Introduction to interfacial waves
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Lecture - 30
Linearised wave equations in deep water: dispersion relation

We had started looking at interfacial waves. Recall that our base state was 1 where the interface was flat and the fluid underneath was quiescent. We had started our analysis by making a number of simplifying assumptions among them was first, we had assumed that the medium would be in compressible.

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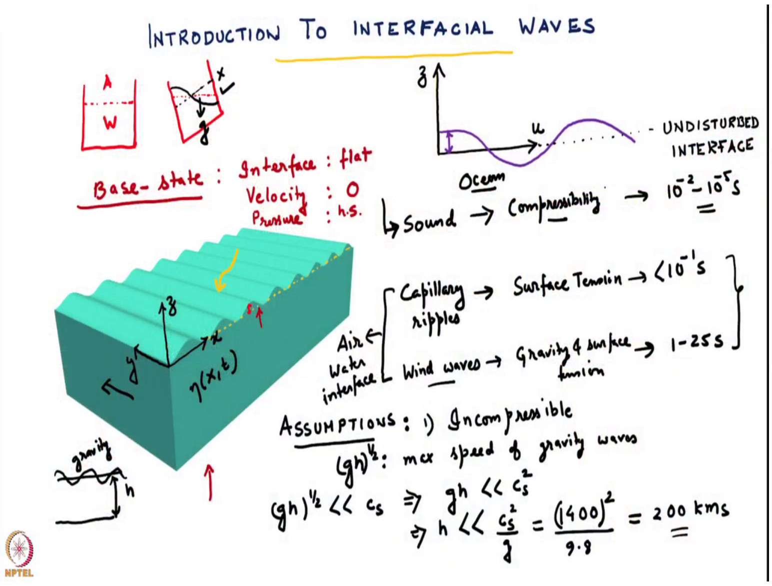

This would be true when the maximum speed of gravity waves, we were looking at surface gravity waves to start with would be much lower than the speed of sound.

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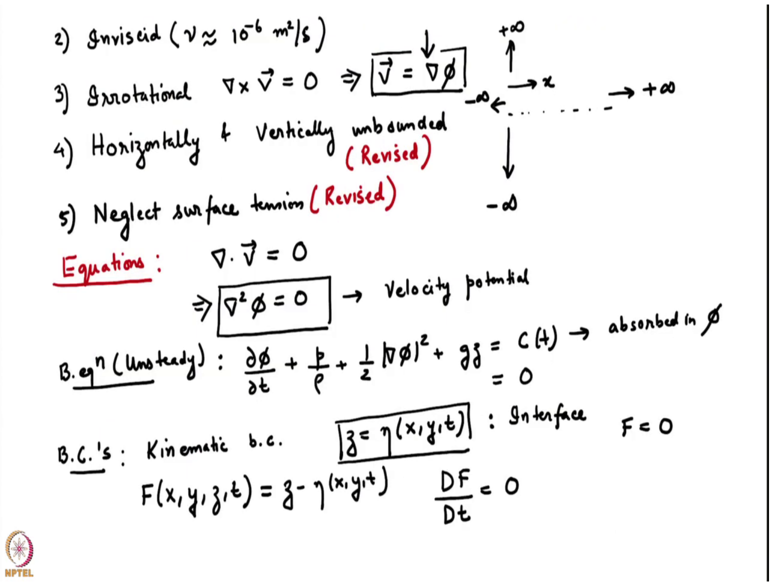

Then, we also looked at the then, we also made the assumption that the medium would be assumed to be inviscid, the motion would be irrotational and the domain would be horizontally and vertically unbounded and as a first step, we would neglect surface tension. We mentioned that we could revisit the horizontal and vertical unboundedness assumption later on and we would also include surface tension later in our analysis.

Now, with all these approximations, we were led to the Laplace equation for the velocity potential phi, the pressure field was governed by an unsteady Bernoulli equation, we also had boundary conditions among them was a new boundary condition that we learnt how to derive, it was called the kinematic boundary condition, it essentially is a statement that the interface or the free surface is a material surface and so, it is an additional statement of mass conservation.

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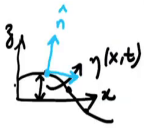
KBC

$$\frac{DF}{Dt} = 0 \qquad F = z - \eta(x, y, t)$$

$$\Rightarrow \left[\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right] F = 0 \quad \text{on } z = \eta(x, y, t)$$

$$\hat{n} = \pm \frac{\nabla F}{|\nabla F|}$$

$$\Rightarrow \frac{1}{|\nabla F|}\frac{\partial F}{\partial t} + \vec{u}\cdot\left(\frac{\nabla F}{|\nabla F|}\right) = 0$$



$\eta(x)$

$$\Rightarrow \frac{1}{|\nabla F|}\frac{\partial F}{\partial t} + \vec{u}\cdot\hat{n} = 0$$

$\leftarrow$ Steady case

$$\Rightarrow 0 + \vec{u}\cdot\hat{n} = 0$$

$$\vec{u}_2\cdot n = \vec{U}_1\cdot n$$

$$\Rightarrow \frac{\partial F}{\partial t} + (\vec{u}\cdot\vec{\nabla}) F = 0 \Rightarrow \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$$

$$\Rightarrow u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = \frac{\partial \eta}{\partial t} \quad \text{on } z = \eta$$

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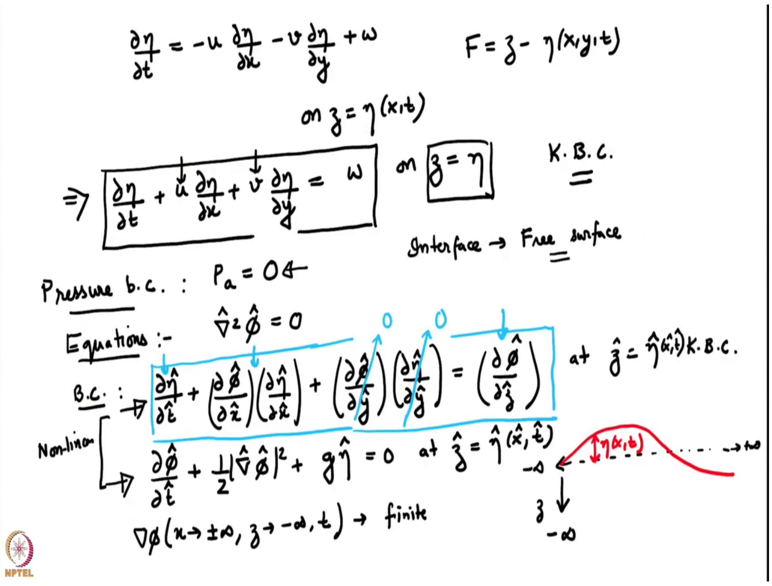

We looked at the kinematic boundary condition and then, we also looked at the pressure boundary condition. Here, we said that we are ignore, we are going to ignore the gaseous medium above, so, we are going to ignore the air as a first step and so, the motion in the air is negligible so, the air is quiescent and it only exerts a pressure on the fluid below it, that pressure can be assumed to be 0 to start with.

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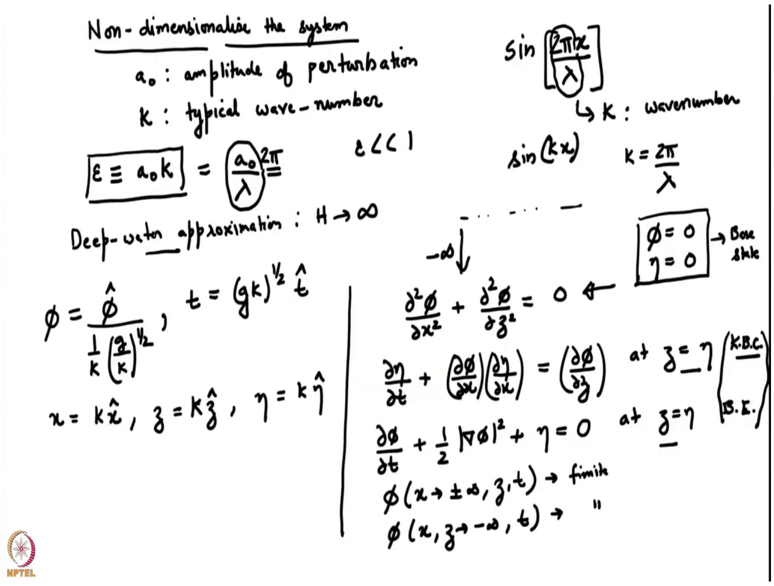

Then, we looked at the boundary conditions and then we non-dimensionalize our system. Our choice of length scale was k inverse, k is a typical wave number of a interfacial perturbation that we would put on the system. Then, we chose the time scale as square root g k to the power half or rather this was the frequency scale and I said that we would revisit these scales later on and try to understand them meaningfully. So, then we plug these scales into the governing equations, and we obtained our non-dimensional set of equations.

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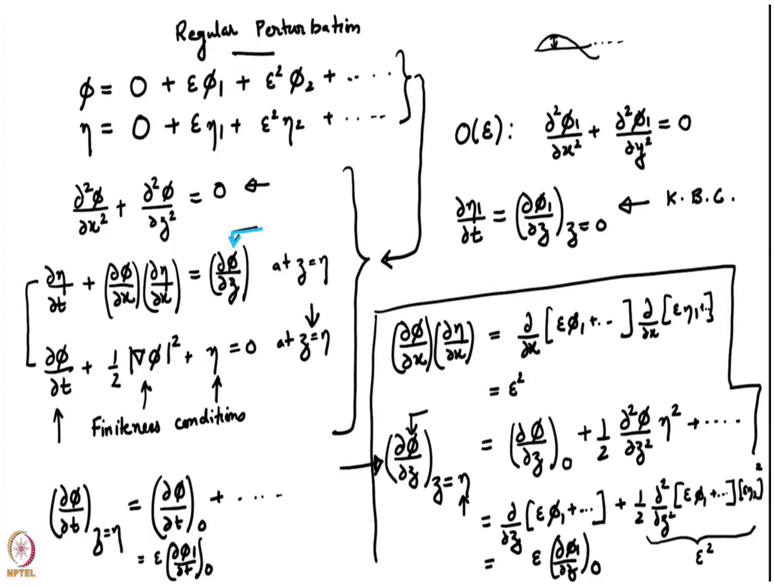

Then, our small parameter here was epsilon, the product of a typical amplitude a naught into the wave number k and we expanded as a regular perturbation epsilon. We plug this in into our governing equations and boundary conditions and we obtained a set of equations at order epsilon.

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$O(\epsilon)$:

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial z^2} = 0$$

$$\left.\begin{aligned} &\frac{\partial \eta_1}{\partial t} - \left(\frac{\partial \phi_1}{\partial z}\right)_{z=0} = 0 \\ &\left(\frac{\partial \phi_1}{\partial t}\right)_{z=0} + \eta_1 = 0 \end{aligned}\right] \longrightarrow \text{Normal mode analysis}$$

Finiteness conditions

$$\left.\begin{aligned} &\phi_1(x, z \to -\infty, t) \to \text{finite} \\ &\phi_1(x \to \pm\infty, z, t) \to \text{finite} \end{aligned}\right] \qquad -\infty < x < +\infty, \quad k > 0$$

Sol^n : Variable separation

$$\phi_1 = X(x)\, Z(z)\, e^{i\omega t} \to \text{non-dimensional}$$

$$\frac{X''}{X} = -\frac{Z''}{Z} = -1$$

$$X'' + X = 0 \Rightarrow X(x) = B_1 \cos(x) + B_2 \sin(x)$$

$$Z'' - Z = 0 \Rightarrow Z(z) = C_1 e^{kz} + \cancel{C_2}^{\,0} e^{-kz}$$

We have to remember here that the order one problem here is trivial because the base state is quiescent. The only non-trivial variable which has an order one contribution is pressure here, nut in the way, we have written down things, we can first solve for velocity potential and the interface and from there, once we have determined these two, we can extract the pressure field because from the unsteady Bernoulli equation.

So, let us proceed with what we had obtained at order epsilon. So, we had obtained these set of equations at order epsilon. In addition, we had also seen that at order epsilon, our boundary conditions were linearized and because in particular, we had two boundary conditions at the interface, one was coming from the fact that the Bernoulli equation. In the Bernoulli equation, we said pressure is equal to 0 at the interface, this led to one boundary condition and the second was the kinematic boundary condition.

Both of these boundary conditions were simplified by the linearized analysis by virtue of the fact that using a Taylor series approximation, we applied the boundary conditions instead of applying it at the unknown boundary z is equal to eta, we applied it at the known boundary as the first term in the Taylor series approximation.

So, you can see that all the terms del phi 1 by del z is applied at z is equal to 0 and del phi 1 by del t is also applied at z is equal to 0. Eta 1 is not a function of z and so, we do not have to worry about the z dependence there. In addition, we also said that we need because our domain is horizontally unbounded, it goes from minus infinity to plus infinity, the variable x and the variable z goes from eta at the top to minus infinity as we go deeper and deeper in the fluid.

So, we have to ensure that when in an unbounded domain, whatever functional dependence we find out, that does not diverge as we go to arbitrarily large depths or arbitrary large horizontal distances. So, we have these finiteness conditions. Now, with having set up all these equations, let us now write down the solution to these set of equations.

So, our first set of equation is the Laplace equation and we are going to use once again variable separation. We have met this technique once when we looked at the 2D vibrations of a rectangular membrane, we have done this earlier in the course. We are going to do the same procedure here and we are also going to do a normal mod analysis. The base state is quiescent so, I am just going to look for oscillatory solutions about the quiescent base state.

Let us first work on the Laplace equation. So, phi 1, I am going to write it as some function capital X of small x some function capital Z of small z into e to the power i omega t. Note that this omega is non-dimensional, this is because t is non-dimensional, we are in a non-dimensional framework here.

So, now, our main purpose is to go back and substitute this form of phi 1 into the Laplace equation. Once we do that, we get back ordinary differential equations for capital X and capital Z. We have seen how to do this before, so, I can write it like this. X double prime by

X all the X dependencies are collected on one side, then we have minus Z double prime by Z, I am just substituting in the Laplace equation. The e to the power i omega t just cancels out, it will not appear anywhere.

And then, I have to put a separation constant. I am going to choose the separation constant to be minus 1. Why am I choosing it to be minus 1? Recall that we are doing this in non-dimensional framework. So, it should be ideally minus some quantity squared here, it would have been minus k squared, k is a typical wave number that we have introduced while doing a non-dimensionalization, we have non-dimensionalized all our length scales by k.

So, you can see that the if we wrote down the corresponding dimensional version of this, it would just be equivalent to setting the separation constant to minus k square. Why in the negative sign? We have met this negative sign before. In this case, you can justify the negative sign from the fact that if you have a negative sign here, then the equation for X would have, would be X double prime plus X is equal to 0. The corresponding equation for Z would be Z double prime minus Z is equal to 0.

You can see why we are taking a negative sign. If I put a negative sign here, then it ensures that the solutions for X are oscillatory. Recall that small x goes from minus infinity to plus infinity. If we choose a positive separation constant let us say plus 1, then the solutions for X would be exponential with a plus X and a minus X.

So, e to the power plus X and e to the power minus X and it would be a linear combination of the two because our domain is unbounded on at both ends so, both the exponentials are going to diverge. So, it is not possible to get a finite solution by eliminating any one of the constants of integration.

In order to prevent that, we look for oscillatory solutions on the in the horizontal direction and in the vertical direction, this choice of separation constant will give us two exponentials. In the vertical direction, the choice of exponential is ok because our vertical domain is semi-unbounded so, we can eliminate one of the exponential which diverges as we go to minus infinity and keep the other exponential which will anyway decay to 0 as we go deeper and deeper in the fluid.

So, with that argument, we know we can now solve X double prime and the solution for X is simple, so, I will call it B 1 cos X plus B 2 sin X, so, this is a function of small x and those are small x's. Similarly, B 1 and B 2 are constants of integration. Small z is equal to C 1 e to the power kz plus C 2 e to the power minus kz.

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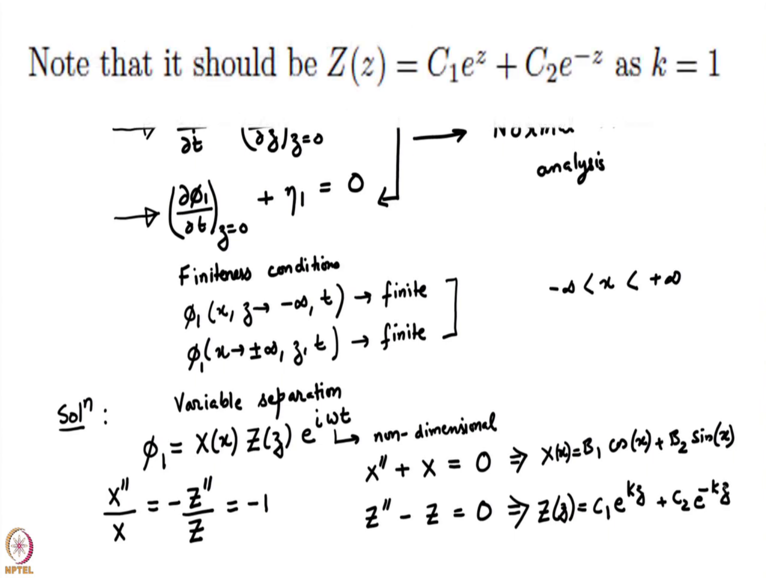

As I argued before we said C 2 to 0 because this term goes to infinity as z goes to minus infinity. So, as we go deeper and deeper in the fluid, recall that our coordinate system is like this and z goes to minus infinity, so, this is x, this is positive direction of z and z goes to minus infinity, we are in the deep water approximation.

Because k is greater than 0, k is a wave number, it is related to wavelength so, k is greater than 0. So, e to the power minus kz for negative z as we make z more and more negative, e to the power minus kz will become larger and larger. So, we have to set the constant of integration to 0 and so, we are left with only a single constant in the z dependence.

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$$\leftarrow \phi_1 = \left[B_1' \cos(x) + B_2' \sin(x)\right] c_1 e^{z} e^{i\omega t} \qquad B_1 = B_1' c_1$$

$$= \left[B_1 \cos(x) + B_2 \sin(x)\right] e^{z} e^{i\omega t} \qquad B_2 = B_2' c_1$$

normal modes

$$\leftarrow \eta_1 = \left[A_1 \cos(x) + A_2 \sin(x)\right] e^{i\omega t}$$

B.C.: $$\frac{\partial \eta_1}{\partial t} - \left(\frac{\partial \phi_1}{\partial z}\right)_{z=0} = 0 \rightarrow (1)$$

$$\left(\frac{\partial \phi_1}{\partial t}\right)_{z=0} + \eta_1 = 0 \rightarrow (2)$$

$$(1) \Rightarrow \quad i\omega\left[A_1 \cos(x) + A_2 \sin(x)\right] e^{i\omega t} + c.c_1 - \left[B_1 \cos(x) + B_2 \sin(x)\right] e^{i\omega t} + c.c_2 = 0$$

$$\Rightarrow \left[(i\omega A_1 - B_1)\cos(x) + (i\omega A_2 - B_2)\sin x\right] e^{i\omega t} + c.c. = 0 \rightarrow (3)$$

Now, let us go further with these solutions. So, we are writing phi 1 is equal to B 1, let me put a; let me put a prime here and I will explain why am I putting a prime because I want to reserve B 1 and B 2 for something else. So, with this approximation, we can write the expression for phi 1 as and now, you can see that I can multiply and then, there is of course, a e to the power i omega t and now, you can see that this constant can be absorbed into B 1 prime and B 2 prime.

So, I am just going to say that B 1 is equal to B 1 prime into C 1 and B 2 is equal to B 2 prime into C 1. So, this is why I introduced to prime so that I can use B 1 and C 1 from here onwards. Now, let us now go back to our equations and try to understand. We have now got a form of phi; we now need to anticipate the form of eta.

You can see that both the boundary conditions, which are indicated by horizontal arrows, one is the linearized kinematic boundary condition, the other is the Bernoulli equation linearized form applied at the interface, at the linearized interface, which is Z is equal to 0.

Both the boundary conditions tell us there is no x derivative in both the boundary conditions. So, they are telling us that the x dependence of phi 1 is the same as the x dependence of eta 1. Now, with that observation, we can anticipate that eta 1 is also going to have the same functional dependence as far as x is concerned. So, eta 1 is also going to be a linear combination of cos x and sin x that is because the boundary conditions do not involve any derivatives with respect to x.

So, we can see this is A 1 cos x. So, I am introducing some instead of writing it is just B 1 cos x and plus B 2 sin x, I am introducing some new constants, but I am keeping the functional dependency the same. So, it is a linear combination of cos x plus sin x, but the constants are now different, A 1 and A 2 for eta 1. Remember that eta 1 is only a function of x and t, there is no z dependence; so, from the normal mode approximation, this e to the power i omega t. So, these two are coming from normal modes.

And our purpose is just like before, we have to go back and substitute this into the equations. We anticipate that they should lead us to an eigenvalue problem and when we plug it in, it should lead us to some kind of a matrix whose determinant is going to determine omega for us. Let us see how.

So, we will write. So, our boundary conditions are del eta 1 by del t, I am just rewriting the boundary conditions, this is the linearized kinematic boundary condition at z is equal to 0 is 0 and then we have; so, I am going to call this equation 1 and then, del phi 1 by del t and z is equal to 0 plus eta 1 is equal to 0, this is 2.

If we now substitute these forms of phi 1 and eta 1 into equation 1 and 2, then we obtain so, 1 implies so, substituting the form of eta 1 and phi 1 into equation 1, we obtain i omega A 1 cos x plus A 2 sin x, thus the derivative of eta 1 into e to the power i omega t plus its complex conjugate, we have to remember to add the complex conjugate because these A 1, A 2, B 1, B 2, these in general are complex constants as we have seen earlier.

So, I will call this complex conjugate 1 because there is one more term which is minus del phi 1 by del z in equation 1. So, when I do the second derivative that just gives me the same because the z dependence is only e to the power z. So, del phi 1 by del z is exactly the same expression plus its complex conjugate, so, this is C C 2. So, C C 2 is the complex conjugate of the term which appears on its left hand side, C C 1 is also similarly ok and this is equal to 0.

I can put them all together and write them as a single equation by collecting all the coefficients of cos x, this whole thing gets multiplied by e to the power i omega t plus the complex conjugate or whatever I have written in the bracket. Only the A 1 and the B 1 will typically give B complex conjugated. You will see that omega in this case will turn out to be a purely real quantity. So, let me call this equation 3.

We have we are now done with equation 1 because we have substituted the forms of phi 1 and eta 1 into 1, let us do the same for equation 2. If we do that by an analogous procedure, you we substitute and we collect all the coefficients of cos x together and cos sin x together.

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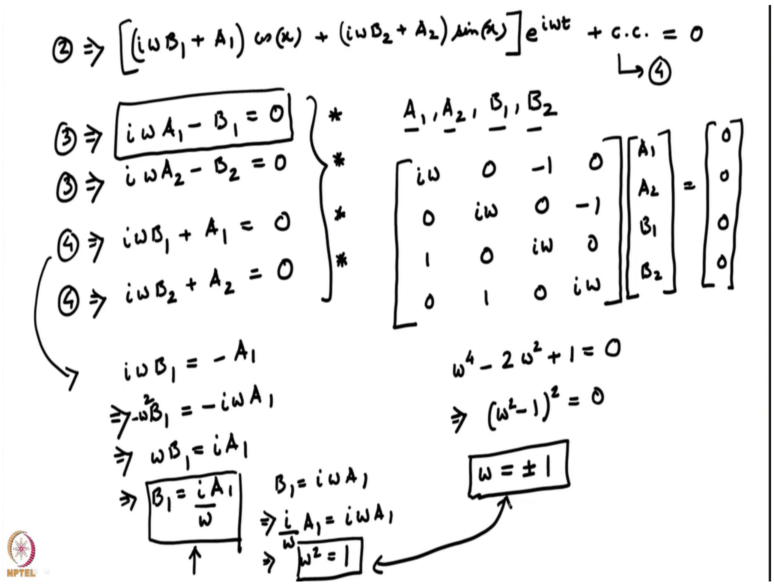

So, 2 implies, it just gives us the following equation. So, we will get i omega B 1, its del phi 1 by del t, so, i omega and the exponential part of z vanishes because there is no derivative with respect to z firstly, and then, it is applied at z is equal to 0. In the previous one also, it was the same argument that we had del phi 1 by del z applied at z is equal to 0 so, the exponential part did not appear in equation 2.

So, similarly we will write one more equation now, i omega B 1 plus A 1 into cos x plus its complex conjugate is equal to 0. You can obtain this very easily by just substituting the forms of phi 1 and eta 1 into equation 2 and then, collecting all the terms together ok. So, let us call this now equation 4. Now, our task is to work on equation 3 and equation 4.

Now, if we look at the equation 3 and equation 4 firstly, it is enough to just work on the term which appears and not worry about the complex conjugate part because that is just the C C of what is written here. So, in both the expressions, you will see that we have a linear combination of cos x and sin x multiplied by e to the power i omega t. We want this to be 0 at all times, the entire expression. E to the power i omega t is not 0 at all times, so, we have to look at the expression inside the square bracket.

So, you can readily see that what is inside the square bracket in both equation 3 and equation 4 is a linear combination of cos x and sin x because cos x and sin x are linearly independent so, we have to set the coefficient of this to 0 in order to satisfy equation 4. So, we are led to four equations.

Equation 3 implies so, each of the coefficients of cos x and sin x are 0. So, equation 3 implies i omega A 1 minus B 1 is equal to 0. Equation 3, this is the coefficient of cos x that we wrote now, let us write the coefficient of sin x that is also 0. Equation 4, the coefficient of cos x is just i omega B 1 plus A 1 is equal to 0. Equation 4, the coefficient of sin x is i omega B 2 plus A 2 is equal to 0.

So, we have four equations, and it is really telling us something very interesting. So, we can for example, use this so, we have four unknowns A 1, A 2, B 1, B 2, you can make this into two unknowns for example, by using this one two of these equations to eliminate so, for example, we can express B 1 in terms of A 1, you can pay attention that this the first and the second equation just tells us that B 1 is equal to i omega A 1 and B 2 is equal to i omega A 2 ok.

You can also see that the last two equations basically tell us so, for example, if I take this equation, then it tells us that i omega B 1 is equal to minus A 1 so, this is really B 1 is equal to minus so, if I multiply both sides by i omega, then I get minus omega square here and minus i omega A 1 and this tells me that omega B 1 is equal to i A 1 and so, B 1 is equal to i A 1 by omega ok.

So, now, you can immediately see that if you compare this with the first equation that I have written here, then you can immediately see that the first equation tells us B 1 is equal to i omega A 1 and this equation tells us B 1 is equal to i by omega into A 1. If I substitute this, then I obtain i omega A 1 is equal to i omega A 1 and this is telling me that omega square is equal to 1. This is the dispersion relation that we are basically finding.

Now, formally you can do this a little bit more formally by not taking any two of these equations but working on all four of them. Now, formally you can do this by taking not two of these equations as we have seen but working on all four of them and convincing yourself that essentially leads to the same dispersion relation.

So, what you can do is you can do this more formally by thinking of these four equations. So, this equation, this equation, this equation and this equation as an equation in A 1, A 2, B 1, B 2. So, I can write this as a set of homogeneous equations in A 1, A 2, B 1, B 2 so, let us do that. So, first we write the coefficient of A 1, then A 2, then B 1, then B 2. So, for the first equation, the coefficient of A 1 is i omega, A 2 is 0, the coefficient of A 2 is 0, then minus 1 and the coefficient of B 2 is 0.

Similarly in the second equation, 0, i omega, 0 minus 1. Third equation, 1, 0, i omega, 0 and fourth equation yeah, 0, 1, 0, i omega, A 1, A 2, B 1, B 2 and this is equal to because this is a homogeneous set so, everything is 0 on the right-hand side. So, that is our metrics whose determinant will determine the frequencies at which the system can oscillate.

If you solve this metrics, you will just find that this is just equivalent to omega 4 or if you write the determinant of this metrics rather, this is just equal to omega 4 minus twice omega square plus 1 is equal to 0. This can be written as omega square minus 1 whole square is equal to 0 and so, we have plus minus 1. So, we find essentially the same answer as we had found by just looking at two of the equations.

Let us now use these to write down the solutions. So, we will have 1 of the frequencies is 1. Remember that this is a non-dimensional omega. So, in scaled units, this is 1. When we dimensionalize our expressions, we will obtain the real dispersion relation. Let us now write it.

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$$\eta_1 = [A_1 \cos(x) + A_2 \sin(x)]\, e^{i\omega t} + c.c.$$

$$\phi_1 = [B_1 \cos(x) + B_2 \sin(x)]\, e^{i\omega t} + c.c.$$

Note the error. $\phi_1 = [B_1 \cos(x) + B_2 \sin(x)]\, e^{z} e^{i\omega t} + C.C$

So, we have found now that eta 1 is equal to A 1 cos x plus A 2 sin x into e to the power i omega t. I am not yet writing that omega is 1, but it is understood that omega is 1. Phi 1 was B 1 cos x plus B 2 sin x e to the power i omega t plus C C.

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$$\eta_1 = \left[A_1 \cos(x) + A_2 \sin(x)\right] e^{i\omega t} + c.c. \qquad \omega^2 = 1$$

$$\phi_1 = e^{z}\left[B_1 \cos(x) + B_2 \sin(x)\right] e^{i\omega t} + c.c.$$

$$= e^{z} i\omega \left[A_1 \cos(x) + A_2 \sin(x)\right] e^{i\omega t} + c.c.$$

$$= e^{z} \omega \left[A_1 \cos(x) + A_2 \sin(x)\right] e^{i(\omega t + \pi/2)} + c.c.$$

$$\eta_1 = \left[(A_1 + \bar{A}_1)\cos(t) + i(A_1 - \bar{A}_1)\sin(t)\right]\cos(x)$$
$$+ \left[(A_2 + \bar{A}_2)\cos(t) + i(A_2 - \bar{A}_2)\sin(t)\right]\sin(x)$$

$$\phi_1 = e^{z}\left[(A_1 + \bar{A}_1)\cos(t + \pi/2) + i(A_1 - \bar{A}_1)\sin(t + \pi/2)\right]\cos(x)$$
$$+ e^{z}\left[(A_2 + \bar{A}_2)\cos(t + \pi/2) + i(A_1 - \bar{A}_1)\sin(t + \pi/2)\right] s$$

We can express using the 4 equation that we have found between A 1, A 2, B 1, B 2, we can express B 1 in terms of A 1 and B 2 in terms of A 2, if we do that, then we get a i omega here and A 1 here, we are just using this equation, B 1 is equal to i omega e 1; A 1 and B 2 is equal to i omega A 2.

To express B 1 and B 2 in terms of A 1 and A 2 and so, we have this. So, I can write this further as we have now omega A 1 cos x plus A 2 sin x e to the power i omega t plus pi by 2 plus C C. I have absorbed the i here in the phi 1 as a phase e to the power i pi by 2 is i alright. Now, can we proceed in real notation? So, let us try to make this into slightly more real notation.

So, if I remember that there are complex conjugates everywhere, then this will become A 1 plus A 1 bar, this is familiar to you from our earlier exercise plus i times A 1 minus A 1 prime, this is cos t and this is sin t, I am now putting omega is equal to 1. So, this we have to recall that omega square is equal to 1 is our dispersion relation. So, omega is equal to 1 and its really plus minus 1, but I am already taking the minus into account in the complex conjugate, we have done this before.

And so, I am writing e to the power i t as cos t plus i sin t and then, we will have a e to the power minus i omega t which is cos t minus i sin t. If you add up everything and collect the coefficient of cos x, you will find that the coefficient of cos x is this and the coefficient of sin x is that.

This is more complicated than the problem that we have done before because here, we have a variable where there are three dependencies, there is a time dependency and there are two space dependencies x, z and then in addition there is a time dependency. In the earlier 2D vibration problem, we also had a similar thing where the displacement of the membrane eta in that case I think, we used as a function of x, y and t.

Here, eta is just a function of x comma t, phi 1 is a function of x, z and t. Again, I am substituting omega is equal to 1 into sin x. Once again you can see that this is completely real because A 1 plus A 1 bar is real, i times A 1 minus A 1 bar is also real and so, this is completely real.

One can similarly write down an expression for phi 1, phi 1 would have a e to the power z so, I have missed a e to the power z here and similarly, A 1 plus A 1 bar and in this case, this would be cos t plus pi by 2 that is because there is a pi by 2 here, omega is again 1 so, it is just cos t plus pi by 2 plus i A 1 minus A 1 bar sin t plus pi by 2 and we can work on those cos t by plus pi by 2 and sin t plus pi by 2 later on, this whole thing multiplies cos x plus again e to the power z into.

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$$\eta_1 = [A_1 \cos(x) + A_2 \sin(x)]\, e^{i\omega t} + c.c. \qquad \omega^2 = 1$$

$$\phi_1 = e^z [B_1 \cos(x) + B_2 \sin(x)]\, e^{i\omega t} + c.c.$$

$$= e^z i\omega [A_1 \cos(x) + A_2 \sin(x)]\, e^{i\omega t} + c.c.$$

$$= e^z \omega [A_1 \cos(x) + A_2 \sin(x)]\, e^{i(\omega t + \pi/2)} + c.c.$$

$$\eta_1 = [(A_1 + \bar{A}_1) \cos(t) + i(A_1 - \bar{A}_1) \sin(t)] \cos(x) + [(A_2 + \bar{A}_2) \cos(t) + i(A_2 - \bar{A}_2) \sin(t)] \sin(x)$$

$$\phi_1 = e^z [(A_1 + \bar{A}_1) \cos(t + \pi/2) + i(A_1 - \bar{A}_1) \sin(t + \pi/2)] \cos(x)$$

$$+ e^z [(A_1 + \bar{A}_1) \cos(t + \pi/2) + i(A_1 - \bar{A}_1) \sin(t + \pi/2)]\, c$$

Notice the error in the expression for ϕ_1. The second term on the R.H.S should be $e^z \left[(A_2 + \bar{A}_2)\cos(t + \pi/2) + i(A_2 - \bar{A}_2)\sin(t + \pi/2)\right]\sin(x)$

And we can simplify this t to the power t plus pi by 2, t plus pi by 2 to obtain the actual answer in terms of real variables.