Introduction to interfacial waves

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Lecture - 29

Linearised wave equations in deep water

We were starting our analysis of Interfacial Waves we had made a number of simplifying assumptions to help with the mathematical analysis. We had assumed that the fluid is incompressible we are also assumed that we are going to use inviscid equations and irrotational we are going to assume that the motion is irrotational.

This allowed us to express the velocity potential as the gradient the velocity as a gradient of a velocity potential and that led us to the Laplace equation from the incompressibility condition.

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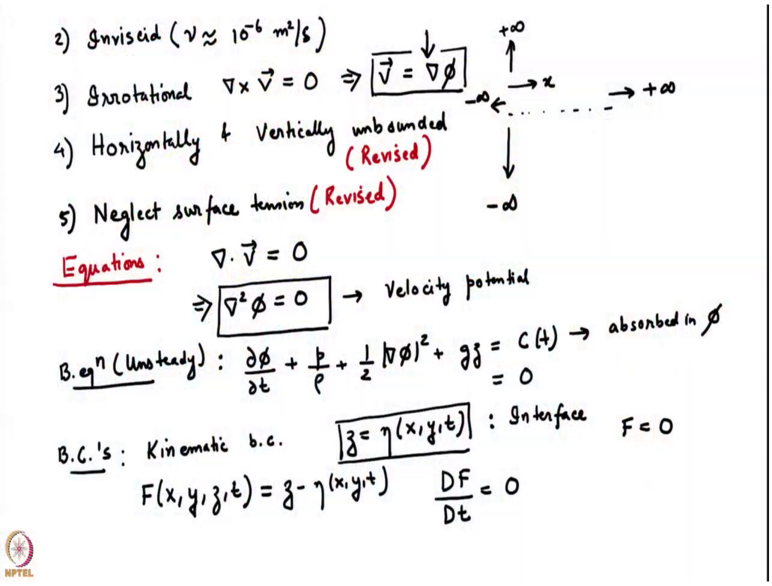

We looked at the equations for pressure and that turned out to be the unsteady Bernoulli equation, we looked at the boundary conditions because we have fluctuating interface here. So, we need one additional boundary condition, we derive the boundary condition by saying that the interface is a material surface this led us to an equation of this form.

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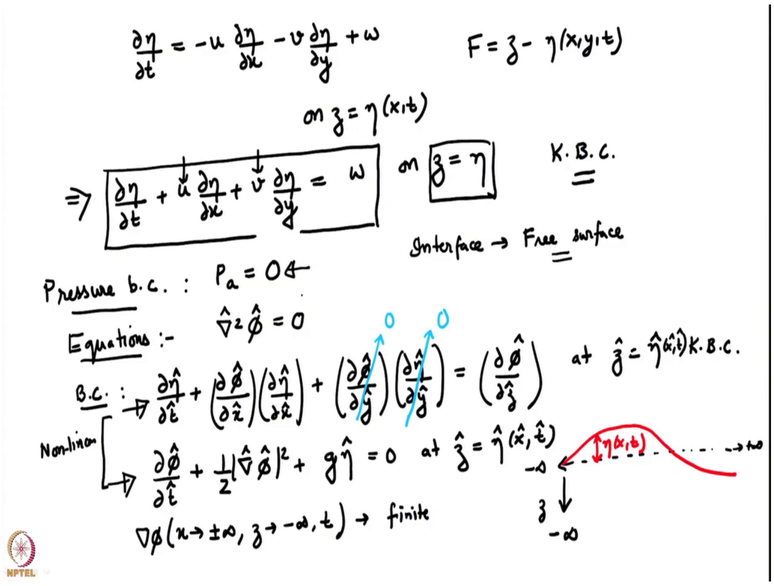

Which I have put in brackets here del eta by del t plus u del eta by del x plus v del eta by del y is equal to w, recall that u v are the x and the y components of velocity, w is the vertical component of velocity along the direction of vertical direction so and this equation is true only on z is equal to eta.

We also saw that if we ignore the motion of the air and assume that air is just quiescent and is not perturb because of the presence of fluctuations on the surface of let us say water. Then we can as a first approximation we can say that the pressure in the air side is just a constant we can take that constant to be 0.

So, with these let us write down what are our equations and what are our boundary conditions. So, we have seen so equations now because I want to use I am shortly going to use regular perturbation I will have to non dimensionalize my equations. So, I will use a hat for every dimensional variable, up till now I have written everything without a hat. But from now onwards whenever things I have dimensional I will put a hat on them, so that I can use non dimensional variables without a hat.

So, we have seen so the Laplacian operator itself is dimensional, so it has a hat. So, we have seen that the equation governing the velocity potential phi hat is the Laplace equation. Then we have a Bernoulli equation, but we are not going to write down the Bernoulli equation because as you will shortly see we can do this analysis without worrying about pressure at first we will obtain pressure at the end.

So, I am not going to write down the Bernoulli equation, but instead what we will do is we will use this pressure condition on the Bernoulli equation to write down a Bernoulli equation which is valid only at the free surface or at the interface.

Here because the we are neglecting air the word interface will be replaced with the word free surface it is called free surface, because it is free of stresses the pressure is 0 there are no tangential stresses ok. So, I am going to use the word free surface when I neglect the dynamics of the fluid the second fluid above. So, at the free surface we are going to apply the Bernoulli equation.

Recall that the Bernoulli equation has a pressure term. So, just setting p to 0 will give me a Bernoulli equation without pressure, but that equation will be true only at the free surface. So, that gives me a boundary condition at the free surface. So, what are the boundary conditions? We have the kinematic boundary condition which we have already derived. So, I am going to replace the u in this formula by del phi by del x and del phi by del y this should be x at z is equal to eta.

Now, I am going to start with the 2D analysis we later we can put 3D, but as we will see putting in 3D just introduces a small slight small more amount of algebra in the calculation. And so it is not so essential to do a full 3D calculation the 2D contains essential features. So, this is so we have to recall that this is the kinematic boundary condition and the 2D approximation implies that all my quantities are not a function of y anymore, so these terms are all 0.

So, I just have 3 terms 2 on the left and one on the right, so this is my kinematic boundary condition. Then as I said before I am going to apply the pressure boundary condition on the Bernoulli equation to get an equation which is true at the interface, if I do that then I just have this is the unsteady Bernoulli equation with the pressure term set to 0. This is g y, but this is being applied at the interface, so y will be replaced by eta is equal to 0.

And we still have to write this is as true at z is equal z hat is equal to eta hat, eta hat now is a function of x hat and t hat only there is no y hat, so here x hat t hat ok. And we have to write this because this term and this term are both functions of z and this is this equation is true only at z is equal to eta.

So, when we replace those terms we have to replace z with eta in those expressions, so we have to remember that. So, you can see that this is a boundary condition; however, my boundary itself is fluctuating in time. So, this is in some sense a time dependent boundary condition.

Now, in addition we also have finiteness conditions. So, we will have because my domain is infinite, we are trying to model a situation where my undisturbed interface is flat goes from minus infinity on this side to plus infinity on that side. I am going to introduce perturbations on it and this quantity is eta and we are not putting a wall below and so my vertical coordinate which is z goes all the way to minus infinity. Now because x and z both go to unbounded in this problem, so we will have to keep be careful.

And so we will have to have some finiteness condition which basically says that u at x goes to plus minus infinity at all times you know. So, let us write it in terms of grad phi, so grad phi is the velocity ok. So, when x goes to plus minus infinity and z goes to minus infinity is finite.

So, divergences are not allowed whether we go to minus infinity whether we go to plus infinity in x or whether we go to minus infinity in z, we are not allowed to find solutions which diverge. This is physically meaningful because if I put a perturbation I would like to look for a wave like perturbation.

The wave like perturbation would be typically a sine or a cosine Fourier mode and that stays finite both at minus infinity as well as plus infinity. You will see that we will get exponential solutions along the depth, but we have to eliminate those exponentials which diverge as you go deeper and deeper.

You will see shortly that these this essentially implies that quantities like pressure perturbations and all decay exponentially at the lowest order up to some distance and beyond that the effect of the surface moving is not felt by the fluid below ok.

So, with those conditions let us start analyzing. Now before we look into the analysis we have made a number of simplifying assumptions it may seem that we have made a lot of assumptions and these are too drastic. However, as you can see even with all these approximations we have a quite complicated problem, we have to determine the velocity potential and the interface how it evolves as a function of time.

Typically we will be solving an initial boundary value problem here I have not yet specified the initial conditions. However, in my immediately next slide I will show that we can do a perturbation approximation on this small parameter comes from initial conditions. Now you can see that these equations are still formidable because there is both the boundary conditions are non-linear so non-linear.

And we have a coupling between phi hat and eta hat. So, the equation for phi hat itself is a linear equation, but then phi hat is coupled to eta hat at the boundary and the shape of that boundary is not known a priori. So, we even with the all the simplification it leads to a set of equations which are mathematically quite hard. So, let us see if perturbation can help us simplify these equations. So, in order to do perturbation we will have to first non dimensionalize our system.

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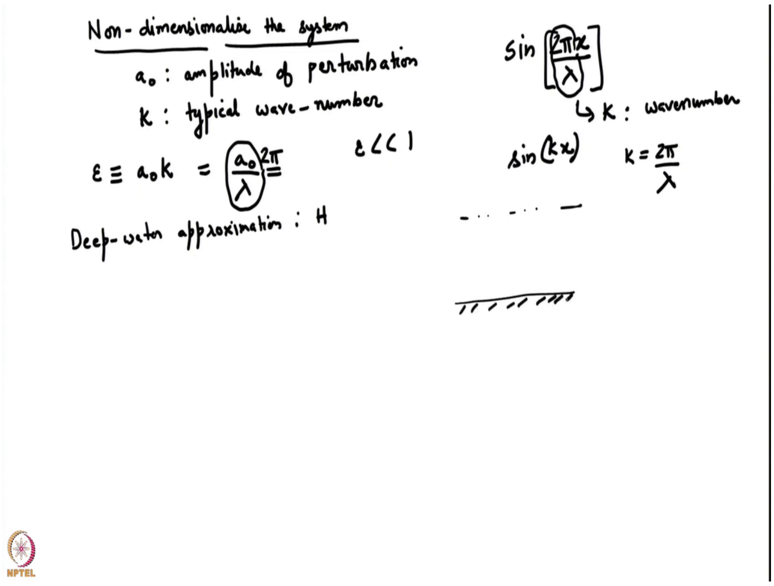

So, I am going to introduce the scales that I am going to use for non dimensionalization so. Firstly, there will be 2 things which will come from initial conditions one will typically put a perturbation, let us say we put a perturbation whose amplitude is a naught, a naught is the amplitude of perturbation and let us say it has a wavelength.

So, it has a wavelength lambda, if you have a wavelength lambda and I put a Fourier mode then I can define something whose x dependence is let us say sin 2 pi x by lambda. So, you can see that this is periodic with a period x is equal to lambda. So, this factor 2 pi by lambda is what is usually referred to as a wave number.

So, k is a wave number and so once you have defined 2 pi by lambda is k this just becomes sin k x. We are extensively going to use this symbol k in the rest of this course, we will not explicitly use lambda you can see that k is related to lambda by definition. So, k has the dimensions of 1 by length, so k is a typical wave number.

We have an expectation that if my amplitude for a given wavelength if my amplitude of perturbation is sufficiently small, then I should be able to solve these systems using perturbation. What would be the small parameter here. So, you can see that epsilon if I define it as a naught into k or this would be a naught into 2 pi by lambda.

So, this 2 pi is not so important it is the ratio of a naught by lambda. So, when I say epsilon is much much less than 1 it essentially means that we are looking at waves whose amplitude is small compared to their wavelength. Note that we have made the what is known as the deep water approximation, we will come back to this later ok. So, we have set the depth of the pool H the undisturbed depth ok.

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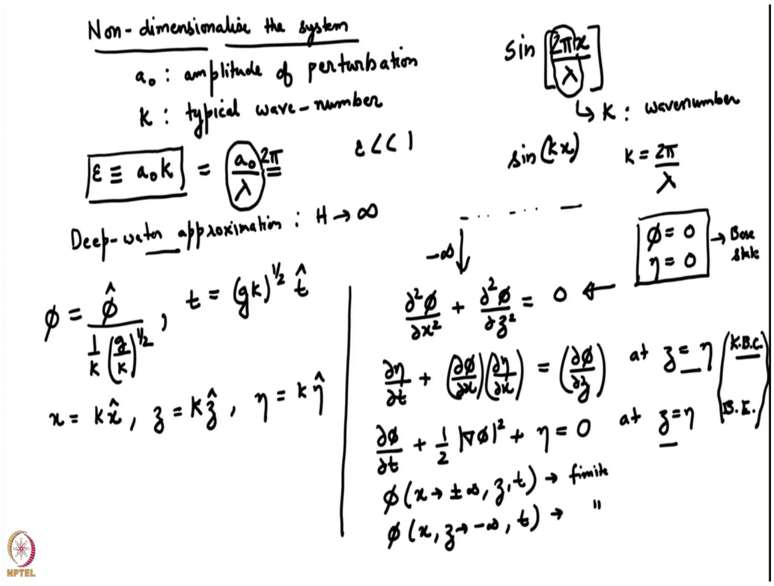

So, we have a pool the interface is flat and we have typically we would have a wall a flat wall at the bottom ok and in this problem meanwhile analyzing this we have set the wall to infinity. So, we have said that it is unbounded vertically. So, this goes to minus infinity. So, H has gone to or H has gone to infinity ok. So, we have made what is known as the deep water approximation the water is sufficiently deep.

We will come back to this approximation later. So, the only length scale 2 length scales are a naught and lambda and so when epsilon is much much less than 1. It means that a naught is much smaller than lambda. So, with that in mind let us now define the non-dimensional scales for our problem.

So, we have to non dimensionalized velocity potential velocity potential has the dimensions of length square by time. So, I have constructed a length scale, so phi hat is dimensional 1 by k g by k to the power half. How these scales are arrived you can think about it later when you calculate the dispersion relation, then it will become clear how did we get these scales right. Now we are choosing something based on the physical parameters of the problem.

We are scaling all lengths by the wave number k a typical wave number that we will introduce in our system ok, you can use k or k naught alright. So, with that let us write down our equations and boundary conditions. You can readily check that after non dimensionalization the form of your equations will remain the same just a little bit of algebra and you will get the same Laplace equation both the terms will have the same non dimensional coefficients and so that is not 0 in general.

So, we get the same Laplace equation as before then we will have the kinematic boundary condition you can check once again. So, the kinematic boundary condition is this and I do not have those del by del y terms. So, you can non dimensionalize the 3 terms the first term the second term and the third term using the scales that I have provided here. And once you do that again with a little bit of algebra it is very easy to show that the form of the equation remains the same.

So, you will just have del eta by del t plus del phi by del x into del eta by del x is equal to del phi by del z and all this is applied at z is equal to eta, the same thing with the Bernoulli equation. The boundary condition which is derived from the Bernoulli equation, so KBC this is KBC and the boundary condition which is obtained from the Bernoulli equation is del phi by del t plus half grad phi square; again all terms have the same coefficient.

So, there was a g in the coefficient of eta. So, the g will get eliminated and you will get this; this is from the Bernoulli equation these are all boundary conditions ok. So, you can see that this is not true everywhere, but only at z is equal to eta that is a governing equation. And we of course have the same requirement that phi when x goes to plus minus infinity at all times at all z and t is finite and written this as a single thing. So, I am writing it properly knows z goes to minus infinity at all time is also finite.

So, that is forms our set of equations that we will need to solve and need to solve perturbatively. The small parameter epsilon is going to come from initial conditions; however, I am not going to straight away solve an initial value problem now. So, I am not specifying the initial conditions, if we specify eta of x comma 0 or eta at initially is some a naught cos k x and if you non dimensionalize eta by this scales you can immediately see that your initial condition would just become eta is equal to epsilon times cos x ok.

So, the small parameter will come from there ok. So, I will assume that there is a small parameter in the problem and then I will expand everything about that small parameter. Once again note that in each of these equations if you substitute phi is equal to 0 and eta is equal to 0 that is a trivial solution to this equations to all the equations that I have written here ok.

This only tells me that phi equal to 0 eta equal to 0 is a base state. So, in the base state there is no velocity and the interface is flat when the interface is flat eta is equal to 0. I am I do not have pressure here the base state profile for pressure is not going to be 0 it is going to be a hydrostatic profile which is linearly varying with respect to depth, we will come to that at the end of the calculation. So, now with these equations let us now set up a Regular Perturbation scheme.

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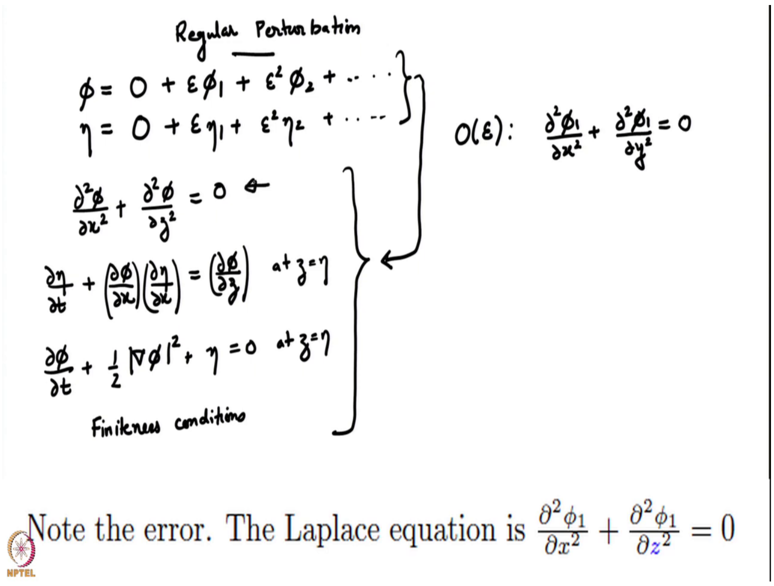

So, a regular perturbation. So, we are going to say that all variables phi and eta right now, we will do this later for pressure also is base state plus some perturbation, eta is also base state plus some perturbation. And we will have to put these into the equations and see what do these equations do and do they lead us to some kind of linear equations which we can then solve analytically let us do that exercise.

So, let us write it so let me write our equations once again. So, our equations where grad square phi is equal to 0 let me write it out in full. Then we had the kinematic boundary condition then we had the Bernoulli condition also at z is equal to eta and then we have the finiteness conditions I am not going to write it this will be important when we actually solve the lowest order system.

So, this is our set of equations we will have to put the perturbations into perturbation expansions into those set of equations and then collect terms at various orders ok like we have done before. So, there is nothing to do at order 1 order epsilon to the power 0 because the base state is trivial here as far as phi and eta are concerned.

So, the lowest order is actually order epsilon here and so you can see that our order epsilon we get the same equation as phi. So, this remains the same in any case the Laplace equation was a linear equation, this is not the source of difficulty in solving these equations it is basically these 2 boundary conditions which is the source of difficulty. Because phi and eta are coupled there and we do not know eta and then these are non-linear boundary conditions. So, now let us collect the various terms that order epsilon from the kinematic boundary condition.

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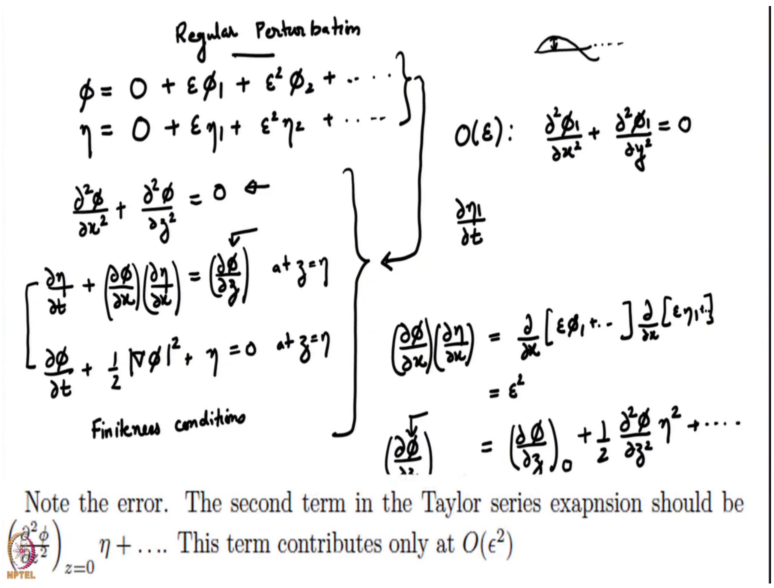

So, we immediately see that at order epsilon we will get the first term del eta 1 by del t ok. Now before we write the next term let me explain what is being done to the next term ok. So, you can immediately see that the next term would be something like this, you can immediately see that it would be del phi by del x into del eta by del x and this term is not going to contributed order epsilon. Because there will be this will be del by del x of epsilon phi 1 plus dot dot dot and this would be del y del x of epsilon eta 1 plus dot dot dot.

And so the lowest order at which it would contribute would be order epsilon square ok. So, we are not going to get a contribution at linear order alright. What about the third term the third the term on the right is so del phi by del z at z is equal to eta. Now we have to be careful here because there are 2 places where the perturbation expansion is going to apply 1 is in phi itself, but another is in the place where you are evaluating the derivative ok.

So, as a first step what we will do is I will replace this with this. So, I am approximating del phi by del z I want to find out del phi by del z at z is equal to eta, eta is the small distance above the flat interface ok. So, I am going to do a Taylor series approximation of this function del phi by del z about z is equal to 0.

So, del phi by del z at z is equal to eta is del phi by del z at z equal to 0 plus del square phi by del z square 1 by 2 into some eta square plus dot dot dot. Now, you can see what is happening now we can substitute the expansions for phi.

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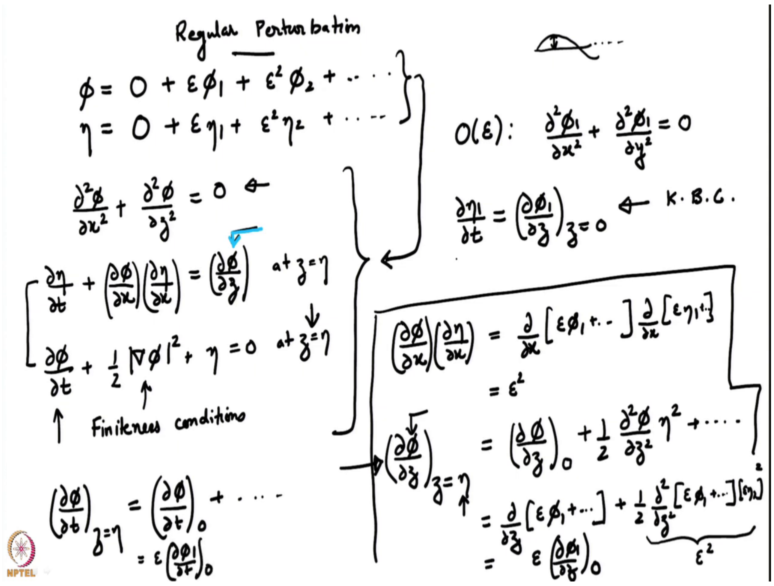

So, if I do that then I will have del by del z of epsilon phi 1 plus dot dot dot plus half del square by del z square epsilon phi 1 plus dot dot dot epsilon eta 1 whole square. You can immediately see that this term is going to be order epsilon square or higher, but this is going to give us an order epsilon term ok.

So, this order epsilon term is going to be del phi 1 by del z evaluated at z is equal to 0, not the unknown interface but at the known base state location. So, we find that so this is all just to justify why I am writing here and so from this term on the right del eta 1 by del t is equal to del phi 1 by del z and now my derivative has to be evaluated not at the unknown interface z is equal to eta, but at the known base state location z is equal to 0.

This is a very important simplification which is has happened because of a combination of the Taylor series expansion and the perturbation expansion, that all my derivatives will be evaluated at the unknown location ok. So, we are going to get this. So, this is the kinematic boundary condition at order epsilon. You can do the same thing to the Bernoulli equation.

In the Bernoulli equation it is the same thing even without doing an expansion you can see that the middle term half grad phi square is not going to contribute at order epsilon. What about del phi by del t del phi by del t once again gets applied at z is equal to eta, you can apply the same logic that I have pointed out here you can do an expansion of del phi by del t at z is equal to eta about del phi by del t at z is equal to 0.

And then you will find that the next higher order terms only contributed order epsilon and this essentially becomes epsilon times del phi 1 by del t evaluated at 0. So, let me write down all my equations at order epsilon.

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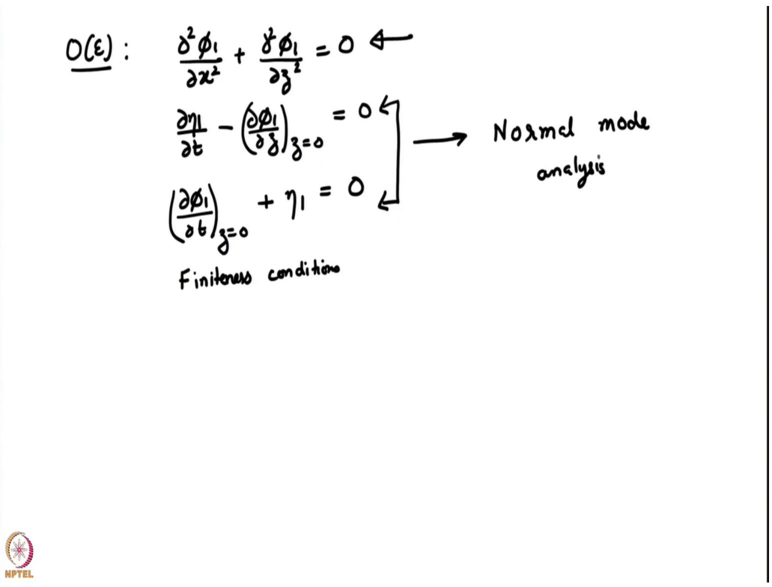

So, at order epsilon we have found that the equations are del square phi 1 by del x square the Laplace equation remains the same. Then we have 2 simplified boundary conditions one of them is del eta 1 by del t minus del phi 1 by del z at z is equal to 0 is 0, I am just shifted it to the left hand side.

And the Bernoulli equation will gave us del phi 1 by del t also evaluated at z is equal to 0 plus eta 1 is equal to 0, this eta 1 comes from the last term in the Bernoulli equation this term at order epsilon it will just be eta 1. So now, this is what we learn. So, what we have what do we have here? We have a linearized set of equations, now our boundary conditions have all become linear.

We can analyze this set of equations and of course we have we will have to remember the finiteness conditions. So, we will have to solve this equation satisfy those boundary conditions and what we are going to do is we are going to apply normal mode analysis. So, we are going to write the solution as some eigen function into e to the power I omega t, you will see that firstly we will have to find out how to solve the Laplace equation in this coordinate system it is very easy.

Then we will have to do a normal mode analysis and then we will have to substitute it into the boundary conditions, when we do this it will give us our frequency relation in this case it will be called a dispersion relation. Once we know the dispersion relation and once we know phi 1 and eta 1 as a function of x z and t our problem is solved we can begin to interpret that problem physically.