Introduction to interfacial waves
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Lecture - 31
Linearised deep water surface-gravity waves (contd..)

We were looking at the order epsilon problem to deep water waves. We had found the following equations. Laplace equations subject to linearised boundary conditions and finiteness conditions.

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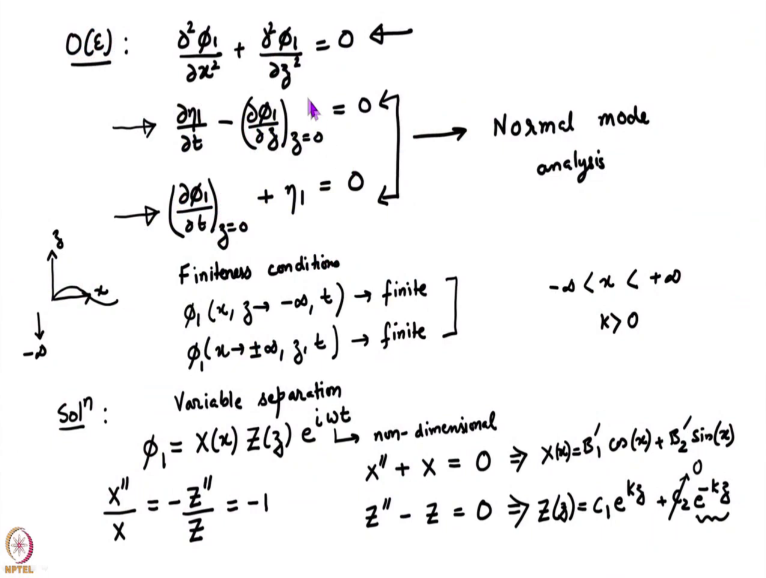

We use the variable separation technique and normal mode analysis, in order to obtain solutions to phi 1 and eta 1. Those look like the following.

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$$\leftarrow \phi_1 = \left[B_1' \cos(x) + B_2' \sin(x)\right] c_1 e^{z} e^{i\omega t} \qquad B_1 = B_1' c_1$$

$$B_2 = B_2' c_1$$

$$= \left[B_1 \cos(x) + B_2 \sin(x)\right] e^{z} e^{i\omega t}$$

normal modes

$$\leftarrow \eta_1 = \left[A_1 \cos(x) + A_2 \sin(x)\right] e^{i\omega t}$$

$$\underline{\text{B.C.}}: \quad \frac{\partial \eta_1}{\partial t} - \left(\frac{\partial \phi_1}{\partial z}\right)_{z=0} = 0 \rightarrow (1)$$

$$\left(\frac{\partial \phi_1}{\partial t}\right)_{z=0} + \eta_1 = 0 \rightarrow (2)$$

$$(1) \Rightarrow \quad i\omega\left[A_1 \cos(x) + A_2 \sin(x)\right] e^{i\omega t} + c.c_1 - \left[B_1 \cos(x) + B_2 \sin(x)\right] e^{i\omega t} + c.c_2 = 0$$

$$\Rightarrow \left[(i\omega A_1 - B_1)\cos(x) + (i\omega A_2 - B_2)\sin x\right] e^{i\omega t} + c.c. = 0 \rightarrow (3)$$

So, in particular phi 1 and eta, 1 look like this.

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$$② \Rightarrow \left[(i\omega B_1 + A_1)\cos(x) + (i\omega B_2 + A_2)\sin(x)\right] e^{i\omega t} + c.c. = 0 \;\hookrightarrow ④$$

$$③ \Rightarrow \boxed{i\omega A_1 - B_1 = 0} \quad *$$
$$③ \Rightarrow i\omega A_2 - B_2 = 0 \quad *$$
$$④ \Rightarrow i\omega B_1 + A_1 = 0 \quad *$$
$$④ \Rightarrow i\omega B_2 + A_2 = 0 \quad *$$

$$\underline{A_1}, \underline{A_2}, \underline{B_1}, \underline{B_2}$$

$$\begin{bmatrix} i\omega & 0 & -1 & 0 \\ 0 & i\omega & 0 & -1 \\ 1 & 0 & i\omega & 0 \\ 0 & 1 & 0 & i\omega \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$i\omega B_1 = -A_1$$
$$\Rightarrow -\omega^2 B_1 = -i\omega A_1$$
$$\Rightarrow \omega B_1 = iA_1$$
$$\Rightarrow \boxed{B_1 = \frac{iA_1}{\omega}}$$

$$B_1 = i\omega A_1$$
$$\Rightarrow \frac{i}{\omega} A_1 = i\omega A_1$$
$$\Rightarrow \boxed{\omega^2 = 1}$$

$$\omega^4 - 2\omega^2 + 1 = 0$$
$$\Rightarrow (\omega^2 - 1)^2 = 0$$
$$\boxed{\omega = \pm 1}$$

And once we imposed the two boundary conditions, we were able to obtain a determinant whose, in order for us to obtain non-trivial values of A 1, A 2, B 1, B 2, the determinant of this matrix was to be 0 and that gave us our dispersion relation, dimensionless variables. Omega is equal to plus minus 1.

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$$\eta_1 = \left[A_1 \cos(x) + A_2 \sin(x)\right] e^{i\omega t} + c.c. \qquad \omega^2 = 1$$

$$\phi_1 = e^{z}\left[B_1 \cos(x) + B_2 \sin(x)\right] e^{i\omega t} + c.c.$$

$$= e^{z} i\omega \left[A_1 \cos(x) + A_2 \sin(x)\right] e^{i\omega t} + c.c.$$

$$= e^{z} \omega \left[A_1 \cos(x) + A_2 \sin(x)\right] e^{i(\omega t + \pi/2)} + c.c.$$

$$\rightarrow \eta_1 = \left[(A_1 + \bar{A}_1)\cos(t) + i(A_1 - \bar{A}_1)\sin(t)\right]\cos(x)$$

$$+ \left[(A_2 + \bar{A}_2)\cos(t) + i(A_2 - \bar{A}_2)\sin(t)\right]\sin(x)$$

$$\rightarrow \phi_1 = e^{z}\left[(A_1 + \bar{A}_1)\cos(t + \pi/2) + i(A_1 - \bar{A}_1)\sin(t + \pi/2)\right]\cos(x)$$

$$+ e^{z}\left[(A_2 + \bar{A}_2)\cos(t + \pi/2) + i(A_1 - \bar{A}_1)\sin(t + \pi/2)\right]\sin(x)$$

And then we also found that eta 1 and phi 1 have this form. We further expressed, in the process of expressing it in terms of real functions, we combined all the complex constants and wrote it like this. Let us now continue from here. So, what we are going to do now is you can see that the expression for eta 1 and phi 1, have these coefficients which are all real, A 1 plus A 1 bar, A 2 plus A 2 bar, i times A 1 minus A 1 bar, and i times A 2 minus A 2 bar. These are all real. So, let us express this in terms of real variables.

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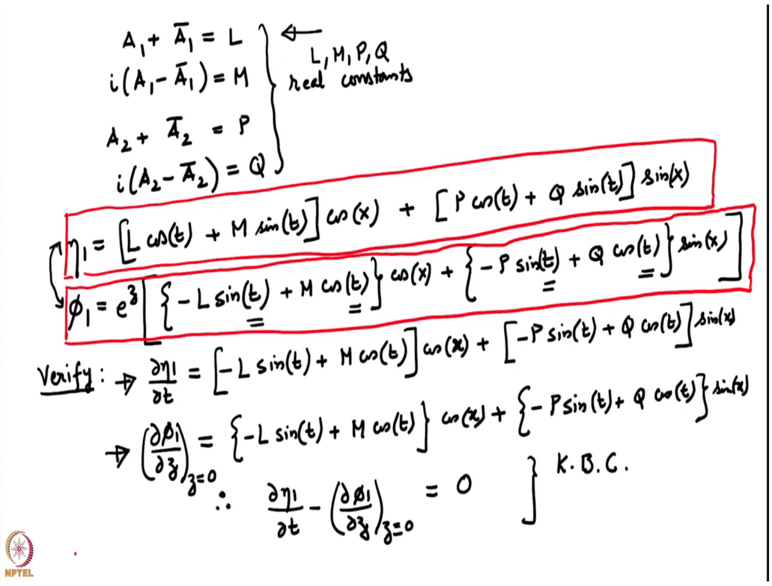

So, I have done that here. So, I have said that A 1 plus A 1 bar is equal to L, some real constant. Similarly, i times A 1 minus A 1 bar is equal to M, A 2 plus A 2 bar is equal to P, and i times A 2 minus A 2 bar is equal to Q. So, L, M, P, Q are all real constants. L, M, P, and Q are all real constants.

In terms of L, M, and P, Q, one can simplify the expressions that we had written. So, all the coefficients get expressed in terms of L, M, and P, Q. Further for phi 1, we had a phase factor of pi by 2, which caused the argument of cos’s and sin’s in the expression for phi 1 to depend on t plus pi by 2. We can simplify all of this using the formula that we know and write our final answer in this form.

So, the first term becomes minus sin t cos t, minus sin t cos t. So, there we obtain the final expressions for eta 1 and phi 1, expressed in totally real notation, ok. So, now, let us quickly verify that these solutions indeed are solutions to our equations. It is easy to verify that the expression for phi 1, so it satisfies the Laplace equation, ok. You can try that for yourself. I will just show you the verification for the boundary conditions with the expression for eta 1 and phi 1 that we have.

The expression for del eta 1 by del t is just given by this. Del phi 1 by del z is just given by that. And so, you can just check from that they are the same expressions. And so, we have verified that del eta 1 by del t minus del phi 1 by del z evaluated at z is equal to 0. So, this derivative was calculated at z is equal to 0, is equal to 0. This takes care of the linearised kinematic boundary condition.

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$$\left(\frac{\partial \phi_1}{\partial t}\right)_0 = -\left[L \cos(t) + M \sin(t)\right] \cos(x) - \left[P \cos(t) + Q \sin(t)\right]$$

$$\left(\frac{\partial \phi_1}{\partial t}\right)_0 = -[L\cos(t) + M\sin(t)]\cos(x) - [P\cos(t) + Q\sin(t)]\sin(x)$$

Let us validate the pressure condition or the Bernoulli equation applied at the interface. So, we have del phi 1 by del t at z is equal to 0. This we need to work out. So, del phi 1 by del t is given by; so, I will ignore the exponential factor because that is just 1, because it is evaluated at z is equal to 0.

And so, we get minus L sin t, and I am going to pull the minus out plus M, sorry, this is going to be L cos t plus M sin t. Overall there will be a factor of minus if you look at the expressions. And similarly, here there will be an overall factor of minus and this would become P cos t minus Q or plus Q sin t. And there is no exponential because that is just 1.

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$$\left(\frac{\partial \phi_1}{\partial t}\right)_0 = -\left[L\cos(t) + M\sin(t)\right]\cos(x) - \left[P\cos(t) + Q\sin(t)\right]\sin(x)$$

$$\eta_1 = \left[\underset{\uparrow}{L}\cos(t) + \underset{0}{M}\sin(t)\right]\cos(x) + \left[\underset{0}{P}\cos(t) + \underset{0}{Q}\sin(t)\right]\sin(x)$$

$$\left(\frac{\partial \phi_1}{\partial t}\right)_0 + \eta_1 = 0 \quad : \text{Lin. B.E.}$$

$$\boxed{L, M, P, Q} : \text{Initial conditions}$$

$$\rightarrow \boxed{\eta_1(x,0) = \cos(x)}, \quad \boxed{\phi_1(x,0,0) = 0} \leftarrow$$

$$\eta_1(x,0) = L\cos(x) + P\sin(x) = \cos(x)$$

$$= (L-1)\cos(x) + P\sin(x) = 0 \Rightarrow L = 1, \; P = 0$$

$$\phi_1(x,0,0) = M\cos(x) + Q\sin(x) = 0 \Rightarrow M = Q = 0$$

$$\boxed{\eta_1 = \cos(t)\cos(x), \quad \phi_1 = -e^{z}\sin(t)\cos(x)}$$

Eta 1 is L cos t plus M sin t into cos x plus P cos t plus Q sin t. There is a sin x here which I have missed, t into sin x. By comparing these two expressions you can immediately verify the linearised Bernoulli equation is satisfied. It, one expression is just negative of the other. So, we have del phi 1 by del t and 0 plus eta 1 is equal to 0. This is our linearised Bernoulli equation, ok.

So, we have verified now that our expressions satisfy the boundary conditions. You can check that the expression for phi, also satisfies the Laplace equation. And of course, everything is bounded when z goes to minus infinity and when x goes to plus infinity as well as minus infinity. So, we have solved our system.

Now, the question arises how do we determine these constants L, M, P, Q. Like before these are determined from initial conditions. Let us take a very simple example. Suppose, we perturbed our interface initially in the form of let us say a cosine mode and we gave an initial impulse to the interface. So, I will specify my initial conditions.

My initial conditions are that eta of x comma 0 is or eta 1 rather is cos x and phi 1 of x, 0, 0. So, this is a surface impulse. We are only specifying the value of phi 1 at the undisturbed surface z is equal to 0. And we are just saying that it is 0. So, you can think of phi 1 as the equivalent of a giving a velocity initially.

So, initially, it is just a displacement of the interface with no velocity anywhere, either on the interface or below. How do these initial conditions determine the value of L, M, P, and Q? Let us first substitute these initial conditions into our expressions. If we do that then we obtain.

So, we first obtain, so eta 1 of x comma 0 is from our expressions we have to put time t equal to 0 in our expression for eta 1 here. You can immediately see some terms go to 0 and some terms do not. So, I am just going to write down the terms which do not go to 0. So, the terms which do not go to 0 are L cos x plus P sin x. By the initial condition this is equal to cos x.

This implies that L minus 1 cos x plus P sin x is equal to 0. Once again because sin x and cos x are linearly independent, their coefficients have to vary, at have to be equal to 0 because in general they are linearly independent. So, their coefficients have to be individually set to 0, in order for to satisfy this expression.

This implies we have L is equal to 1, P is equal to 0. So, I have used the interfacial displacement condition and I have obtained the value of L and P, P is 0, L is equal to 1. What about the value of M and Q? Those will come from the second condition. So, once again phi 1 of x at z is equal to 0, at time equal to 0, let us obtain. So, for that we just go to the other expression, this expression, and we substitute z is equal to 0 and t is equal to 0 in this expression.

So, once again e to the power 0 will just be unity and when you substitute time equal to 0, some terms will go to 0. We will write down what is left. So, it would be M cos x plus Q sin x is equal to 0. This basically once again by the same arguments M is equal to Q is equal to 0. So, this is telling us what are the values of L, P, M, Q.

Let us substitute and find out. If we substitute then we find eta 1 is L is equal to 1, and M and P are 0. So, this is 0, M is 0, this term is 0, P is 0, and Q is also 0. So, we just have cos t cos x. Phi 1 is equal to with the exception of L everything is 0. So, we go back to this expression and we find that with the exception of L all the other terms are 0. So, it is just minus e to the power z, L is 1, sin t cos x.

Minus e to the power z sin t cos x. This is our solution to the initial value problem where the initial conditions are those, this and that. And our initial conditions have determined the value of L, M, P, Q for us. Obviously, this is not the most general solution. We will write the most general solution shortly.

And we will look at this process of obtaining the most general solution in terms of initial conditions also in some detail. But let us dimensionalize these expressions in order to get a physical field for what is the frequency and what do the velocity potentials and the interface displacements look like in the general case. So, let us redimensionalize our expressions.

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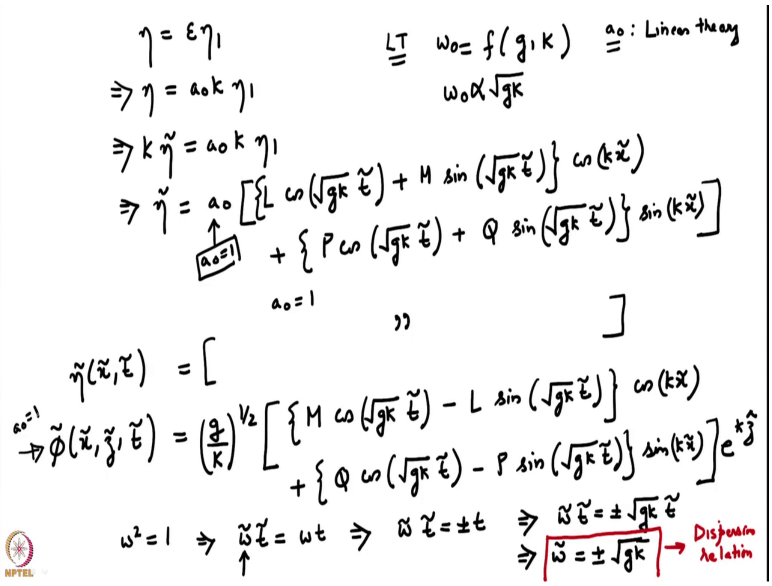

So, we know that we had written while in perturbation we had written eta as epsilon eta 1 plus epsilon square eta 2. Because this is an order epsilon calculation, so eta is just epsilon eta 1. And then, eta is equal to epsilon is a 0 into k into eta 1, and eta was k, eta tilde, eta tilde is the dimensional eta. And so, this is equal to into eta 1, one k cancels out. And so, we have a 0. And we have just found out what is the expression for eta 1.

So, eta 1 is L cos t or L cos, let me write it in terms of dimensional quantities. So, the, so t is square root gk into t tilde, where t tilde is now dimensional plus M sin square root gk t tilde, this whole thing multiplies cos k x tilde plus P cos, the same thing. Now, you can see that all these variables are going to get multiplied by this a naught.

So, I can L is a real quantity L, M, P, Q, these are all real constants a naught is an amplitude it is also a real constant. So, I can multiply this and I can set up new constants instead of doing that I will just set a naught equal to 1. And it will just give me my expression, it is just the expression in the bracket. So, I will not write it again, it is just the expression in the square bracket. So, this is my expression for eta tilde as a function of x tilde and t tilde, with some unknown constants sitting in the expression.

Similarly, we can obtain expression for phi tilde as a function of x tilde, z tilde, and t tilde. And you can go and see yourself that it just turns out to be g by k to the power half; so, k x tilde plus and of course, the whole thing gets multiplied by e to the power k z tilde. So, these are our expressions for the interface displacement as a function of time and the velocity potential.

One can also express the dispersion relation. So, our dispersion relation non-dimensionally was omega square is equal to 1 or omega is equal to plus minus 1. We know that omega tilde which is a dimensional omega into t tilde should be equal to the non-dimensional omega into the non-dimensional t.

Why? Because the argument of exponential is always non-dimensional. So, whether we write it as e to the power i omega t or whether we write it as e to the power i omega tilde t tilde it is the same. So, the product of omega into t is non-dimensional. Now, if omega is dimensional, then t is dimensional. If omega is non-dimensional, t is also non-dimensional, ok.

So, and we now want to find out what is the dimensional omega tilde, so we just use that. So, so this is t tilde is equal to plus minus t because omega is plus minus 1, and the relation between t tilde and t is just this. So, the t tilde cancels out on both sides, so we get our dispersion relation gk. I will put this in a red box.

We have determined the expressions up to order epsilon for eta, phi, and omega. The dispersion relation tells us that we can choose any k we want, k is related to the wavelength of the perturbation that we put at the surface. Now, we can put any k we want, k actually goes from 0 to infinity and any k is allowed because the domain is horizontally unbounded.

However, if we put a given k, the system vibrates at only these frequencies which are given by omega is equal to plus minus gk. You can see that square root gk is the has the dimensions of 1 by time. So, this is a frequency. Now, you could have argued this from dimensional arguments also, ok. So, you can see that this frequency one can compute a time period corresponding to this frequency. So, the time period is just omega tilde is equal to 2 pi by t tilde and you can get the time period.

Now, you can see that if you say that the time period of a surface gravity wave depends; what can it depend on? So, you can see that the restoring force is because of gravity. So, the time period must be a function of gravity. In general, you would expect it to depend on wavelength, so the lambda is also there.

And so, and there is another length scale which is a naught. This is a linearised theory, and so, in a linear theory we do not expect the amplitude of perturbation to depend to affect the time period, ok. So, a naught would drop out. And so, you can ask yourself that the time period, if it is a function of g and k or lambda whichever you want to write it as, k is just 2 pi by lambda.

So, you can if you want, so let us write it in terms of omega naught. So, the frequency of the wave, if it is just a function of g and k, it cannot depend on a naught because this is a linear theory. We are borrowing ideas that we learnt earlier. We saw that whenever we look at linearised oscillations the amplitude of perturbation does not appear in the expression for the frequency.

However, there is a non-linear correction where it does appear. The same will turn out to be true here also. When we do this later for a stokes wave, we will find that this omega tilde is equal to plus minus gk, is actually just an approximation, and there is a correction and the correction depends on this a naught. This a naught determines the amplitude of the surface perturbation that we are putting.

So, by a linear theory you would expect this. And you can see that the only quantity that we can calculate which has the correct dimensions as omega 0, would be just square root gk. So, one would expect omega 0 to be proportional to some square root gk, ok. So, the constant of proportionality in this case turns out to be unity, ok.

So, now, we can go further, and we can ask ourselves what is the pressure field when we have a small amplitude wave at the surface. We have already worked out the velocity potentials, so from this, we can anticipate what is the velocity field we just have to take the differentiation of these expressions with respect to x and z. And we can choose simple initial conditions like the way we did earlier.

So, we have let us say a single cosine mode initiated at time t equal to 0, with no impulse at the surface. Then, we have eta 1 and phi 1 which are given by these simplified expressions. You can dimensionalize these expressions and then find out the velocity fields under this wave. Let us now calculate the pressure field.

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Pressure: $\tilde{p}_b = -\rho g \tilde{z}$ $\qquad p_b = \dfrac{\tilde{p}_b}{\left(\frac{\rho g}{k}\right)}, \; z = k\tilde{z}$

$\Rightarrow p_b = -z$

$\phi = 0 + \epsilon \phi_1$

$\eta = 0 + \epsilon \eta_1$

$p = -z + \epsilon p_1$

perturbation pressure field

B.E.

$p + \dfrac{\partial \phi}{\partial t} + \dfrac{1}{2}|\nabla \phi|^2 + z = 0$

$\Rightarrow O(\epsilon): \; p_1 + \dfrac{\partial \phi_1}{\partial t} = 0$

$\Rightarrow p_1 = -\dfrac{\partial \phi_1}{\partial t}$

$p = -z + \epsilon p_1$

$$\tilde{p} = -\rho g \tilde{z} - \rho g e^{k\tilde{z}}\left[\left\{L\cos(\sqrt{gk}\,\tilde{t}) + M\sin(\sqrt{gk}\,\tilde{t})\right\} \times \cos(k\tilde{x}) + \left\{P\cos(\sqrt{gk}\,\tilde{t}) + Q\sin(\sqrt{gk}\,\tilde{t})\right\} \times \sin(k\tilde{x})\right]$$

Note that in the second term on the R.H.S for $\tilde{p}$, a_0 has been set to unity

So, we will come to pressure now. Recall that in the base state pressure is the only quantity here which is non-trivial. The pressure variable in the base state is hydrostatic. So, if I use a tilde for the base state variable, dimensional base state variable, then this is minus rho g into z tilde. This minus is just to take into account that as z becomes more and more negative, the pressure increases linearly with distance.

We will define, we will non-dimensionalize this, and so our non-dimensional variable will be p b without the tilde and that will be using a hydrostatic pressure variation. So, rho g by k. 1 by k has the dimensions of length, so this is just a hydrostatic pressure field. So, if I substitute this z is anyway non-dimensionalized as the following like before.

And so, if I substitute this, then my non-dimensional base state variation just turns out to be minus z. So, we will have to do a similar expansion for pressure just as we did for everything else. For phi and for eta, our base state contribution was 0 and 0, and then we had an epsilon phi 1 and an epsilon eta 1.

Similarly, pressure will also have a similar expansion. However, we have to remember that the expansion has to start at minus z because in the base state the pressure field is not 0. If we do that, then there is epsilon p 1. And our intention is to determine this pressure field p 1, this is the perturbation pressure field. We have already determined the perturbation phi 1 and the perturbation eta 1. So, we now want to determine p 1.

Let us look at it. That is very easy. We will just use the Bernoulli equation. So, we have the Bernoulli equation is p plus del phi by del t. Now, we are writing the Bernoulli equation anywhere in the field. We are not necessarily at the surface. Plus half grad phi square plus z is equal to 0.

This is our Bernoulli equation that we wrote earlier. If I substitute this expansion here, then you can immediately see all the 3 expansions have to be substituted, then you can immediately see that the minus z will cancel out the plus z. And so, I will have at order epsilon, I will have just p 1, and there will be a contribution from del phi by del t which will be this.

This term will contribute only at order epsilon square. It will not contribute at order epsilon as we have seen before, and this term is cancelled out by the base state contribution. So, our perturbation pressure field is just expressed like this p 1 is equal to minus del phi 1 by del t. And so, if we know phi 1, we can just differentiate it with respect to time and get p 1. You can immediately see that because phi 1 is depends exponentially on z, p 1 will also depend exponentially on z.

What this implies is that, that at linear order the perturbation pressure decays exponentially. This is different from the hydrostatic variation that we have seen until now in the base state. So, if you have a, if you have a wave in deep water, then typically the scale up to which the perturbation pressure is felt is of the order of the wavelength of the wave. This is also one of the reasons these are called surface waves because their effects do not permeate deep into the fluid.

Now, you can express this. And I am just going to write down the full dimensional expression. It is the same procedure. We write p as minus z plus epsilon p 1, and then dimensionalize. If we do that, then you will get p hat or rather p tilde minus rho gz tilde minus rho g, there will be a naught, but I am setting a naught to 1. I have done the same thing in the expression for phi also. There was an a naught here and just like eta I said that a naught is equal to 1 because the a naught can be absorbed in these constants L, M, P, Q.

So, similarly in the expression for phi, I have said a naught equal to 1, because it is equivalent to absorbing it in M, L, P, Q. So, a naught times M is another M like that. So, I am not going to write down the a naught. I said the a naught equal to 1. And then we have rho g exponential of k z tilde into L cos root gk t tilde plus M sin root gk t tilde.

I have just taken this differentiation with respect to time and this whole thing multiplies cos k x tilde plus P cos Q sin square root gk t tilde. And this whole thing multiplies sin k x tilde, then close the square bracket. So, this is our expression for the total pressure field.

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Pressure: $\tilde{p}_b = -\rho g \tilde{z}$

$$p_b = \frac{\tilde{p}_b}{\left(\frac{\rho g}{k}\right)}, \quad z = k\tilde{z}$$

$$\Rightarrow p_b = -z$$

$$\phi = 0 + \varepsilon \phi_1$$

$$\eta = 0 + \varepsilon \eta_1$$

$$p = \underset{\uparrow}{-z} + \underset{\uparrow}{\varepsilon p_1}$$

perturbation pressure field

B.E.

$$p + \frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + z = 0$$

$$\Rightarrow O(\varepsilon): \; p_1 + \frac{\partial \phi_1}{\partial t} = 0$$

$$\Rightarrow p_1 = -\frac{\partial \phi_1}{\partial t}$$

$$p = -z + \varepsilon p_1$$

$$\tilde{p} = \underline{\underline{-\rho g \tilde{z}}} - \boxed{\rho g e^{k\tilde{z}} \Big[\big\{ L\cos(\sqrt{gk}\,\tilde{t}) + M\sin(\sqrt{gk}\,\tilde{t}) \big\} \times \cos(k\tilde{x}) + \big\{ P\cos(\sqrt{gk}\,\tilde{t}) + Q\sin(\sqrt{gk}\,\tilde{t}) \big\} \times \sin(k\tilde{x}) \Big]}$$

There is a base state pressure field here and this entire thing is the perturbation pressure field. So, this completes our solution to the problem.