Introduction to interfacial waves

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Lecture - 27
Floquet analysis of the Mathieu equation

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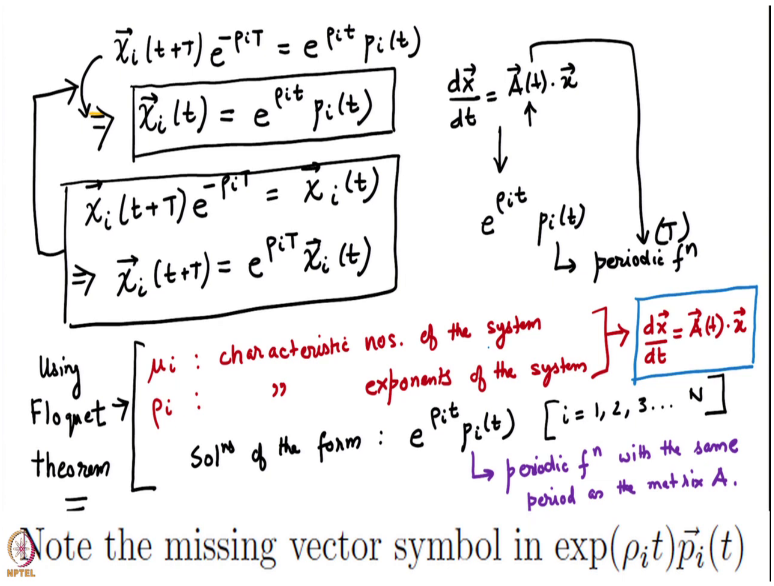

We were looking at the Floquet theorem to analyze the solutions to the system d x by d t is equal to A of t into x, where A is a time periodic matrix in general of size N by N and we had found using Floquet theorem that the general solution to this equation can be written in the form that the typical solution to this equation can be written in the form e to the power rho i T into p i t and rho i is basically defined as using the characteristic numbers of the system mu i. These were related to the eigenvalues of the matrix C that we had seen earlier.

And so, we had found that the solutions to the system can be expressed in this form. So, now, let us use this and apply this to the Mathieu equation that we had found earlier. Now, before we before we do this, there is one more theorem that is necessary. I will just state it without proof. This theorem is also not very difficult to prove. It can be proven in a few lines.

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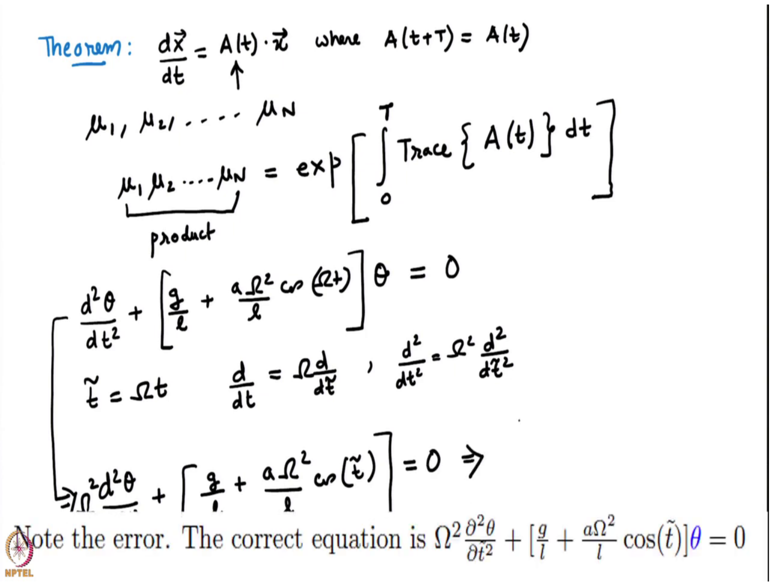

So, the theorem says we are going to use this theorem to make qualitative conclusions about the solutions to the Mathieu equation. So, the theorem once again says is a statement about the characteristic numbers to the system that we are looking at A of t into x. So, N by N system, where A of t plus T; A is a periodic matrix.

So, if the characteristic numbers of the system are mu 1, mu 2 up to mu N, then the theorem says that the product mu 1 mu 2 up to mu N. So, this is a product. The product of all the characteristic numbers is given by exponential of integral 0 to T Trace of the matrix A of t d t.

Recall that the trace of A matrix is the sum of all the diagonal elements of the matrix. So, you have to take the matrix, the coefficient matrix of your system, add up all its diagonal elements, plug this, that will give you in general a function of time and plug this to this integration and take the exponential of it and that will give you the product of your characteristic numbers of your system.

Note that the upper limit of integration is capital T and so, the product will just be a constant. Now, let us we will need this theorem in addition to what we have concluded from Floquet theorem for analyzing the Mathieu equation.

So, let us write the equation that we had found. So, we recall that we are analyzing the motion around the lower fixed point for the Kapitsa pendulum. So, our equation was d square theta by d t square plus g by l plus a omega square by l cos omega t into theta. We had linearized about the lower fixed point. So, psi was 0 plus theta and then, we retain the first term in the Taylor series approximation. Theta is any way non-dimensional.

So, lets non non-dimensionalize time also. So, we define a non-dimensional time which is just omega t. Capital omega is the frequency with which the point of suspension of the pendulum is being oscillated and the amplitude of that oscillation is this quantity small a ok. So, with this, you can immediately see that d by d t is equal to d by d t tilde into omega and similarly, d square by d t square is equal to omega square by d t tilde square.

So, if you plug that in into this equation, I want to express all derivatives in terms of t tilde and so, this equation just becomes d square theta by d t tilde square. There will be an omega square here plus g by l plus a omega square by l cos omega t or cos t tilde is equal to 0.

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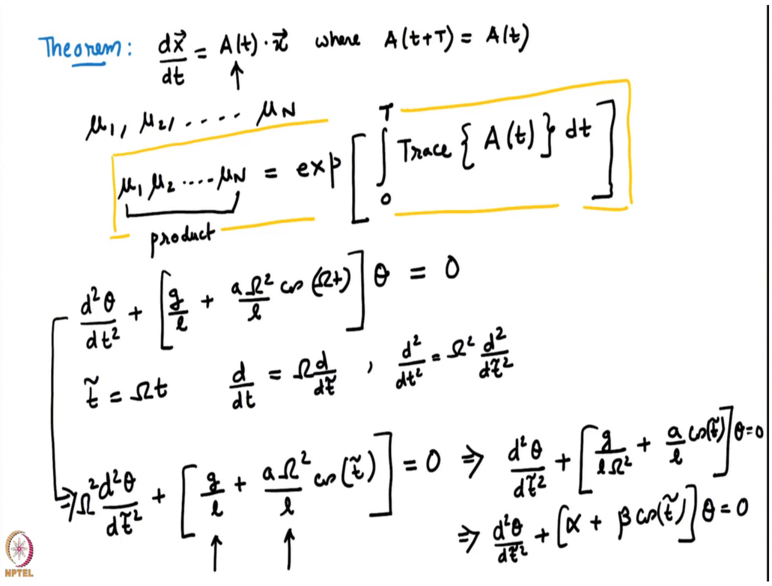

This, I am going to write it as d square theta by d t tilde square plus if I divide throughout by omega square, then I get g the first term inside the bracket I get, g by l omega square. I will call that note that is a non-dimensional number. So, that is g by l omega square plus here I will divide by omega square again and so, I will get again a non dimensional number a by l cos t tilde into theta is equal to 0.

That is my Mathieu equation, whose coefficients are not non-dimensional. I will define them as alpha and beta. So, theta is now a function of t tilde.

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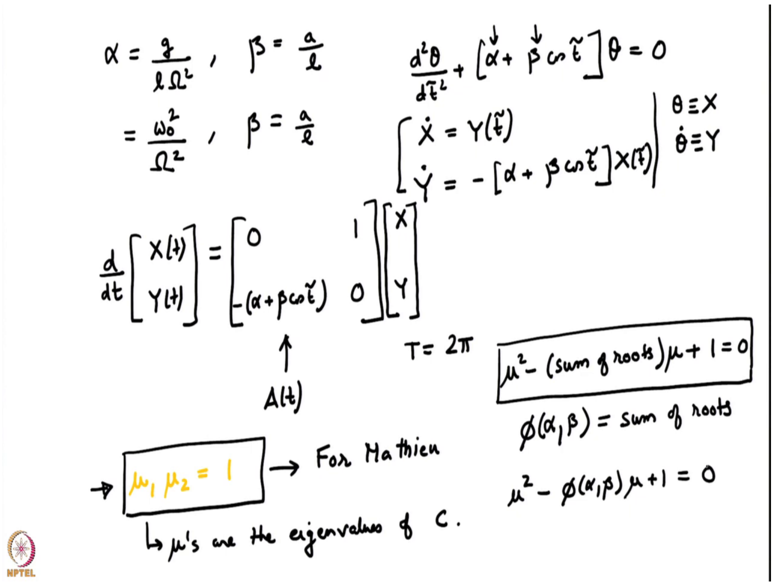

So, alpha is g by l omega square and beta is a by l. You can interpret these non-dimensional variables easily g by l recall is the natural frequency of the pendulum, it is the square of the natural frequency of the pendulum, if the point of suspension is not oscillating. So, this alpha is just a measure of the natural frequency of the pendulum, square of the natural frequency to the forcing frequency.

Similarly, beta is a non-dimensional measure of the amplitude. We are moving the pendulum point of suspension up and down with an amplitude small a. So, it measures how far has the pendulum gone compared to its string of length l, which is treated as inextensible here. So, beta is a by l. So, that is the physical meaning of these two non-dimensional numbers.

So, our system now becomes write it again here. Let us convert it into a set of two first order ordinary differential equations, the way we had done it before. So, if I put that here, then X dot is Y and Y dot is minus alpha plus beta cos t tilde X. So, we have to remember that theta is defined as X and theta dot is defined as Y, these are definitions.

So, with those definitions, I get d x by d t tilde is Y. This is a function of t tilde and d y by d tilde is X which is also a function of t tilde with a coefficient which is again time dependent. So, now, I can write this as a first order system.

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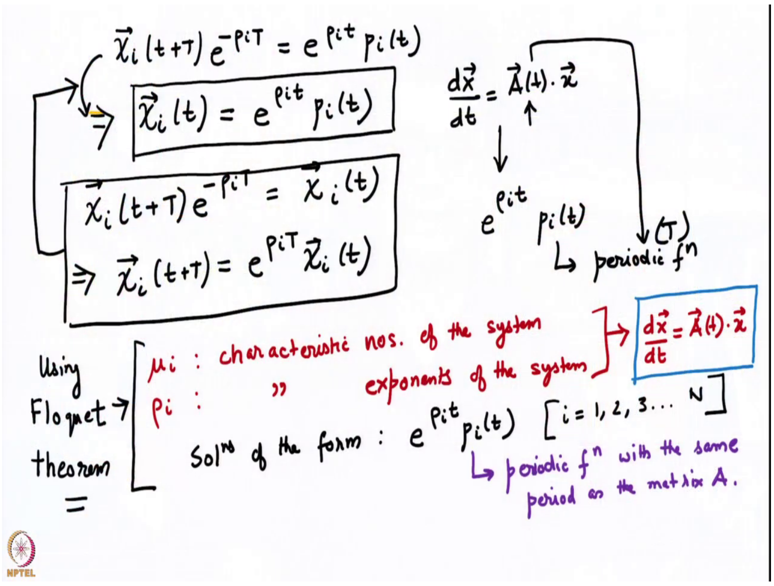

What I want to do is I want to make conclusions about the solution to this first order system, from whatever we have concluded earlier about from Floquet analysis. So, we can write this first order system as d by d t of X and Y and the coefficient matrix here A of t is a time dependent matrix as you can see 0, 1; this will be time dependent. So, let me put the 1 slightly away and then, this is 0.

I am just rewriting these two equations in matrix format and then, this has to be multiplied by X and Y. So, now, you can see that this is my A matrix; A of t. you can also see that A is a periodic matrix; it has a time period 2 pi. So, T in this case is 2 pi because alpha and beta are just constants. So, after 2 pi cos t will repeat itself, so alpha plus beta cos t will also repeat itself.

So, now, let us apply the theorems that we have encountered, one we have proved the Floquet theorem which says that the solution to this system can be written as a product of e to the power some characteristic exponent into a periodic function. The periodic function will have the same period as the matrix A of t. So, in this case, the periodic function will have a period of 2 pi.

We have one more theorem which we have written without proof which says that the product of the characteristic values of the system can be obtained by this formula. Now, note that the A matrix has trace 0 for our Mathieu system. So, the trace here is just 0 in this formula on the right. So, the integral just evaluates to 0, exponential of 0 is 1, the our Mathieu system is a 2 by 2 system.

So, we have only two characteristic values. So, we immediately conclude that mu 1 into mu 2 is equal to 1 for our Mathieu system. There are only two mu's; mu 1 and mu 2 and the product of them is 1 Told you before they need not be necessarily real numbers, they can also be complex numbers ok; but their product is always 1. So, with that and recalling that mu is the mu; mu's are the eigenvalues of the matrix C, that we have encountered while proving flow case theorem.

So, now, let us see without actually working out the matrix C and without actually calculating what is, what are the values of mu 1 and mu 2; what are the qualitative conclusions that we can draw about the solution to the Mathieu equation, using what we know so far. So, recall that mu's are the eigenvalues of C. So, mu must come from a characteristic equation, C in this case would be a 2 by 2 equation and if mu is the eigenvalue, then mu are the roots of the characteristic equation for C.

So, this that characteristic equation will be a quadratic equation in this case. So, it will be of the form mu square sum of roots; the roots are basically the eigenvalues into mu plus product of roots. The product of roots in this case is 1. So, I know that this term is 1. So, mu must be the root of an equation whose form is this. This we already know.

Let us just call the sum of roots as some function phi. Phi in general, you can see that what will determine phi? There are only two parameters in our problem; alpha and beta. So, by changing alpha and beta, I should be able to tune the roots of the system. So, phi in general is expected to be a function of alpha and beta.

So, the sum of roots is let us say phi and now, let us explicitly because this is a quadratic and because we can write down the roots of the quadratic, let us write down the. So, now, we have our quadratic takes the form phi, alpha, beta mu plus 1 is equal to 0. Let us write down the roots of this quadratic.

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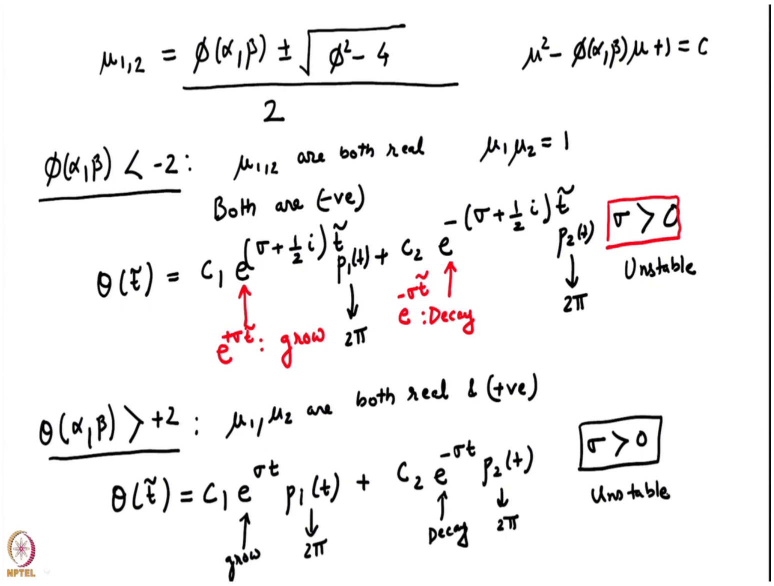

So, this will have two roots mu 1 and mu 2 and this will be of the form phi minus b plus minus root over b square minus 4; a and c are 1. So, let me make this shorter and then, divided by 2. For reference, I will write the quadratic again here. This is the quadratic whose roots are mu alpha and beta and this is the formula for those roots.

Now, we obviously, do not know what is phi as a function of alpha and beta; otherwise, we could have determined mu analytically. But you can immediately see that a number of cases arises which will decide the structure of mu and in turn, decide the structure of the solution to that equation.

The first thing that you should notice is that that here unlike all the problems that we have found, the solution to the equation in this example as I have said before has exponential rho i and rho could be purely imaginary, it could be purely real or it could be in general a complex number.

In particular, if it is purely real and if it is if rho is positive, then you can see that you your solution has two parts an oscillatory part which is the periodic function p i of t. But it also has a pre-factor which is exponential and if the exponent is positive, then that pre-factor will grow. What that means is that the Mathieu equation can have solutions which grow in time and which grow exponentially in time.

Of course, the periodic part will cause them to oscillate, but the amplitude will grow with every oscillation. We have not encountered instability in all the examples that we have seen until now, we have only encountered oscillatory behavior. This is the first example, where we will find that depending on the value of alpha and beta, we can have oscillatory solutions.

They may or may not be periodic, but we can also have growing and decaying solutions In particular, the growing solutions will be of interest because they signify instability of the system ok. So, we can distinguish now depending on phi. So, you can see that whether mu is complex or not depends on whether phi is greater than 2 or less than minus 2 or in between the 2 ok.

So, we can we can distinguish four separate cases and I am just going to write down the qualitative aspects. Until now, we have not determined phi analytically; but you can you can conclude this. So, phi is a function of alpha beta and if phi is less than minus 2, this is one case. So, you can see that phi square minus four is real and so, mu 1, 2 are both real and because the sum phi represents the sum of mu 1 and mu 2, they are both real, their product we have seen is 1 which is positive.

So, we cannot have one of them positive, one of them negative. So, in this case both are negative. So, that the product is positive and the sum is less than minus 2. One can use a similar reasoning to show that in this case the general solution to our equation theta of t tilde is c 1 e to the power sigma plus half i t tilde plus c 2. This is the general structure of the equation, when phi is less than minus 2 and sigma is greater than 0.

Note that this is an unstable solution. There is there are two parts to it. So, e to the power half i that is an oscillatory part. So, that is not going to diverge in time. However, there is a e to the power sigma t tilde and e to the power minus sigma t tilde. If sigma is greater than 0, then this part is going to decay in time; e to the power minus sigma t tilde is going to decay, but e to the power plus sigma t tilde is going to grow and it will grow in an oscillatory manner. There is also p 1 here p 2 here and these are 2 pi periodic as we have seen from Floquet theorem.

So, now, you can see that this solution when theta is less than minus 2, we are going to have growing solutions; but the growth is going to happen in an oscillatory manner. In particular, you can see that this half indicates will grow in time; but the oscillatory part will have half the frequency of the forcing. Now, similarly, one can also write down the solutions for theta greater than plus 2; then again, we have unstable behavior. In this case, again mu 1 and mu 2 are both real and positive and the general solution looks like this. So, this is p 1.

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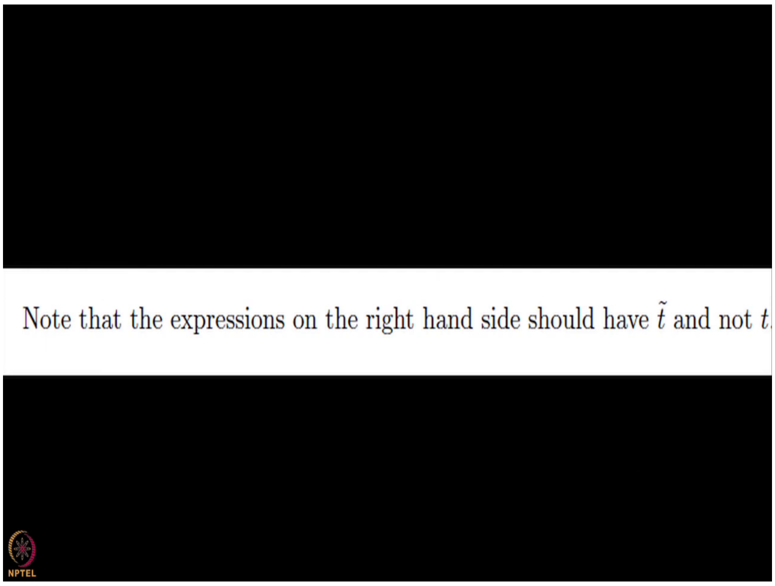

And so, this like usual is a 2 pi periodic function. This is another 2 pi periodic function and sigma also here is greater than 0. So, again, this will grow, this will decay and so, once again, you have an oscillatory response and the oscillatory response now, an oscillatory response which whose amplitude grows with time and in this case, the frequency of the oscillation will be the same as the frequency of the forcing.

So, here this response, this is an unstable response, this is also an unstable response; but both of them are oscillatory unstable responses. In the first case, the frequency is one-half the forcing frequency. In the second case, the frequency in the oscillatory part is the same as the forcing frequency ok.

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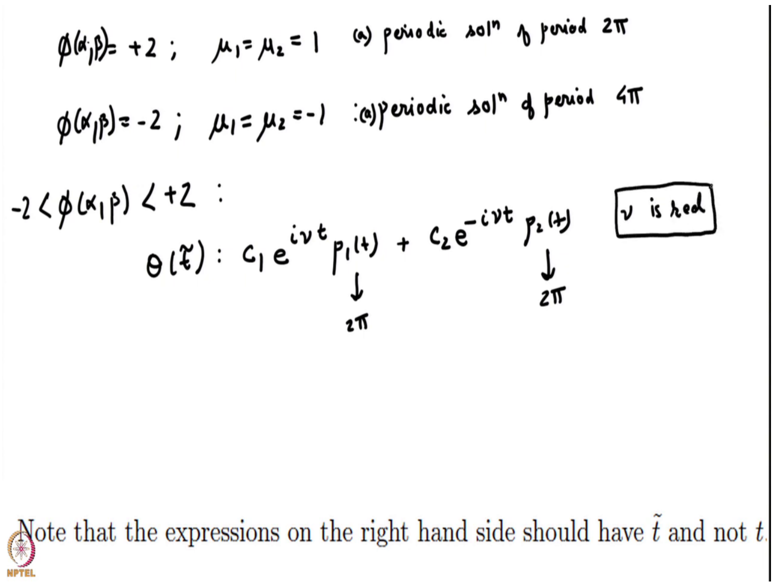

In addition, we also have bounded oscillatory solutions to this. These occur when theta is equal to plus 2. In that case, mu 1 is equal to mu 2 is equal to 1. It is not theta, its phi; phi, alpha, beta. So, we have looked at the range where phi is less than minus 2, greater than plus 2 and so, in this case we will; so, there are three separate regimes now. So, phi is equal to plus 2 mu 1 mu 2 1, then phi is equal to minus 2. In this case, mu 1 equal to mu 2 is equal to minus 1. In this case, there is a periodic solution, periodic solution of period 4 pi.

So, there are two solutions; one is periodic and the other one is unstable. Here also there are two solutions; one is periodic with period 2 pi, periodic solution of period 2 pi; while the second solution diverges in time. We have left behind one more case which is phi of alpha beta is between minus 2 and plus 2.

Here also you get bounded oscillatory solutions and the general solution here in this case is of the form theta of t tilde c 1 e to the power i nu t p 1 of t plus c 2 e to the power minus i nu t p 2 of t. This is 2 pi and this is 2 pi periodic ok and nu is real.

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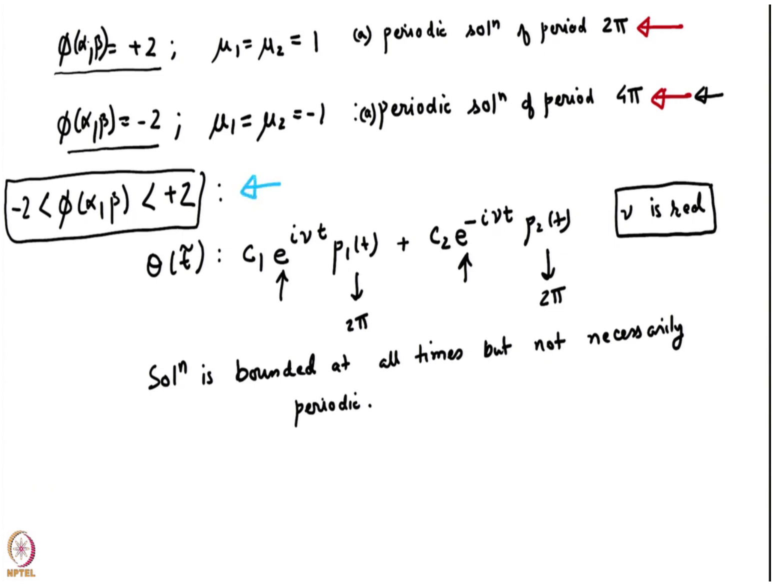

So, this in this regime, the solution is bounded. You do not have any growth because you can see that there is no quantity, none of these terms grow in time. This one oscillates, this also oscillates and p 1 and p 2 are anyway periodic functions of time. So, solution is bounded at all times, but not necessarily periodic.

So, without even finding phi as a function of alpha and beta, we are able to conclude about the qualitative behavior of the different kind of solutions to the Mathieu equation and we have found out that there are various all of this can be thought of on the alpha-beta plane.

So, on the alpha-beta plane, at every point, you will have a given value of alpha beta. Now, depending on whether we know what is phi as a function of alpha and beta, we can demarcate these various regions. So, for example, this will be a set of curves, this will be another set of curves, this will be a region on the alpha beta plane and similarly, the other two inequalities phi less than minus 2 and phi greater than plus 2 will also be regions because these are inequalities.

So, what these do is these, these split up the alpha beta plane into various parts. In each part depending on which regime we are in, we may have bounded oscillatory solutions or we may have exponentially growing solutions. But they will typically grow in a oscillatory manner and we know what is the frequency of that oscillation. In general, one can plot those curves numerically.

There are analytical, there are perturbative ways of also plotting those curves. We will not go through those. But I will just tell you the qualitative nature of these curves on the alpha-beta plane. So, the qualitative nature of the curves on the Mathieu equation.

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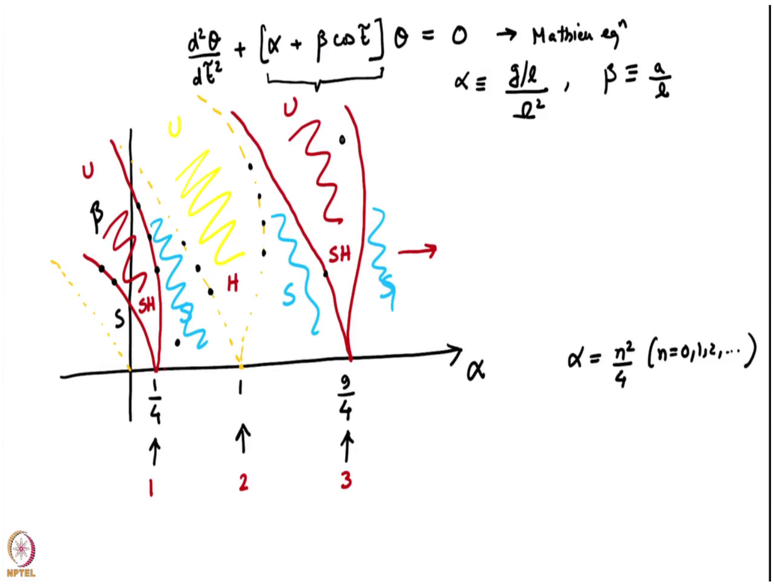

So, again for reference, I am writing down the equation d square theta by d t tilde square plus alpha plus beta cos t tilde into theta is equal to 0. This is the Mathieu equation. Using Floquet theorem, we have found that in general, the solution to this equation can be written as exponential to some exponent into t multiplied by a periodic function. This generalizes what we knew from normal modes.

If this part was constant, then we could have just done exponential into lambda t. Now, we have to do exponential into some mu into t multiplied by a periodic function of t. We have also used Floquet theorem to deduce qualitative nature of the solutions on the alpha-beta plane.

So, let me draw the alpha-beta plane. So, recall that alpha is here, the natural frequency of the pendulum, square of the natural frequency to the forcing frequency and beta was the amplitude to the length, the amplitude of oscillation to the length. So, on the alpha-beta plane, so we will put beta in the vertical axis and alpha here. We will get various curves depending on those inequalities that we solve for. In general, the qualitative nature looks like this ok.

So, we will get these tongue-shaped structures and there will be a single yellow ok. So, let me 1 this point is 1 by 4, this point is 1, this point is 9 by 4. So, in general, at every alpha is equal to n square by 4, where n goes from 0, 1, 2, 3 and so on. So, you can see that this is n is equal to 1, n is equal to 2, n is equal to 3 and so on ok. Now, what do those these regions imply?

So, inside each of these regions, so suppose you are inside this region, suppose you are inside this region, then we will observe unstable behavior. Suppose, you are inside the red region or this region, we will again observe unstable behavior, unstable behavior.

There will be a qualitative difference between unstable behavior in the yellow region and the red region. In the sense that in both cases, we will see exponential growth with respect to time, but the oscillatory part in one case will have the frequency of the oscillatory part will be one-half the forcing frequency; in the other case, the for the frequency of the oscillatory part will be the same as the forcing frequency.

We will get similar such tongues, if you keep going here. So, beyond this, so this is 1, this is 2, this is 3 and say even at 4, you will get another such tongue. And the first tongue is where you get unstable behavior and the frequency of oscillation is one-half the forcing frequency. So, it is called sub-harmonic.

The next tongue is a harmonic tongue, you which we see growth exponential growth, but the pre factor is an oscillatory function, whose frequency is the same as that of the forcing frequency. So, this is the same as the forcing frequency. So, this is harmonic. The first one is sub harmonic; the third one is again sub-harmonic. So, alternatively, we will see harmonic, sub-harmonic, harmonic, sub-harmonic like that alternative behavior.

On these tongues, one will see the behavior that we have outlined here this and that. So, on these tongues, there will be periodic solutions. Again, there will be two solutions; one will grow in time diverge whereas, the other will be periodic. So, on the sub harmonic tongues the on; so, if you are at a point on the tongue; so, if you have a point on the tongue.

So, on the boundary, not inside, then one would see a periodic solution, whose period is sub harmonic. So, 4 pi ok. So, this corresponds to this solution, the lower one, the lower solution. On the harmonic tongue, if you take a point on the harmonic tongue, we will see another periodic solution and that would be period 2 pi. So, here if we take points, so this is again a sub harmonic tongue. So, let me take here. So, if you take on this on the boundary, then there are periodic solutions; but there are also diverging solutions on the boundary ok. What about the region in between these tongues?

So, these regions are stable. This is stable. Again, there will be a tongue which will come out when n is equal to 4, but between the n is equal to 9 by 4 tongue and the n is equal to 4 square by 4 tongue, there will be another stable region. This region, so this if you choose alpha beta in this region.

In this region and in this region in between the tongues, then you will get stable bounded oscillatory behavior that corresponds to this this solution. Once again this is an inequality. So, we are getting regions on the alpha-beta plane. Every equality produces curves, there are many such curves, there are many such tongues and every inequality will produce a region.

So, we have regions of stable behavior, we have regions of unstable behavior and the boundary between them there are periodic solutions; but they are also diverging solutions on those same boundaries. So, this is the qualitative behavior. There is also another a single line which comes out, I will just put it here out of 0 ok. And this region is stable, this region is stable.

So, this is what we infer about the qualitative nature of the solution to the Mathieu equation. This equation can be easily solved numerically on the computer using a suitable package like Mathematica or MATLAB. I encourage you to try this. You can choose, take this equation, choose simple initial conditions, you can choose theta of 0 is 1, theta dot is 0, it is a second order equation. Solve it by choosing a certain value of alpha and beta.

If you have a stability chart like this, you can find out whether the choice of alpha and beta that you have made corresponds to points inside those tongues or corresponds to those regions which have indicated in the light blue line. If you choose a point of alpha comma beta. So, if you choose something which if you choose a point here, then you would obtain bounded oscillatory behavior.

If you choose a point there for example, you would obtain exponential growth; but it will oscillate and grow and the oscillation frequency will be sub-harmonics. So, it will be half the forcing frequency.

So, like that, one can go to separate different parts of the region and get different kinds of behavior; stable or unstable. Physically it implies that the pendulum, the oscillating Kapitsa pendulum that we are seeing, the lower point is not necessarily a stable point. It depends on alpha and beta. So, by suitably choosing alpha and beta, one can get oscillations of increasing amplitude.

Of course, when the when the amplitude gets much larger, one has to take into account the non-linear terms and so one the exponential growth that one finds here in the unstable regions, may be cut off by higher order non-linearities. Similarly, that our top most fixed point and that is the more interesting part of the Kapitsa pendulum; the topmost fixed point in the absence of forcing, we know is an unstable point.

If you keep the pendulum like this, it is going to fall. However, by adjusting alpha and beta suitably, one can render the topmost point stable, at least for small amplitudes. There are some very interesting videos which demonstrate this experimentally. I encourage you to look up in google. Look for stabilization of the inverted pendulum.

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