Introduction to interfacial waves
Prof. Ratul Dasgupta
Department of Chemical Engineering
Indian Institute of Technology, Bombay

Lecture - 26
Floquet theorem (contd.)

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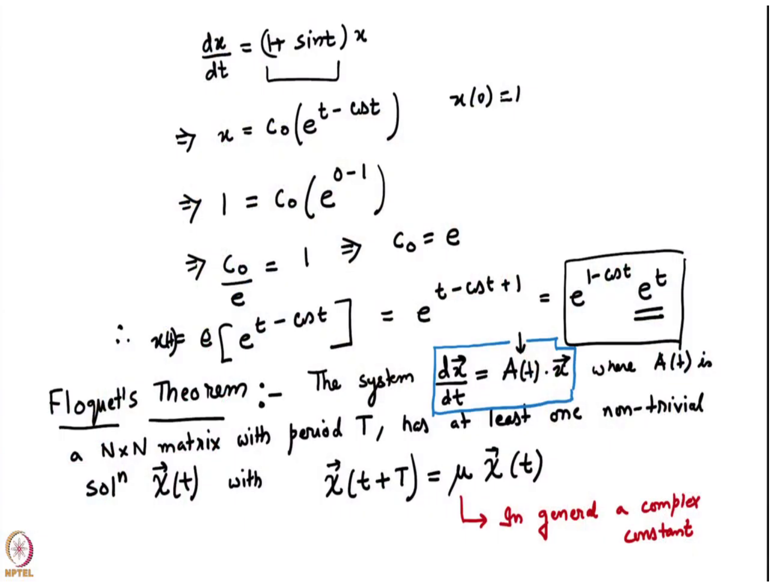

We were looking at the proof of Floquet's theorem. Recall the Floquet's theorem says that if you have a first order system dx by dt is equal to a into x, where a is a N by N periodic matrix with period t then this system has at least one non trivial solution chi of t, where chi of t satisfies this relation that chi of t plus the time period of the matrix capital T is equal to mu into chi of t.

Note that mu in general is a complex constant. We had started by looking out at the proof of this theorem.

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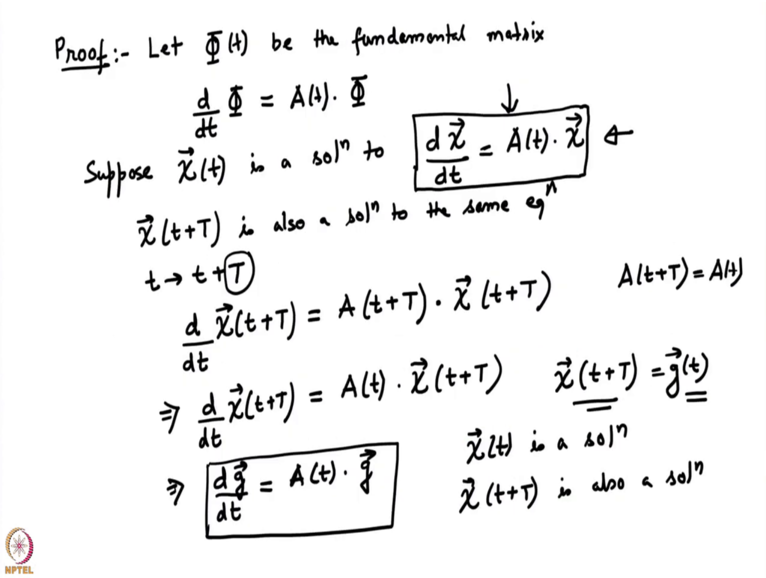

So, for that we introduced the notion of a fundamental matrix of a system and then we showed that if chi of t is a solution to this equation then chi of t plus T is also a solution. So, now, recall that the definition of the fundamental matrix of our system is just that if we have N linearly independent solutions if we place them column wise side by side in a matrix then that gives us the fundamental matrix of our system.

So, now let us look at these solutions chi of t and chi of t plus delta T.

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N L.I. sol^{ns} $\vec{\chi}_1(t),\ \vec{\chi}_2(t) \dots \vec{\chi}_N(t)$

$$\Phi(t) = \begin{bmatrix} \chi_{11}(t) & \chi_{12}(t) & \dots & \chi_{1N}(t) \\ \chi_{21}(t) & \chi_{22}(t) & & \chi_{2N}(t) \\ \vdots & \vdots & & \vdots \\ \chi_{N1}(t) & \chi_{N2}(t) & & \chi_{NN}(t) \end{bmatrix}$$

$$\vec{\chi}_1(t+T),\ \vec{\chi}_2(t+T) \dots \vec{\chi}_N(t+T)$$

$$\Phi(t+T) = \begin{bmatrix} \chi_{11}(t+T) & \dots & \chi_{1N}(t+T) \\ \chi_{21}(t+T) & \dots & \chi_{2N}(t+T) \\ \vdots & & \\ \chi_{N1}(t+T) & \dots & \chi_{NN}(t+T) \end{bmatrix}$$

$$\underline{\underline{\Phi}}(t+T) = \underline{\underline{\Phi}}(t)\cdot \underline{\underline{C}} \leftarrow \text{must be true}$$

So, suppose we have found N linearly independent solutions N linearly independent solutions and I will call them chi 1 of t, chi 2 of t and so on up to chi N of t. We can form a fundamental matrix of our system and this would be phi of t which would basically just be all the chis arranged side by side as a column.

So, chi 1 if I write it chi 1 written out as a matrix we will have N rows and one column. So, I will write it write out the elements of chi 1, so, chi. So, this is all elements of chi 1. So, this will be chi 11. So, it is all the second element of chi 1 and then the nth element of chi 1.

Similarly, chi 2 the first element, chi 2 the second element and then nth element and then the last one chi N, the first element the second element of chi N and the nth element of chi n. So, we have our N by N system and this is the fundamental matrix of our system. we also know that if chi 1, chi 2, chi 3 up to chi N are solutions to our system we have just proved then that chi 1 of t plus T, chi 2 of t plus T up to chi N of t plus T each of these is also a solution to our system.

We can form one more fundamental matrix and we will call this t plus T and this will be arranged by arranging each of those as columns just as we had done earlier. We took chi 1 chi 2 chi N up and arrange them as columns. Now, I will take chi 1 at t plus T chi 2 at t plus T each of these are also solutions. So, I will arrange them as columns. So, these are all at t and the corresponding ones here would be at t plus T.

I am not writing the second one and so, if this is a fundamental matrix then so, is this because by definition the fundamental matrix consists of all the solutions to the system all the linearly independent solutions to the system arranged as columns. Now, you can see or I will assert that the relation between phi of t plus T is phi of t dotted with some N by N matrix c, this must be true.

Why should it be so? Let us take one simple example to understand and you can immediately generalize that. So, I will take a simple 2 by 2 example and then we can immediately generalize them to this N by N case. So, let us take a 2 by 2 example.

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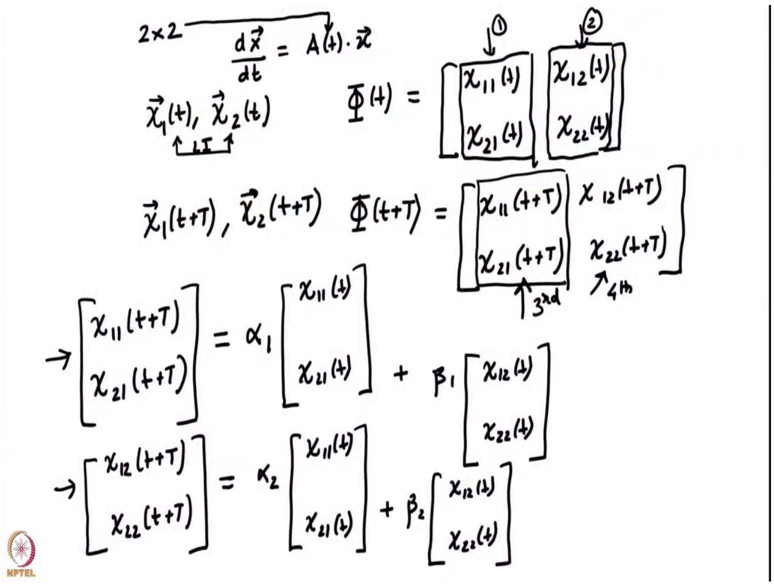

So, my system is 2 by 2. So, then we will have so, d x by d t is equal to A of t dot x where A of t is a 2 by 2 matrix ok. So, there will be two linearly independent solutions to this system in general. So, I will call them chi 1 of t and chi 2 of t. The fundamental matrix phi of t for this system would be just a 2 by 2 matrix like before and we know that if chi 1 chi 2 are solutions then so, are chi 1 at t plus T and chi 2 also at t plus T. So, I can again get one more fundamental matrix at t plus T which is just the same things this one.

So, you can see that these two are linearly independent ok. So, I can write the column. The first column of chi phi t plus T, it is basically chi 11 t plus T and chi 21 t plus T. Now, this is some vector. It is a solution vector to the 2 by 2 system and because this is a solution it can be expressed as a linear combination of the two linearly independent solutions that we have.

I expect this is a 2 by 2 system. There can be at most two linearly independent solutions. I have now three solutions. One solution is this and another two linearly independent solutions are this and that. So, I can express this 3rd one as a linear combination of the 1st and the 2nd linearly independent solutions that must be true. So, I will write it slightly write it down because there is no space.

Similarly, I can take the 4th one and I should be in general again be able to express because this is also a solution, this 4th column of the equations. So, in general I should be able to express it as a linear combination of 1 and 2, but the coefficients will in general be different.

So, this would be some alpha 2 into again chi 11 of t into chi 21 of t plus some beta 2 into chi 12 of t into chi 22 of t. What does this tell us? This tells us that I can place these two columns side by side and create the matrix phi t plus T. And I can write the right hand side of these two equations as the product of two matrices. You can verify that this is true.

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$$\begin{bmatrix} x_{11}(t+T) & x_{12}(t+T) \\ x_{21}(t+T) & x_{22}(t+T) \end{bmatrix} = \begin{bmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix}$$

$$N \times N : \quad \boxed{\Phi(t+T) = \Phi(t) \cdot C}$$

$$\mathrm{Det}(A \cdot B) = \mathrm{Det}(A) \cdot \mathrm{Det}(B)$$

$$\mathrm{Det}\,|\Phi(t+T)| = \mathrm{Det}\,|\Phi(t)| \cdot \underline{\underline{\mathrm{Det}\,|C|}}$$

$$\mathrm{Det}|C| \neq 0$$

$$C = \Phi^{-1}(t) \cdot \Phi(t+T) = \Phi^{-1}(0) \cdot \Phi(T)$$

You can verify that in this simple 2 by 2 example this is true. I am just placing the two column vector side by side and creating a square matrix on the left and then on the right what I have is just this. And the matrix that you will have to multiply it with is alpha alpha 1 beta 1 alpha 2 beta 2.

You can check that this is indeed what is the what you would get if you multiplied these two matrices and you would recover the equation that we had written in the last slide. So, this you can immediately generalize this thing to the N by N case and so, I had asserted earlier that I will be in general able to write a relation like this which connects the phi the fundamental matrix at t plus T to the fundamental matrix at t and it multiplies it by a square matrix whose dimensions is N by N.

So, now you can immediately see that phi in the N by N case phi at t plus T we will in general be able to write it as phi of t dotted with a matrix C. You can see that because the determinant follows the rule the determinant of A dot B is determinant A into determinant B. Therefore, you can see that determinant of phi of t plus T is equal to determinant of phi of t into determinant of C.

And you can see that because this is a fundamental matrix and this is also a fundamental matrix their determinants cannot be equal to 0. Therefore, determinant C in general is not equal to 0. Now, what did we learn, what do we gain by doing this? You will find that the solutions to this equation the solutions to our first order system are related to the Eigen values of C. Let us see how.

So, we can see that from this matrix equation involving fundamental matrices we can obtain that C is equal to because phi is the has a non zero determinant; that means, it is invertible. So, I can in general write it as this is phi inverse t dotted with phi of t plus t. And if I choose the time this is this is true expected to be true at all times. So, if I choose small t is equal to 0, then this just becomes this.

And if you can adjust the initial conditions, so that phi inverse of 0 becomes the unit matrix then the C matrix is just obtained by taking the fundamental matrix and evaluating it at the time period of the coefficient matrix. Now, let us move onwards to completing the proof of the Floquet theorem. We need this in order to prove the Floquet theorem.

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Let μ & $\vec{s}$ be eigenvalue & eigenvector of C

$$\boxed{C\cdot\vec{s} = \mu\vec{s}}$$

$\boxed{\Phi(t)\cdot\vec{s}}$ is a sol^n to our system $\dfrac{d\vec{x}}{dt} = A(t)\cdot\vec{x}$

$$\begin{bmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

$$= s_1\underbrace{\begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix}}_{\text{sol}^n \text{ to our system}} + s_2\underbrace{\begin{bmatrix} x_{12}(t) \\ x_{22}(t) \end{bmatrix}}_{\text{sol}^n \text{ to our system}}$$

$\therefore$ $\boxed{\Phi(t)\cdot\vec{s}}$ is a sol^n

$\therefore$ we can set

$$\vec{x}(t) = \boxed{\Phi(t)\cdot\vec{s}}$$

$$\Rightarrow \vec{x}(t+T) = \Phi(t+T)\cdot\vec{s}$$

$$\Rightarrow \vec{x}(t+T) = \left(\Phi(t)\cdot C\right)\cdot\vec{s}$$

$$= \Phi(t)\cdot\left(C\cdot\vec{s}\right)$$

$$= \Phi(t)\cdot\left(\mu\vec{s}\right)$$

$$= \mu\,\Phi(t)\cdot\vec{s} = \mu\vec{x}(t)$$

Let mu and S be eigen value and eigen vector of the matrix C this implies that C dotted with S is equal to mu dotted with S or mu S because mu is a scalar. Note that phi the fundamental matrix evaluated at time t dot S is a solution to our original system. The system is d x by d t is equal to A into x.

How do we see that? Let us say let us take our simple two dimensional example. So, the two dimensional example is just the 2 by 2 example is just this this is the fundamental matrix into let us say the component of the eigen vector in this 2 by 2 system is just S 1 into S 2.

If you multiply this I will in general get a column matrix, but note that I can write this multiplication as S 1 times the first column of the square matrix plus S 2 times the second column. Note that it is equivalent to writing like this. What does this give us? This tells us that this product of phi t dotted with the S vector is just a linear combination of those two vectors.

What are those two vectors? Each of them is a solution to our system; solution to our system by definition. We had placed each solution as a column in the fundamental matrix. So, each of them is a column in the fundamental matrix and what is this is showing us is that this is just a linear combination of the columns. If I have two solutions and if I take the linear combination then what I get is again a solution. This is a property of the system.

So, this is if I go back and substitute this will be a solution to my set of equations. So, therefore, phi t the fundamental matrix dotted with S is a solution to our system. So, therefore, we can set chi of t, we had said that chi was a symbol for solutions to our system. We have found once a set of solutions. So, I can set chi of t to be equal to phi of t dotted with S.

So, this implies chi of t chi of t plus T is equal to phi of I am just replacing with t plus T. And this implies if I work on the right hand side, so, phi of t plus T is we have shown into the matrix C this we have shown earlier here we have shown. So, I am just using that.

So, this phi of t plus T is being replaced by phi of t into C and this is chi of t plus T. And now I can do this matrix multiplication later and put C inside the bracket and multiply it with S. C dot S we have set is equal to S is by definition an eigen vector of the matrix C with eigen value mu. So, this is equal to phi of t dotted with mu of S and this because S mu is a scalar, so, I can write this as phi of t dotted with S and from here we know that phi of t dotted with S is nothing but chi of t. So, this is mu times chi of t.

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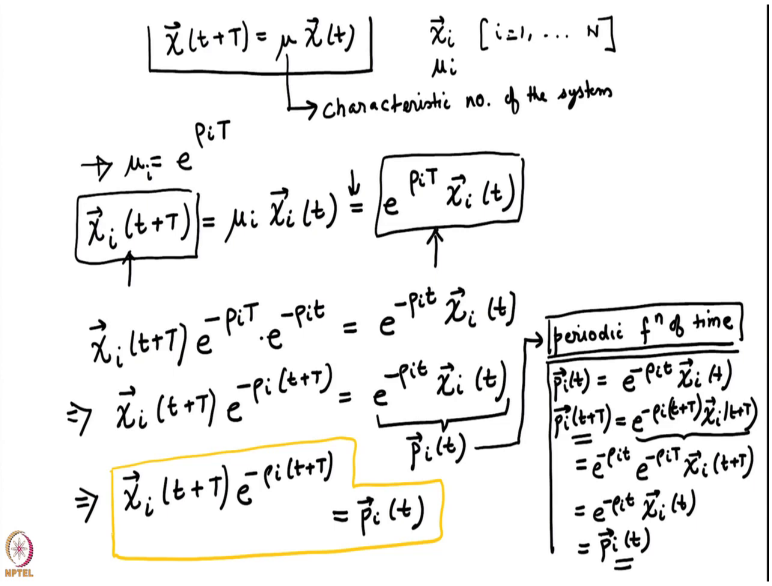

So, we have proved the Floquet theorem. We have shown that if you have a solution to the linear homogeneous time periodic first order system dx by dt is equal to ax then if chi of t is a solution then it satisfies chi of t plus T is equal to mu times chi of t. This was the assertion and we have just proved it. You can go over the various steps. In case things are not clear you can go over it once more ok.

So, this number mu which I had said could be possibly complex is called a characteristic number of the system. Now, what can we use the Floquet theorem for? We can use the Floquet theorem to come with the structure for what would be the solutions to a equation whose coefficients are time periodic. Let us see how.

So, because mu is a characteristic number of the system we will set mu is equal to e to the power rho I into T. In general I would have N different solutions. So, there would be chi i, i going from 1 up to N. So, for each of such chi there would be a corresponding mu. So, mu i this is just a definition of rho i. We are doing this because we want to introduce an exponential in our final answer.

So, I define this rho i by this equation that e to the power rho i into the time period of your coefficient matrix is equal to mu i or in other words rho i is equal to 1 by T log mu i ok. So, how what do we gain by doing this? So, let us say we have a solution to our first order periodic system and we know from Floquet theorem that it satisfies this.

Now, this implies I can write this as by the definition of rho i, I can write this as chi i into T. Now, if I take this equal to that, so, I am basically using this equality and if I multiply both sides, so, you can see what I am multiplying with. So, I am multiplying both sides by e to the power rho i T and then I am also multiplying minus rho i small t ok. So, I will get e to the power minus rho i t chi i of t.

Now, I can write this as we will show very easily that this is a function. Let us call it p i this is a vector. So, this is a function of time and we can show easily that p i of t is a periodic function of time. How do we see that? It is easy to see p i of t is just e to the power minus rho i of t chi i of t. So, p i of t plus T is equal to e to the power minus rho i t plus T chi i of t plus T. I am going just going to work on the right hand side, so, this part.

So, I can write this as e to the power minus rho i t into e to the power minus rho i capital T into chi i of t plus capital T. And chi i of t plus capital T is just. So, chi i of t plus capital T is just e to the power plus rho i of capital T into chi i of t. If you substitute it here you will see that the plus rho i into capital T cancels out the minus rho i into capital T and we are left with e to the power minus rho i t into chi i of t and this by definition is just p i of t.

So, we have just managed to show that p i of t plus T is the same as p i of t. So, p i is a periodic function of time. So, what do we gain by this? We gain that chi i of t plus T into e to the power minus rho i t plus T is equal to p i of t. Let us take this further.

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$$\vec{\chi}_i(t+T)\, e^{-\rho_i T} = e^{\rho_i t}\, p_i(t)$$

$$\Rightarrow \vec{\chi}_i(t) = e^{\rho_i t}\, p_i(t)$$

Note the missing vector symbol on the R.H.S. The equation should be $\vec{\chi}_i(t) = \exp(\rho_i t)\, \vec{p}_i(t)$

I can write this as chi i of t plus T into e to the power minus rho i T is equal to e to the power rho i t p i t. I have just multiplied both sides by e to the power plus rho i T. I have taken this equation. I have taken this equation and I have multiplied both sides by e to the power plus rho i into small t. This is what I have done. So, you will get this equation, but what is the left hand side? So, you can readily see that the left hand side is nothing but chi i of t and this is equal to rho i t p i t.

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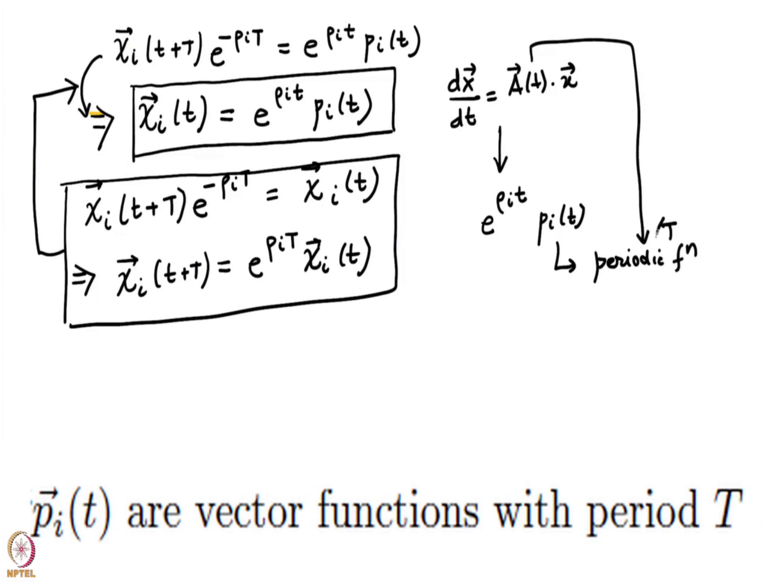

The left hand side, so, we are basically using the fact that how do we know this? We know that chi i of t plus T is equal to mu times chi i of t and mu we had written it as e to the power rho i of T. So, this is. So, this is how we are able to go from here to here. So, this is being used there. Now, what did we gain by doing all this? We gain something very interesting.

We recall that chi i is a solution to our first order system d x by d t is equal to A into x and so, chi i is one typical solution of this system. This analysis and this Floquet theorem is telling us at least one solution to this set of equation can be written in this form ok. Now, what is this form?

This tells us that the solution to this set of equation can be written as e to the power rho i into t into p i into t, where p i is a periodic function with the same period as the matrix A, the same period t. Next we will see how to apply this theorem to Mathieu equation and what do we conclude about the solutions to the Mathieu equation based on this theorem.