Introduction to Interfacial waves
Prof. Ratul Dasgupta
Department of Chemical Engineering
Indian Institute of Technology, Bombay

Lecture – 03
Coupled, linear, spring-mass systems (continued..)

In the last class, we had started by looking at oscillations in mechanical systems with a finite number of degrees of freedom. Particularly, we had looked at some simple examples of one and two degrees of freedom, and we had learned how to use complex exponential notation to examine the stability of the base state.

So, we had perturb about an equilibrium base state, and we had learn the method of normal modes and using that we had found out how to write down the most general solution to the governing equations.

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$z(t) = Ae^{i\omega t}$ } Normal mode assumption

A : complex constant

$[-m\omega^2 + k]A = 0 \qquad A \neq 0$

$\Rightarrow \boxed{\omega^2 = \frac{k}{m}}$ Frequency of vibration

$\omega_1 = \left(\frac{k}{m}\right)^{1/2}$

$\omega_2 = -\left(\frac{k}{m}\right)^{1/2}$

$\underbrace{z(t)}_{\text{real}} = A_1 e^{i\omega_1 t} + A_2 e^{i\omega_2 t}$

$= A_1 e^{i\sqrt{\frac{k}{m}}t} + A_2 e^{-i\sqrt{\frac{k}{m}}t}$

A_1, A_2 : initial conditions

$z(0) = z_0$ & $\dot{z}(0) = v_0 \rightarrow$ 1

$A_1 + A_2 = z_0 \rightarrow$ 2 $\qquad i\sqrt{\frac{k}{m}}\left[A_1 - A_2\right] = v_0 \rightarrow$ 3

So, recall that we had first looked at a single degree of freedom system.

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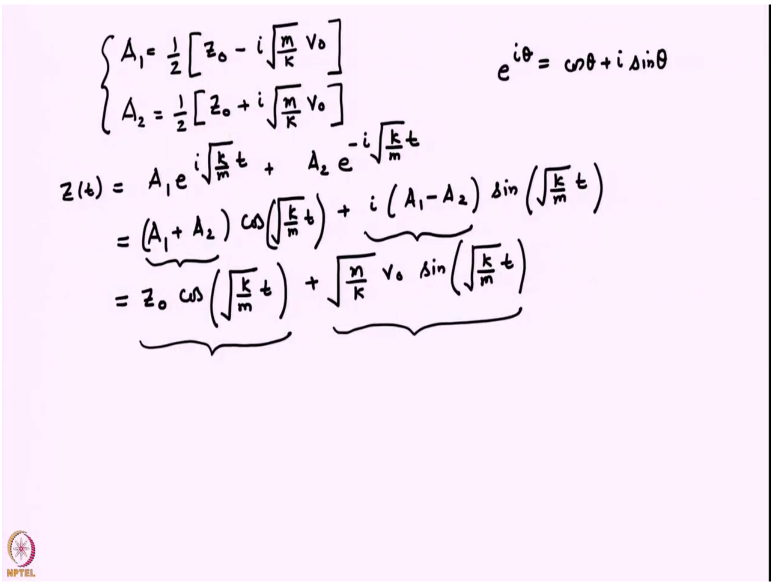

And then, we had written down the answer using we started with complex exponential notation and then finally, the answer was written in real notation.

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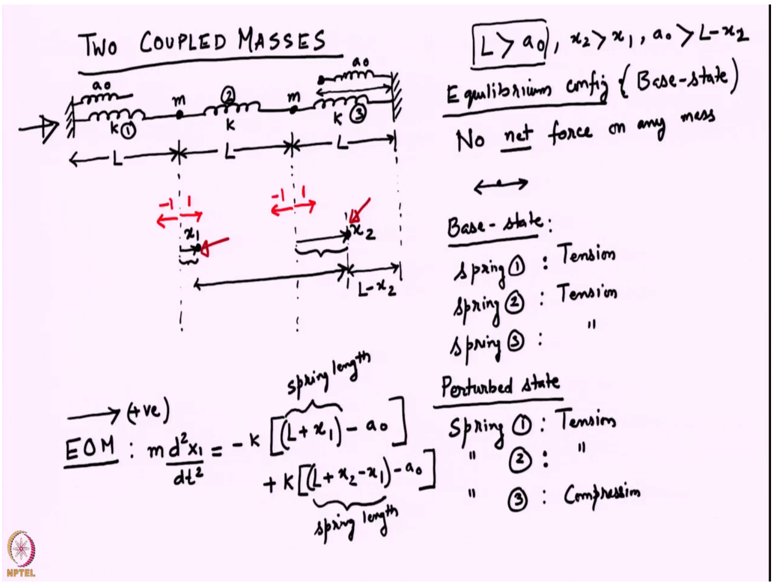

Similarly, we started with two coupled masses connected through 3 springs.

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$$m\frac{d^2x_2}{dt^2} = -k\left[(L+x_2-x_1)-a_0\right] - k\left[a_0-(L-x_2)\right] \rightarrow (1)$$

$$m\frac{d^2x_1}{dt^2} = -k\left[(L+x_1)-a_0\right] + k\left[(L+x_2-x_1)-a_0\right] \rightarrow (2)$$

$$(1) \Rightarrow \quad m\ddot{x}_1 = -kx_1 + k(x_2-x_1) \rightarrow (A)$$

$$(2) \Rightarrow \quad m\ddot{x}_2 = -k(x_2-x_1) - kx_2 \rightarrow (B)$$

$$m\frac{d^2}{dt^2}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \vec{X} \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \boxed{m\frac{d^2\vec{X}}{dt^2} + \underset{\uparrow \text{Symmetric}}{K}\cdot\vec{X}(t) = 0} \qquad \boxed{m\frac{d^2x}{dt^2} + kx = 0}$$

$\hookrightarrow$ Governing eq^n

$$x = Ae^{i\omega t} \qquad \omega = \pm\sqrt{\frac{k}{m}}$$

$$\vec{X} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{i\omega t} \Big\} \text{ Normal modes}$$

And then, we wrote down the governing equations. These turned out to be coupled linear ordinary differential equations. We analyze the same using method of normal modes.

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$$-m\omega^2 A_1 = -kA_1 + k(A_2 - A_1)$$

$$-m\omega^2 A_2 = -k(A_2 - A_1) - kA_2$$

$$\begin{bmatrix} 2k - \underbrace{m\omega^2} & -k \\ -k & 2k - \underbrace{m\omega^2} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{Non-trivial} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

For non-trivial sol^{ns}, $\det|\ \ | = 0$

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} = 0 \Rightarrow (2k - m\omega^2)^2 = k^2$$

$$\Rightarrow 2k - m\omega^2 = \pm k$$

$$\Rightarrow \omega^2 = \frac{k}{m} \text{ or } \frac{3k}{m}$$

And then, we found that the frequencies are given by the eigenvalues of a certain matrix which depends on the properties of the system. And once we found out those eigenvalues we computed the corresponding eigenvectors.

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Eigenvectors $\omega^2 = \frac{k}{m}$

a)
$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow A_1 - A_2 = 0$$

$A_1 = A_2 = 1 \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to $\omega^2 = \frac{k}{m}$

b) $\omega^2 = \frac{3k}{m}$

$$\begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow A_1 + A_2 = 0$$

$A_1 = 1,\ A_2 = -1$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ corresponding to $\omega^2 = \frac{3k}{m}$

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$$\vec{X}(0) = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix} \qquad \dot{\vec{X}}(0) = \begin{bmatrix} v_1^{(0)} \\ v_2^{(0)} \end{bmatrix}$$

$$\left.\begin{aligned}
x_1(0) &= (c_1 + c_2) + (c_3 + c_4) = x_1^{(0)} \\
x_2(0) &= (c_1 + c_2) - (c_3 + c_4) = x_2^{(0)} \\
\dot{x}_1(0) &= i\sqrt{\frac{k}{m}}\,(c_1 - c_2) + i\sqrt{\frac{3k}{m}}\,(c_3 - c_4) = v_1^{(0)} \\
\dot{x}_2(0) &= i\sqrt{\frac{k}{m}}\,(c_1 - c_2) - i\sqrt{\frac{3k}{m}}\,(c_3 - c_4) = v_2^{(0)}
\end{aligned}\right\}$$

$$c_1 - c_2 = -\frac{i}{2}\sqrt{\frac{m}{k}}\left[v_1^{(0)} + v_2^{(0)}\right] \qquad c_1 + c_2 = \frac{1}{2}\left[x_1^{(0)} + x_2^{(0)}\right]$$

$$c_3 - c_4 = -\frac{i}{2}\sqrt{\frac{m}{3k}}\left[v_1^{(0)} - v_2^{(0)}\right] \qquad c_3 + c_4 = \frac{1}{2}\left[x_1^{(0)} - x_2^{(0)}\right]$$

And then, we also interpreted these physical meaning of these eigenvectors.

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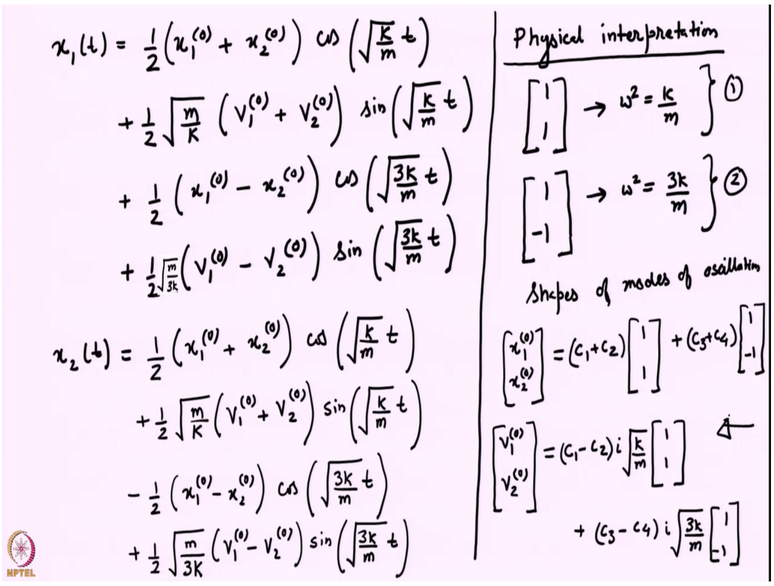

We also learnt to write down the most general solution to this system in real notation. This is provided here now. We also understood how to physically interpret the eigenvectors. I also told you that the physical interpretation of the eigenvectors is that in this particular example, we have 1, 1 and 1, minus 1, this essentially implies that any initial condition can be expressed as a linear combination of these eigenvectors. You can see that here on the right hand side where I have expressed x 1 0, x 2 0 and v 1 0, v 2 0 in terms of the eigenvectors. You can see that these are expressed as a linear combination.

So, the values of c 1, c 2 or rather c 1 plus c 2 c 3 plus c 4 c 1 minus c 2 and c 3 minus c 4 will be decided by initial conditions. If you choose your initial conditions to have such that the initial conditions have projections only along one eigenvector, then the system will oscillate only along that particular mode of oscillation. So, if you choose for example, these initial conditions have projection only on 1, 1, the eigenvector 1, 1, then the system will oscillate purely in mode 1.

Similarly, if you choose your initial conditions to, so that they are along the second eigenvector 1, minus 1, then the system will purely vibrate in 1, minus 1. In general, you did not choose initial conditions which have projections only along 1, you could choose initial conditions for example, which are projections along both. In that case, the system will move in a combination of normal modes.

The resulted motion will be oscillatory, but not necessarily periodic. This is got to do with the fact that this these two frequencies of the system are not rational multiples of each other, ok. So, we will with that quick summary we will now continue to our next topic wherein we will introduce N-coupled masses. Now, one essential difference between this example that we are going to do now compared to what was done in the previous class, is that there in the previous class or the equations of the motion were inherently linear. Here we will see their equations of motion are inherently non-linear, and we will have to do some linearization about the base state in order to render them linear.

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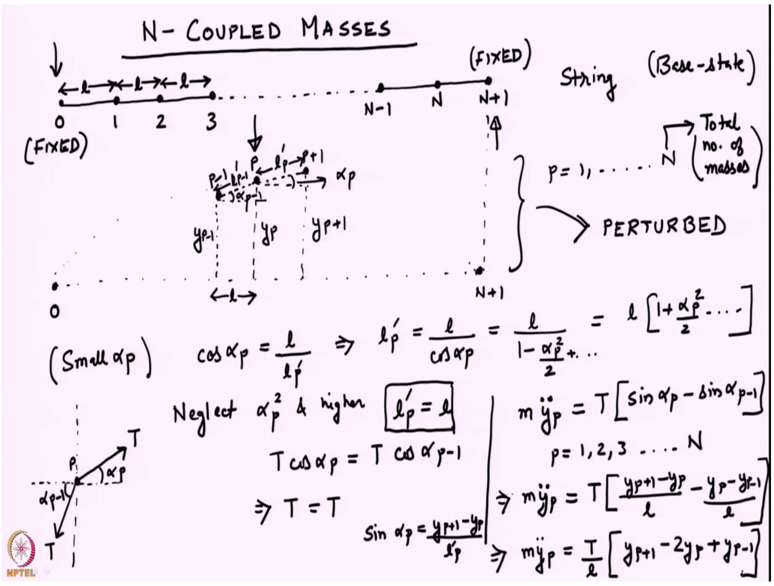

So, let us look at N-coupled masses. So, what we have in mind is N masses. So, label them 0, 1, 2, 3, and then maybe N minus 1, N and N plus 1. Now, in the base state once again they are in equilibrium and the distance between them. So, this is you can think of each mass being connected now through a string. And for ease of visualization instead of perturbing them along the horizontal direction I am going to perturb them vertically.

So, what does that mean? So, let me so each of them have are equidistant in their base state or equilibrium configuration. So, now I am going to draw the perturb state. So, in the perturb state it will look like this. I should mention here that the 0 and the Nth N plus 1th mass do not move. So, these are fixed. So, these play the role of walls.

So, we have chosen this particular boundary condition. So, which essentially implies that I am only going to solve for the motion of masses 1 to N. There will be no motion of the 0th mass which is this, and there will be no motion of the N plus 1th mass, both of them will have 0 vertical displacement at all times.

Now, let us draw a typical let us say the pth mass. So, let me draw the mass here and then we have; so, we would expect we would expect a profile like that you know I we are perturbing something like this, and let me not make it at the maximum. So, let us say the p minus 1th mass is here and the p plus 1th mass is here. So, this is p, these are all at different angles with respect to the horizontal.

So, this angle is alpha p, this is p plus 1 this is p minus 1. p is a general index which can take any value. So, p could be any number between 1 to N. So, it could be 1, it could indicate the 1th mass, the 2nd mass, the 3rd mass and so on. So, this angle is alpha p minus 1.

Now, we are interested in the vertical displacement of the masses. So, let me indicated the vertical displacement by the symbol y and this is for the p minus 1th mass. So, this is y p minus 1, this is y p, and this would be y p plus 1. So, now, you can see that in the base state or the equilibrium state, the entire mass distance of the string was l.

Now, in general these this string is going to get stretched. So, I am going to indicate the stretched distance by l p or rather, so this is l prime p minus 1 and this distance is l prime p. And we can see from the geometry of the triangle that cos alpha p is equal to l divided by l p prime. Now, this I can use to express l p is equal to l by cos alpha p.

Now, whatever we are going to do here is going to be valid. This is where the linearization comes in. You can see that cos alpha p is a non-linear function of alpha p, and so, I would like to make a small alpha p approximation. So, we know that the first term, so if I expand cos alpha p in a Taylor series about alpha p equal to 0, then we know that the first term is 1 and then we have alpha p square by 2 plus dot dot dot. And if we just use the binomial theorem then this is 1 plus alpha p square by 2 minus dot dot dot.

Now, you can see that since we are doing a small alpha p approximation, so we are going to neglect all terms which are quadratic and higher in alpha p. So, this is a linear alpha p calculation. So, you can immediately see that at this order of approximation l p prime is the same as l. So, the string in the perturb configuration. So, this is the perturb configuration. The string does not get stretched.

Now, remember that in the base state the string is under some kind of tension. We will also assume that this tension will not change even in the perturbed state. So, if the tension in the base state is T, the tension in the perturb state is also remains T. Under that approximation let us draw the free body diagram of the pth mass. So, this is the mass whose free body diagram I am drawing.

So, this mass obviously is being acted upon by two forces, one from the left and one to the right. So, let me draw it here. So, there is a tension T exerted by the string on the right of the pth mass, so this is the pth mass and this tension makes an angle alpha p with the horizontal.

Similarly, there is another tension which is acted upon to the left and that tension, I am exaggerating the angle a little bit and that tension is this is the angle alpha p minus 1. And now we have to do our force balance on this pth mass. Now, you can see that we are mostly interested in the vertical force balance because we are going to write down an equation for the vertical coordinate y p.

However, it is instructive to also write down the horizontal force balance. So, because this is a small angle of approximation, we are essentially saying that T cos alpha p is equal to T cos alpha p minus 1. Now, this in general cannot be true exactly, but if you make the small angle approximation then the first term in the Taylor series of cos alpha p and cos alpha p minus 1 is just 1, which ensures that at this level of approximation T is just equal to T. It is automatically satisfied.

So, at this level of approximation, the mass is in horizontal equilibrium. There is no horizontal acceleration. What about the vertical direction? So, the vertical is this. So, we are applying Newton second law of motion. So, m y p double dot. So, the mass is not under vertical equilibrium, there is a net force acting on it which produces acceleration.

I am writing mass into acceleration on the left hand side and on the right hand side we will have the net force. The net force as you can see is T sin alpha p minus sin alpha p minus 1. Now, this is an equation which governs the motion of the pth mass. But this is the same equation which would be true for every value of p from 1, 2, 3 all the way up to N, where capital N represents the total number of masses. It is a positive integer. We will keep capital N arbitrary and you will see the advantage of doing so shortly.

Now, under the small angle approximation, I can write the above expression as T and from this geometry of this triangle you can immediately see that is sin alpha p is y p plus 1 minus y p divided by l p prime, but l p prime under our approximation is equal to l. So, I am just going to replace l p prime in the denominator by l.

This is equivalent to the usual approximation that we do where we say that sin theta for small angle is just theta which is also equal to tan theta, ok. So, y p prime plus y p plus 1 minus y p divided by l minus y p minus y p minus 1 also divided by l, and this can be simplified to T by l to that.

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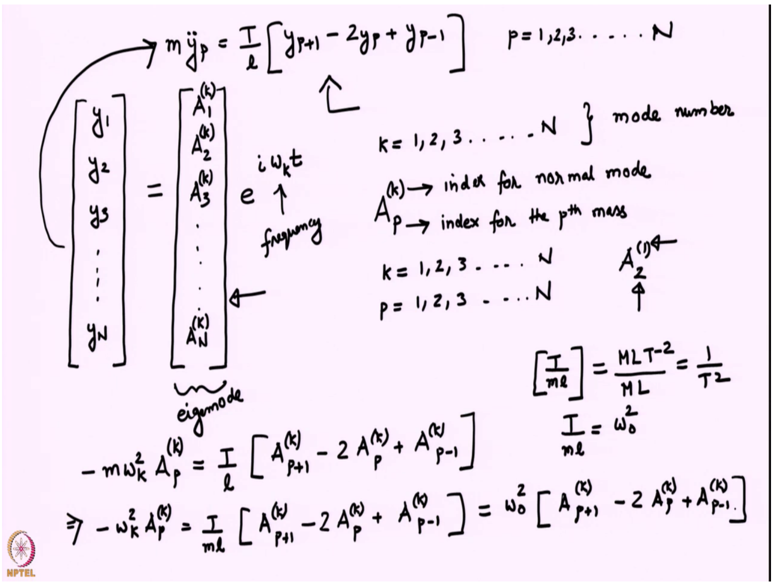

Now, this equation, I am going to write this equation once again, actually represents N-coupled ordinary differential equations. The reason is that p goes from 1, 2, 3 all the way to capital N. Now, for each value of p you will write one equation like this. You can immediately see that this is a linear equation, and they are all coupled to each other.

So, depending on how many total masses you have, if you have 10 masses you will have 10 such equations, if you have 100 masses you will have 100 such equations they are all coupled to each other. Once again we are going to use the method of normal modes to solve this, but because we are our number of total number of masses is an unknown quantity we are keeping it in symbolic form as N.

We are not going to use the matrix form because the matrix the dimension of the matrix will be decided by N and capital N is arbitrary here, it is finite but arbitrary. So, we are going to bypass the matrix notation that we introduced in the last class and we are going to introduce another way of looking at these coupled linear ordinary differential equation solutions using the method of normal modes, but without having to write down matrices and calculating determinants.

So, let us now impose the normal mode analysis. So, I would say that y 1, y 2, y 3 all the way up to y N, these are the displacements of each mass and these would be A 1, A 2, A 3, A N e to the power i omega t. But you see now we have some experience with doing this, and so we know that in general if we have 10 masses we would expect a 10 by 10 matrix and 10 different frequencies and 10 different eigenvectors.

If we have 100, we would have 100 such frequencies and 100 such eigenvectors. So, we would expect that we would have if you have capital N number of masses, you would have capital N number of modes capital N number of normal modes. So, let me indicate.

So, this normal mode analysis is going to is going to be true for every specific mode where the entire system vibrates at one frequency, and so I am going to put a index k put a bracket around it, so that it is not mistaken as a power and k can take the values 1, 2, 3 all the way up to N. And k will indicate mode number which normal mode is the system vibrating it. If I have capital N number of degrees of freedom as I have here, I would expect capital N number of different modes of vibration or in other words we would have capital N number of linearly independent eigenvectors in general, ok. And similarly that many number of eigen frequencies or frequencies of motion.

So, now in order to understand this in symbolic notation, whenever you see the symbol, so it is clear that this is the eigen mode for the kth mode and this is the frequency. So, when I take one element of this column vector, I would write it as, in general if I take the pth element of this column vector then I would write it as this.

You should note that there are two indices and I will reassert what do these indices stand for. The index at the top indicates which mode is the system vibrating. The index at the bottom indicates index for the pth mass both of these go from 1, 2, 3 up to N, but indicate different things.

So, for example, A 1 2 would represent the vertical displacement of the second mass when the entire system is vibrating in the first mode of oscillation, so on and so forth when we have change the values. I hope this is clear. So, now, we are going to substitute this the usual procedure into our equation of motion, and once we get that equation it will converted it into an algebraic equation like before. Let us write that down.

So, we obtain; so, I am going to apply it and write it for the pth equation, this is the pth equation or rather the equation for the pth mass, and so vibrating in the kth normal mode and if we shift the mass to the right hand side we get a coefficient T by m l. The reason for doing that is as follows note the dimensions of T by m l, T by definition is a tension, it is a force.

So, force has dimensions ML T to the power minus 2, and what is at the bottom is ML. So, we have 1 by T square or in other words T by m l has the dimensions of frequency squared. So, I will call T by m l as some omega 0 squared. It is some characteristic frequency of the system. So, I am going to rewrite this as the right hand side of this as omega 0 square into the same thing.

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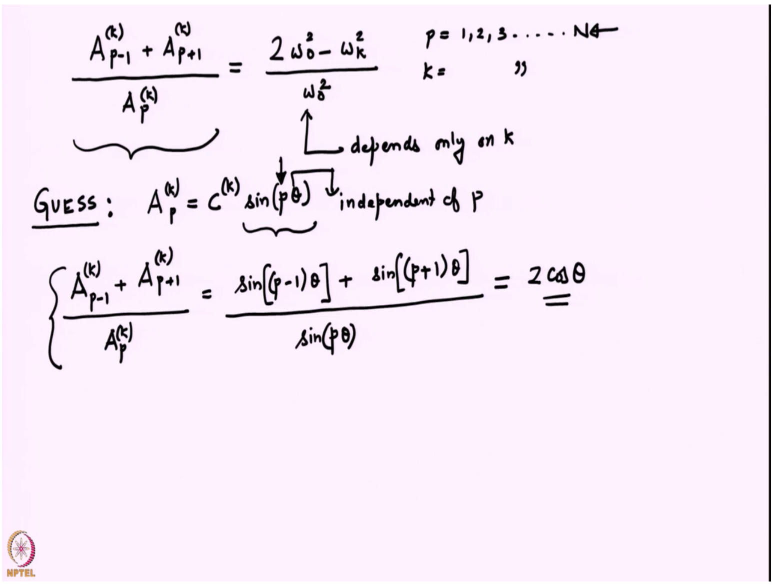

And if you do a little bit more algebraic manipulations on this equation, you can see that we can write it as; and once again to remind us that p goes from to N, k also goes from 1, 2, 3 up to N; p is an index for the displacement for the mass that we are talking about, k is an index for the normal mode.

Now, why do we write it like this? Notice first thing that until now we have not written any matrices. We are trying to calculate eigenvalues and eigenvectors, but because our N is arbitrary, the total number of masses; we cannot write down our matrix of arbitrary dimensions and then calculate the eigenvalues and eigenvectors. So, this is an alternative way of doing this exercise without explicitly writing down the matrices.

So, now, why do we write it like this? You can immediately see that the way we have written it, the right hand side depends only on k, the left hand side depends both on p and k. Now, this should remind you of variable separation when you have studied variable separation ways of solving.

For example, the Laplace equation, where we a function of x on the right hand side, a function of y on the left the other side and then you set it equal to a constant. This is similar to that, but not exactly the same. You have some function of p and k on the left hand side, you have some function only of k on the right hand side. How do we solve this? So, a guess is this is the case. Why do we make this guess?

You can partly see it from the fact that A p k, which is the amplitude of motion of the pth mass vibrating in the kth normal mode will depend on both p and k. But this ratio although A p k depends both on p and k, this ratio of the sum of the two divided by the divided by A p k is independent of p that is because the right hand side depends only on k, and so this ratio also depends only on k. So, we can see that is one guiding factor for making this guess. We are going to verify this shortly.

So, let us check this. The c k gets canceled out in the numerator and denominator, so I am not going to write it. If you expand out the numerator using the formula of sin a minus b plus sin a plus b, you will see that some terms cancel out and you will be left with only 2 cos theta. I urge you to try this on your own. This is a very simple exercise.

So, as expected our guess is correct in the sense that A k p is 2 cos theta, and so this ratio is independent of p provided theta is independent of p. So, the only p dependence lies in here. You can also justify this guess slightly more intuitively by looking at the boundary conditions.

Recall that we had fixed boundary conditions, our there was a wall on the left most point which is the 0th mass and the wall at the right most point. You can see that this guess respects that. You could have guessed, we could have also guessed cos p theta, and cos p theta would also be independent of this ratio would still be independent of theta, but cos p theta would not have satisfied some of our boundary conditions.

So, this is the reason why we are guessing it to be sin because sin will vanish at the left most point and the right most point can be chosen in such a way that such sin again vanishes there. So, this is the basic motivation for this.