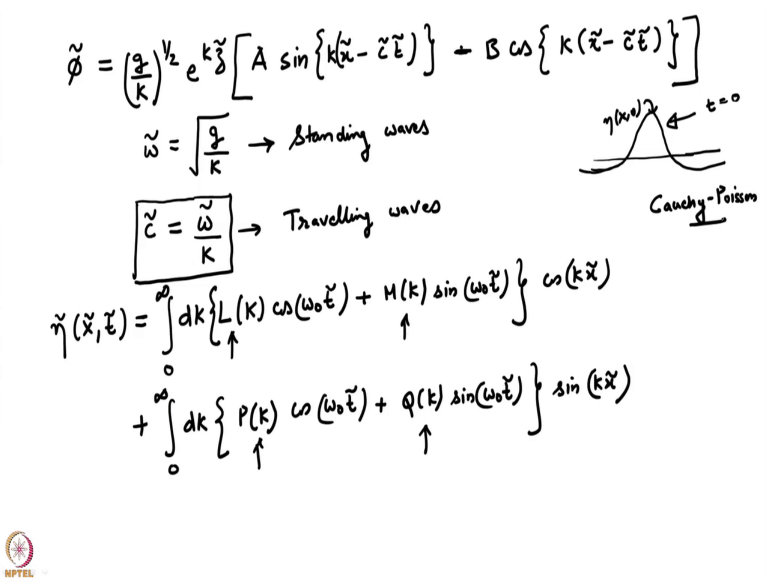

So, eta of x comma 0 is like this. I do not give it any initial velocity, but I deform the surface like this. We will soon see that using this framework of superposition and Fourier transform, it is possible to answer the question. In particular, whatever we have done until now will turn out to be a special case of the general framework that we are going to come up with. This framework is called the is this solution is called the Cauchy-Poisson problem, after the name of the two people who first propose the solution. So, we will go to the Cauchy-Poisson problem next.

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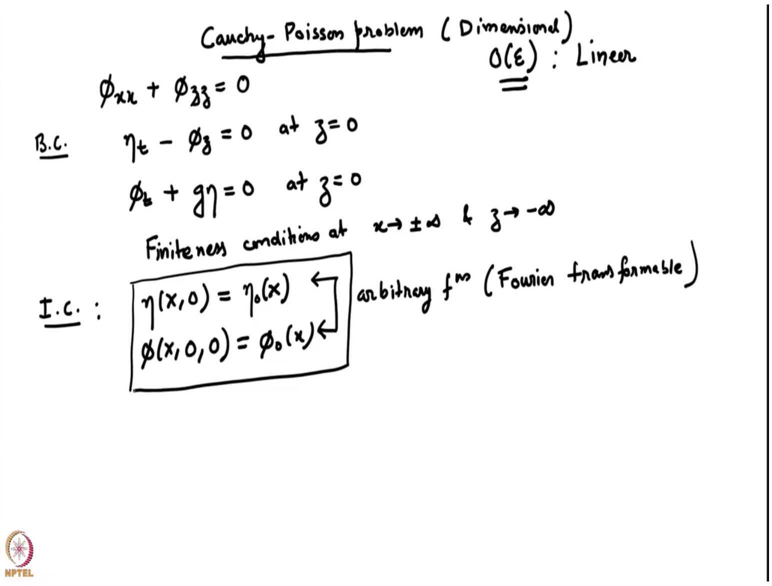

So, we will start with the Cauchy-Poisson problem. So, the Cauchy-Poisson problem is the name of an initial value problem. It basically says that if we put an initial deformation of the interface and if you put an initial impulse at the interface in the form of velocity potential, then what happens to the surface at later times, how does the surface behave at later times and what are the velocity fields which are created.

This is also famously referred to as the 'pebble in the pond problem'. You can see this very easily, if you throw a stone into a pond which is still, it creates circular ripples which spread out. You will soon see that this framework allows you to visualize the structure of those ripples and how they spread out.

To start with, we will do this problem in Cartesian geometry. The pebble in the pond problem is actually the corresponding cylindrical geometry problem, I will write down the solution to that problem later on. It also has the same analytical structure. So, once again, we are looking at waves in deep water in Cartesian geometry.

However, our initial conditions are more complicated. So, we have initial conditions in mind, where there is a surface hump or there could be a impulse velocity potential at the surface, it could be in general some arbitrary function of x. And so, we need to use the principle of superposition to write down the solution. So, our boundary conditions like before are I am going to use subscripts now onwards to indicate partial derivatives.

So, del eta by del t is just eta t minus phi z equal to 0, this is at z is equal to 0 that is my linear kinematic boundary condition and phi eta t plus just for the sake of simplicity, I am going to write down the governing equations. So, what I am going to write down are the corresponding dimensionalized expressions so for this system.

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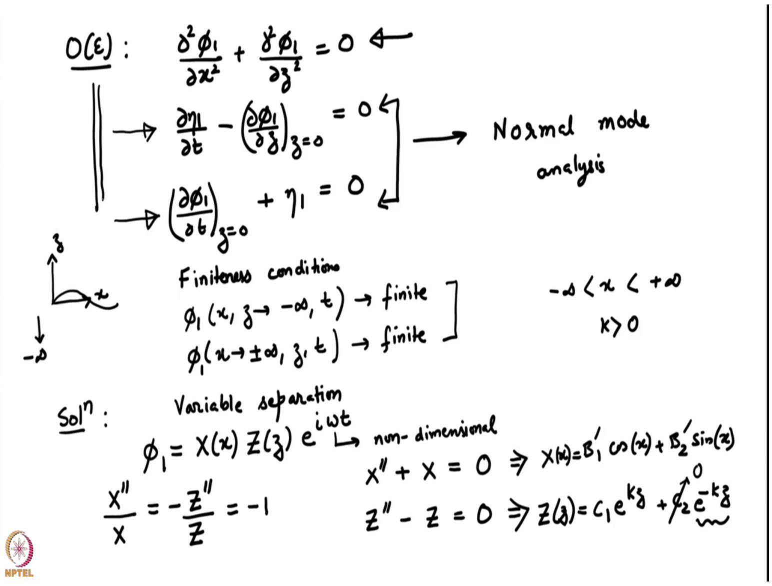

So, we had written down non-dimensional equations here, I am just going to dimensionalize it and write down and hence forward, we are not going to explicitly non-dimensionalize. What has to be done, when we are solving or at order epsilon will become clear as we go along. So, we will straight away solve the dimensional equations.

We linearize them, apply Taylor expansion and take the first term and we will just solve the dimensional equations, it will help us move faster. So, the linearized Bernoulli equation is just this, this is dimensional. So, I am solving all dimensional. So, the whole problem is being solved. You can think about it as being solved at order epsilon. So, this is linear.

So, all non-linear terms are ignored, all quantities which are to be evaluated at the unknown boundary are evaluated at z is equal to 0; we have seen this earlier and of course, we have the finiteness conditions. So, we will have to solve the equations with some initial conditions. As I said we are going to take the initial conditions to be general.

So, our initial conditions will be eta of x comma 0, instead of going to be a Fourier mode, a single Fourier mode is going to be some function f of x phi of x 0 0 at the surface and at time t equal to 0, we will specify a phi and this will be some phi naught of x. Let me call this eta naught of x. So, these two are some arbitrary, but well behave functions. In particular, they need to be functions which can be Fourier transformed in x ok.

So, Fourier transformable. We will use the method of Fourier transforms to solve for these initial conditions. We will find that the answer can be written as an integral, these integrals were first as Fourier integrals. These integrals were first written down by Cauchy and Poisson in a very famous solution to this problem and we see that whatever we have done until now, are just special cases of those integrals. We will continue in the next video.