Introduction to Interfacial Waves
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Lecture - 25
Introduction to Floquet theory

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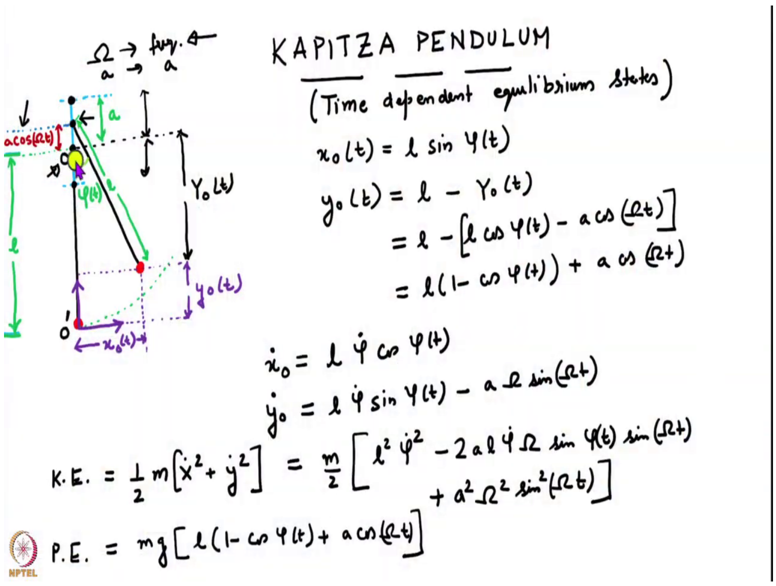

We were looking at the Kapitza Pendulum, where we had introduced a modification to the simple pendulum, the point at which the pendulum is suspended was being oscillated vertically in the same direction as acceleration due to gravity with a frequency capital omega and an amplitude small a. So, we had analyzed this, this system and we had written down an equation of motion for the same.

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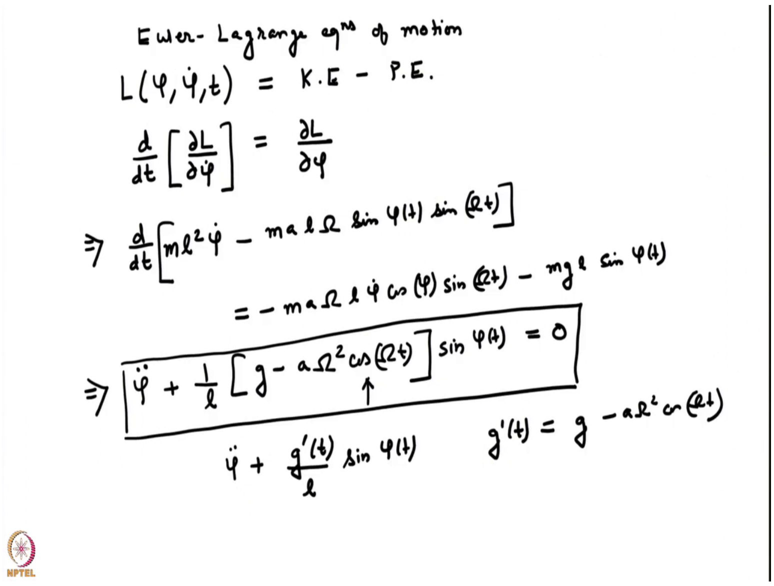

In particular, we had found that the equation of this pendulum actually behaves as if gravity becomes a function of time, an oscillatory function of time and the effective gravity g prime is a difference between two terms, the gravity if the pendulum was not in motion and then, an additional quantity a omega square cos capital omega into t.

Now, our task is to analyze this equation. So, as a first step, let us write down the fixed points of this ordinary differential equation. Notice that this is a non-linear ordinary differential equation, when we switch off the oscillatory motion of the point of suspension; then, it reduces to that of the simple pendulum. So, this is a non-linear ordinary differential equation. Let us write down its fixed points.

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$$\varphi = X \qquad \dot{X} = Y$$
$$\dot{\varphi} = Y \qquad \dot{Y} = -\frac{1}{l}\left[g + \Omega^2 a \cos(\Omega t)\right] \sin X$$

Note the error: The sign of the second term inside the bracket i.e. $\Omega^2 a \cos(\Omega t)$ has been made positive. This will not change any of our conclusion as this sign change is equivalent to assuming that at $t = 0$, the point of suspension of the pendulum is at $-a$ instead of $+a$ assumed earlier

So, as usual, we define, we write it as a as two first order ordinary differential equations and so, we define the angle psi as X and the angular velocity psi dot d psi by dt as Y and then, in terms of the these variables by definition, X dot is equal to Y and Y dot is just the equation of motion which in this case is I am shifting everything to the right hand side. So, this is g plus omega square a cos omega t sin X.

As I had mentioned earlier you can readily see that the fixed points of the system, in this case the fixed points are nothing but the equilibrium states, the equilibrium states are exactly the same as that of a regular pendulum that we had studied earlier.

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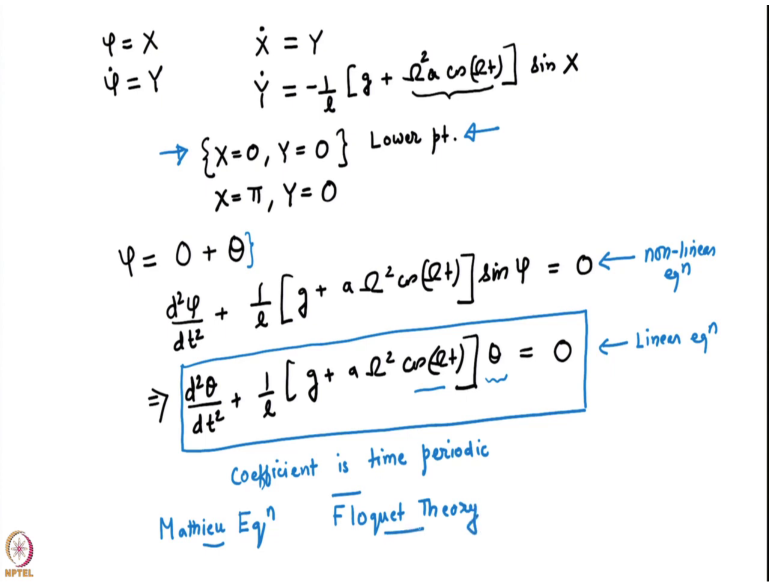

So, you can see that X is equal to 0, Y is equal to 0 causes the right hand side of thus two first order ordinary differential equations to vanish. Similarly, X is equal to pi; Y is equal to 0 is another fixed point of the system. The main difference is that, that the right hand side is a function of time. In our earlier case, this additional term was not there and so, the right hand side was not a function of time.

You can readily see that in this condition, in this fixed point the pendulum is either vertically upwards or vertically downwards and you can think of it as if the effective gravity becomes an oscillatory function of time. So, these are these fixed points represent fixed points, where the base state or the equilibrium state is actually time dependent.

The tension in the string of the pendulum would have to instantaneously adjust to balance gravity, if you are in the oscillating frame of reference. So, now, let us analyze the motion of this pendulum by looking at one of the fixed points. So, let us look at the fixed point which is below. So, that is represented by this. So, the lower point. So, this is theta is equal to or psi is equal to 0. So, this is the fixed point.

So, if you leave the pendulum at theta equal to 0 with a psi is equal to 0, with 0 velocity the pendulum will stay there; even though, the point of suspension is moving vertically up and down with a certain frequency and amplitude. So, now, let us substitute in this equations and so, what we will do is, we will perturb about the fixed point. So, we will say that the angle psi is.

So, we are taking these two fixed points. So, the angle psi is some 0 plus some theta. Now, recall that our original equation was; that our original equation was d square psi by dt square plus 1 by l g plus a omega square cos omega t into sin psi is equal to 0 and if I substitute psi is equal to 0 plus theta, then it just becomes d square theta by d t square plus 1 by l; this part remains the same and then, I have sin 0 plus theta. If I express that in a Taylor series about the point 0, then this just becomes the first term in the Taylor series approximation is just theta. Now, this is an important equation.

Notice that the original differential equation which was derived was a non-linear equation. We have linearized it about one of the fixed points. So, this is the lower fixed point, about which we have linearized this equation or in other words, we are giving the pendulum, we are introducing the pendulum at its fixed point which is the lowermost point.

The point of suspension is going up and down at a certain frequency omega and amplitude small a and then, we give it a small perturbation which is this theta and we ask what is the equation which governs theta, if we retain only terms which are linear in theta in the resulting expression. So, we have to linearize this. So, if I linearize, then I only retain the first term in the Taylor series expansion of sin theta about theta equal to 0 and so, this is the linearized, this is a linear equation.

Now, this equation the first thing to notice is that this equation unlike the previous pendulum equation that we had, the linearized pendulum equation is an equation whose coefficient is time periodic. You can recover the original linearized pendulum equation by just setting small a equal to 0; but notice that equation, we could have solved it by the method of normal modes.

This equation has time periodic or time dependent coefficients in general. This is the time periodic part and so, this we cannot do normal modes on this; we cannot just say that theta is equal to some constant into e to the power some lambda into t. This is that form is not going to work here, although this is a linear equation.

So, this equation is a very well-known equation, it is called the Mathieu equation. Again, named after the a French mathematician, who studied it. It frequently shows up in when we analyze stability of time dependent base states. Here, as I told earlier, the base state is time dependent, so when we are perturbing about that base state, so this is the lower fixed point is the base state here.

So, the lower when we are perturbing about it, the resultant linearized perturbation is governed by the Mathieu equation and since, this is not a constant coefficient equation. The coefficient actually depends on time and particularly, they are time periodic. So, the solutions of this require some more effort than what we have done until now. In particular, we will learn something called Floquet theory, which helps us analyze these kind of systems.

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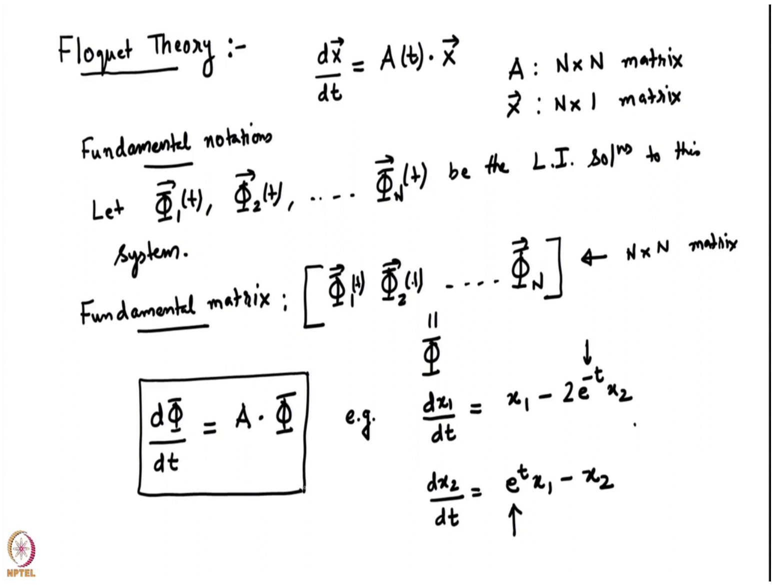

So, now, let us start with Floquet theory. Now, before we get into Floquet theory, we will have to do a little bit of linear algebra while doing Floquet theory. So, I would request all of you to brush up your linear algebra fundamentals before going through this part. Now, what does Floquet theory do? Floquet theory helps us in understanding the solutions to equations, whose coefficients are time periodic.

Recall that we have a second order equation, whose coefficient is time periodic. We have a cos omega t in the equation that I just showed you, the Mathieu equation. So, I can use Floquet theory to analyze the equation because the coefficient of the Mathieu equation is also time periodic.

Now, while drawing face portraits, I have told you that we can express any nth order differential equation as a set of n first order ordinary differential equations. So, it is enough to understand how does Floquet theory deal with a set of first order coupled ordinary differential equations.

So, for that let us write down. So, what I am going to do is mostly going to apply to first order ordinary differential equations; but we have to understand that we can convert a second order namely the Mathieu equation into this form.

So, let us first write down some of the things. So, we want to understand the solution to this first order system. Here, A is a N by N matrix and X which I am representing as a vector is actually a in matrix notation, it will be represented by a column matrix. So, it is a N by 1, N rows into 1 column matrix. Now, we have seen we know how to solve this when A is not a function of time, when the matrix A is a constant matrix.

We now will learn how does that method extend when A becomes a time periodic matrix ok. So, now, let us introduce some fundamental notation. So, in general, this is a linear system; this is a linear first order system because A is a N by N matrix, I expect N linearly independent solutions.

So, let us call those linearly independent solutions as so let phi 1 of t phi 2 of t phi N of t be the linearly independent solutions to this system. So, now, we will introduce a matrix, which is called a fundamental matrix of the system and the fundamental matrix is obtained simply by writing all the linearly independent solutions phi 1, phi 2. These are all functions of time side by side in the matrix.

So, each of them is a column. So, the first column of this matrix is the first linearly independent solution, the second column is phi 2, the third column is phi 3 and so on; each of them, I will write as columns. And because there are N of them, I will get N rows. So, I will get N columns and because each solution has N elements in it. So, I will get N rows. So, I will get a N by N matrix.

So, this represents a N by N matrix. Now, of what use is a fundamental matrix? The fundamental matrix satisfies the equation. So, if I call this fundamental matrix as let us say I will call this matrix as phi. So, then, the fundamental matrix satisfies the equation d phi by d t is equal to A which is itself a N by N matrix dotted with phi. One can easily check this. So, let us take an example.

So, suppose I have this set of equations. So, these are my coupled set of linear ordinary differential equations and I can readily see that I can the A matrix here is time dependent. So, if you collect the coefficients of x 1 and x 2 in both the equations and put them in a matrix, you can see that they are time dependent because of this term and because of that term.

So, now, let us we can solve this set of equations easily. These are coupled linear ordinary differential equations with time dependent coefficients. This particular case can be solved simply. So, one solution is 2 e to the power t.

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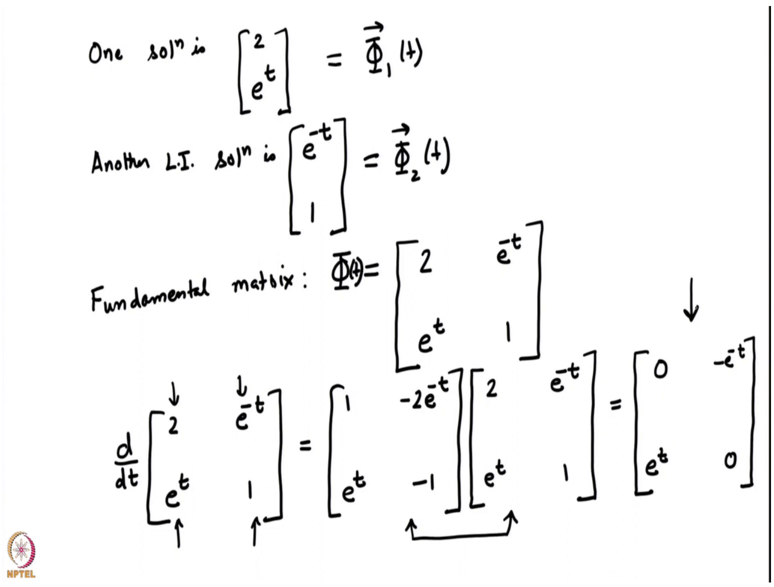

So, you can check that this solution satisfies the equations. Similarly, another solution linearly independent solution e to the power minus t 1. You can think a little bit about how did we obtained these solutions. If you think a little bit, it will become clear to you how does one obtain these solutions. So, these two are my phi 1 and phi 2 by previous notation. So, how do we set up the fundamental matrix?

The fundamental matrix is phi and phi is just phi 1 and phi 2 written side by side. So, phi 1 is 2 e to the power t and phi 2 is e to the power minus t 1 and so, there we have our fundamental matrix. Now, you can immediately. So, the fundamental matrix is also function of time and now, you can readily see that d by d t of this satisfies.

Our original matrix would be the matrix on the right hand side A of t for this case would be 1 minus 2 e to the power minus t e to the power t minus 1, 2 e to the power minus t e to the power t 1. You can check that this is true; just multiply the two matrices and you will find that the product of these two matrices will give you a square matrix. So, for example, and so, if you take the differentiation of each of those terms, you will see that I get that matrix ok.

So, now, we have checked that the fundamental matrix, we have verified that in our example, the fundamental matrix satisfies the equation that I have written. This is the equation. Now, let us use these ideas to go over to Floquet theorem. Now, before we go to Floquet theorem, it is important to realize the following thing that a differential equation with periodic coefficients need not have solutions which are themselves periodic in time ok.

So, I will repeat that a differential equation with periodic coefficients, time periodic coefficients need not have solutions which are periodic in time.

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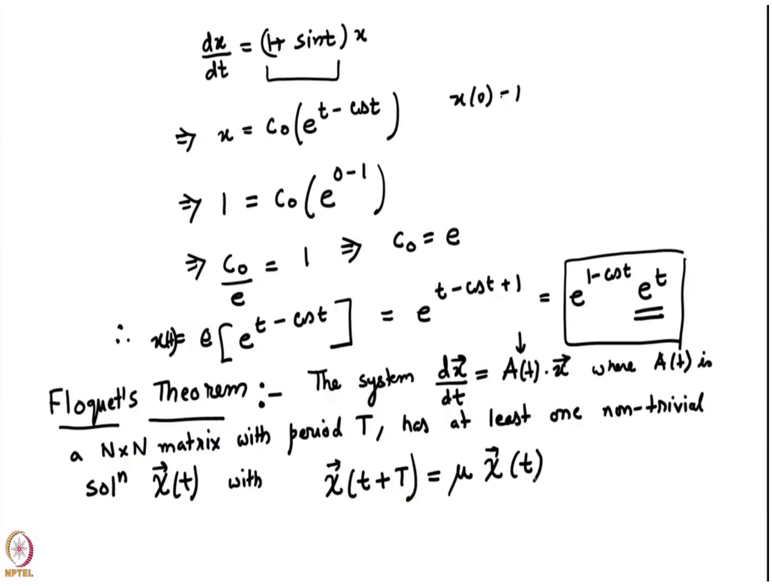

Let us look at an example. So, I will take a simple example dx by dt is equal to 1 plus sin t into x. You can clearly see that this is a periodic function, it is a time dependent coefficient and it is a periodic function. So, this is a differential equation with time periodic coefficients can be easily integrated and the solution is x is equal to c 0 e to the power t minus cos t and if I choose x of 0 is equal to 1, so then, we get. So, this is 1, this is c 0 e to the power 0 minus 1. So, this is c 0 by e is equal to 1, this implies c 0 is equal to e.

So, therefore, x is equal to e e to the power t minus cos t ok and so, you can write this as e to the power t minus cos t plus 1 and you can readily see that this is not a time periodic function. In fact, you can see that if I write this as e to the power 1 minus cos t into e to the power t that part diverges in time alright. So, now, we have seen that the solutions of systems, whose coefficients are time periodic need not themselves be periodic ok. With that background, we will now move over to Floquet's theorem.

So, Floquet's theorem says the system dx by dt is equal to the matrix A of t into x, where A of t is a N by N matrix with period T has at least one non-trivial solution. I will call the solution as a column vector. So, its a I am indicating it with the vector symbol chi of t and chi of t satisfies this relation, that chi of t plus the time period of the matrix t is equal to some constant mu into chi of t. This is Floquet's theorem.

It is useful and interesting to look at the proof of this theorem because it will tell us about the proof of this theorem contains some details, but those details are useful when we later look at the Mathieu equation and applies Floquet theorem on the Mathieu equation. Recall that the Mathieu equation is a second order equation with time periodic coefficients, I can write it as a set of two first order equations.

So, Floquet's theorem will apply to the Mathieu equation because the matrix A of t in the Mathieu equation will also turn out to be a time periodic matrix. So, let us now go over to the proof of Floquet theorem.

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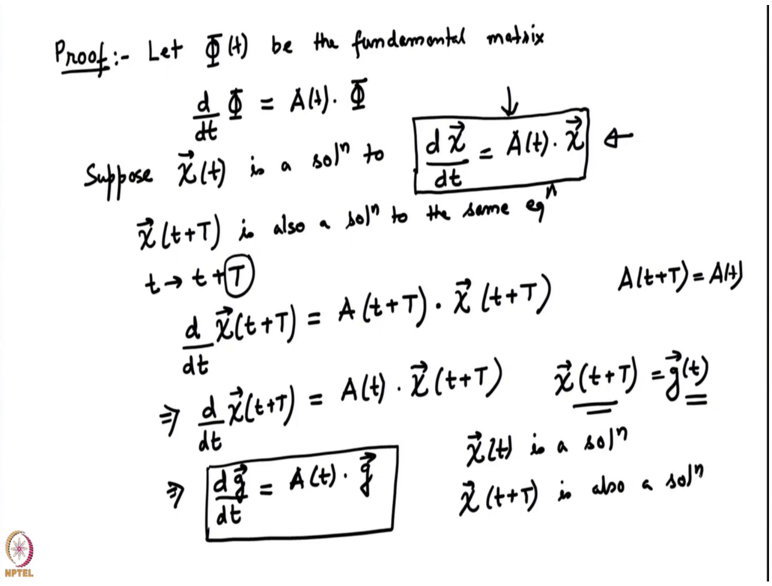

So, let phi be the fundamental matrix of the system. By the way, we have just seen that this implies that d by d t of phi is equal to A of t dotted with phi; A is a matrix, phi is also a matrix; both are N cross N matrix in general and we note the following that suppose chi of t is a solution to d chi by dt is equal to a of t dotted with chi. This is nothing but this system is nothing but our system of equation that I have introduced earlier.

So, I have introduced this system earlier on which I had written the Floquet theorem. So, I have said that chi is a solution. So, I have replaced x by chi and I have rewritten the same equation. So, chi must satisfy this equation. So, now, note that if chi is a solution to this and we replace then chi of t plus T is also a solution to the same equation. How do we see that? We just replace t by t plus T in the equation which is satisfied by chi.

So, you can see that because capital T is a constant. So, the differentiation with respect to small t, when I replace small t by small t plus capital T, the denominator remains the same. This actually becomes t plus T now and this matrix becomes t plus T and this chi also becomes t plus T. But we know I will work on the right hand side, but we know that the matrix is a time periodic matrix.

So, A of t plus T is equal to A of t. So, I can replace this with this. If I replace chi of t plus T with some function g of t, its important to remember that chi is not necessarily periodic. I had told you earlier that the solution to a periodic set of equations need not be itself periodic. Chi is a solution to a system, whose coefficient matrix is periodic.

But chi itself need not be periodic. So, chi when you substitute in the argument of chi t plus capital T, it did not come back to itself. So, in general, I will indicate chi of t plus T with some different function ok. So, g of t ok. So, this implies dg by dt is equal to A t dotted with g and thus, we find that if chi is satisfies this equation, then chi of t plus T which I have indicated with the symbol g of t also satisfies the same equation.

So, we conclude that if chi of t is a solution, then chi of t plus T is also a solution. This tells us that if we discover one solution to this set of equation, then I if I take that solution and replace the argument with argument plus the time period of the coefficient matrix, then what I will get is one more solution. This is what it demonstrates.