Introduction to Interfacial Waves

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Lecture - 24
Kapitza pendulum

We were looking at the solution to the duffing equation using the method of multiple scales. We had said that we will go up to order epsilon, but in order to determine one particular unknown we had to actually go up to order epsilon square.

At order epsilon square the algebra becomes a little bit lengthy and we have four terms on the right hand side. I had worked out two of these terms and then for the other two I have told you how to do it and I given you the final expressions. Now let us look at all these terms added, because we have to add these terms 1 plus 2 plus 3 plus 4.

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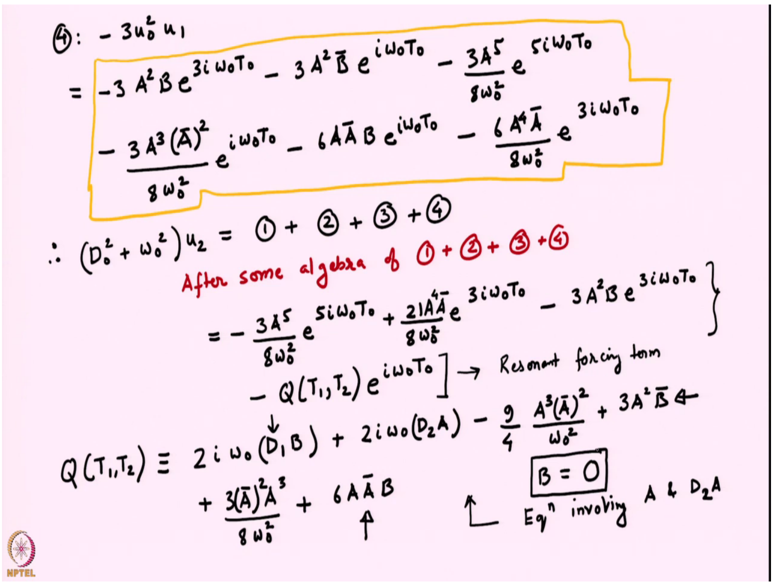

So, when we add these terms. So, I have provided the fourth term also. So, let me put this in, so the fourth term is this. And the first second and the third term have been provided to you earlier. Now after doing some algebra on these terms; one can simplify it and obtain simpler expression for this. It can be written in this manner, where as I said before our interest is only in this term.

Why so? Because as I said before we do not want to go up to order epsilon square, recall that we wanted only the variable B. B was a function of T 1 and T 2 we are not going to go up to T 2.

So, we wanted to determine B as a function of T 1. For that we need to look find out what are the resonant forcing terms at this order. The resonant forcing term is only this term. So, this is the resonant forcing term. And because the coefficient of this term is lengthy, so we have written it as Q some function of T 1 and T 2 where Q is given by this expression.

Now, you can show all of this by just putting together all the four terms that we have given; 1 2 3 4 adding them up and doing a little bit of algebra that is all. The algebra is very straight forward. Now we are interested in this resonant forcing term only because this is a solution to the homogeneous equation. So, we have to set in order to eliminate this we have to set the coefficient to 0, or in other words we have to set Q to 0.

Now if you look at Q there are a number of terms in Q 1 way of eliminating Q is to say that B is equal to 0. Notice what will happen if we say B equal to 0, this term will go to 0, this term will go to 0 and this term will go to 0 and what will be left is only terms which contain A or A bar. Now we have already determined the dependence of A on T 1; however, in our earlier slides we had done that.

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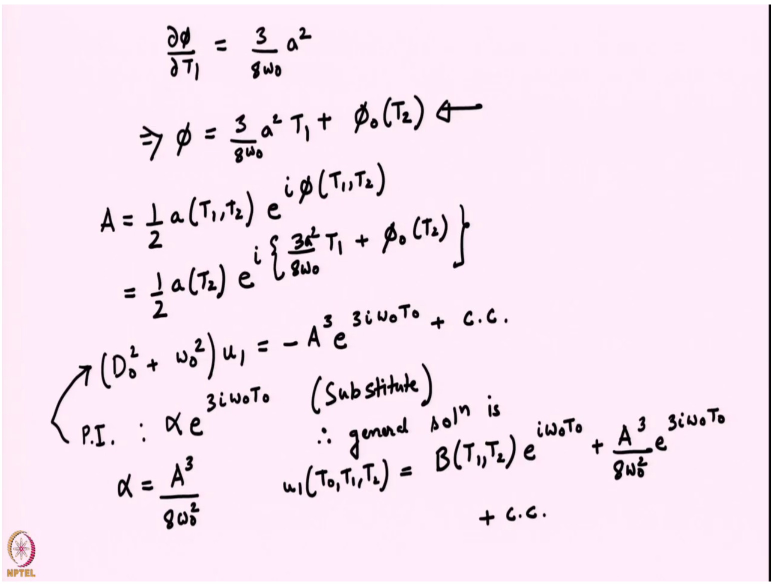

However, there is a dependence on T 2 and we said we are not going to determine that, so we will treat these small a and small phi naught which are functions of T 2 as constants at this order at order epsilon. So, we do not have to be worried about the T 2 dependence of A 1.

So, you can see that if I set B to 0 then the rest of the equation which survives is an equation involving A 1 or A rather and derivatives with respect to A with derivatives of A with respect to T 2. So, we are not going to determine the dependence of A on T 2 and, so we do not have to worry about that equation. So, the solution that we really are after is just this that B is 0. That is enough to write down the solution to the duffing oscillator up to order epsilon.

Let us proceed from here.

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$$u = u_0 + \varepsilon u_1$$

$$= A e^{i\omega_0 T_0} + \varepsilon \left[\cancel{B} e^{i\omega_0 T_0} + \frac{A^3}{8\omega_0^2} e^{3i\omega_0 T_0} \right]$$

$$A = \frac{1}{2} a\, e^{i\left\{\frac{3}{8\omega_0^2} a^2 T_1\right.}$$

It should be $A = \frac{1}{2} a e^{i\left\{\frac{3}{8\omega_0} a^2 T_1 + \phi_0\right\}}$

So, we have determined now. So, up to order epsilon we will write it as u 0 plus epsilon u 1. And u 0 we found was A e to the power i omega 0 T 0 plus epsilon. u 1 was B e to the power i omega 0 T 0 plus a particular integral, we have already seen this earlier and we just now concluded that B is 0. So, this gives us this expression for u.

Now we have also seen that A is half a; this small a we had written it as a function of T 2. Now we will treat this as a constant into e to the power i 3 by 8 omega 0 square into a square into T 1.

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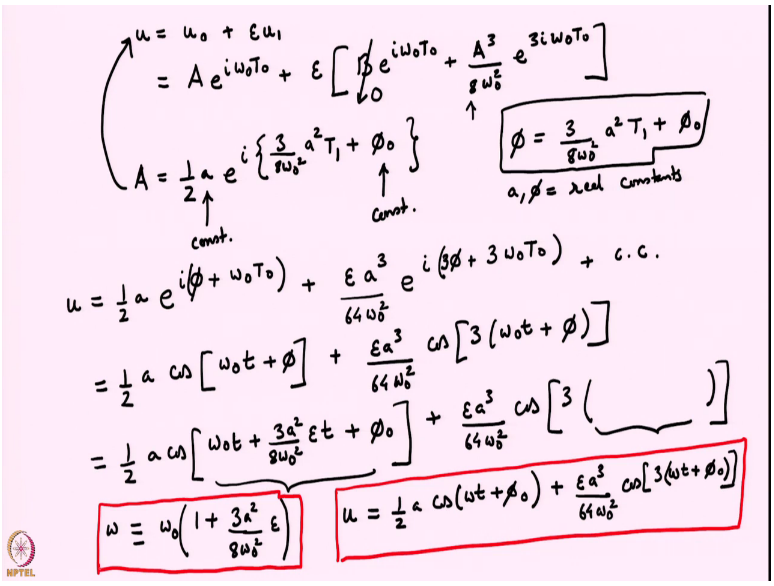

Let me write this 0 here, plus a phi naught we have said that this phi naught is a function of T 2, at this order it is just a constant. So, this is a constant, this is also a constant and these are real constants ok.

So, with that let us go back and plug this expression of A into the expression of u and write our final answer. So now, we are trying to write down an answer which is completely in in terms of real functions.

So, we will have half a e to the power where phi is just this, phi is just 3 by 8 omega 0 square a square T 1 plus phi naught plus omega 0 T 0 plus epsilon small a cube. So, I have 1 by 2 cube which gives me 1 by 8 and then there is already a 1 by 8 here, so I get a 64 in the denominator omega 0 square. And then I have e to the power i phi plus thrice omega 0 T 0 plus of course, the complex conjugate.

And then now if I shift to real notation remember that a and phi naught are real constants. So now, we will have a cos omega naught T naught is just T and then I will write this as phi plus epsilon a cube by 64 0 square cos the third harmonic, and this comes because there is a cubic non-linearity in the duffing equation. So, this should be a 3 phi a cube yeah this should be a 3 phi.

And we do not need to add any complex conjugate now. And we now only need to substitute the value of phi and, so we have a cos omega 0 T plus 3 by 8 omega 0 square, we have the value of phi here and capital T 1 is just epsilon into small t. So, that I will do the replacement now.

Epsilon T plus phi naught plus epsilon a cube by 64 omega 0 square cos 3 times the same thing. Whatever I have written here the same thing goes here. In fact, if I write I can write this in compact notation as if I define a omega which is omega 0 into 1 plus 3 a square. Let me write the epsilon. If I define an omega like this then I can write the final answer as half a cos omega T plus phi phi naught plus epsilon a cube by 64 omega naught square into cos 3 omega T plus phi naught.

And so this is the solution to my duffing oscillator up to order epsilon. So, I have put the first non-linear correction, you can see that a harmonic of the primary has appeared. The linearized solution we just oscillate at omega naught. This should be omega naught and this is my correction to the dispersion relation or connection to the frequency relation.

You can see that the frequency of the oscillator depends on epsilon, so if you change epsilon the frequency of the non-linear oscillator will be slightly different from that of the linear oscillator. This is what we expect to find and this is what we had found earlier in the simple pendulum also, where the non-linearity was slightly different in the sense that it had a cubic term, but with a minus sign. And we have solved this using the method of multiple scales.

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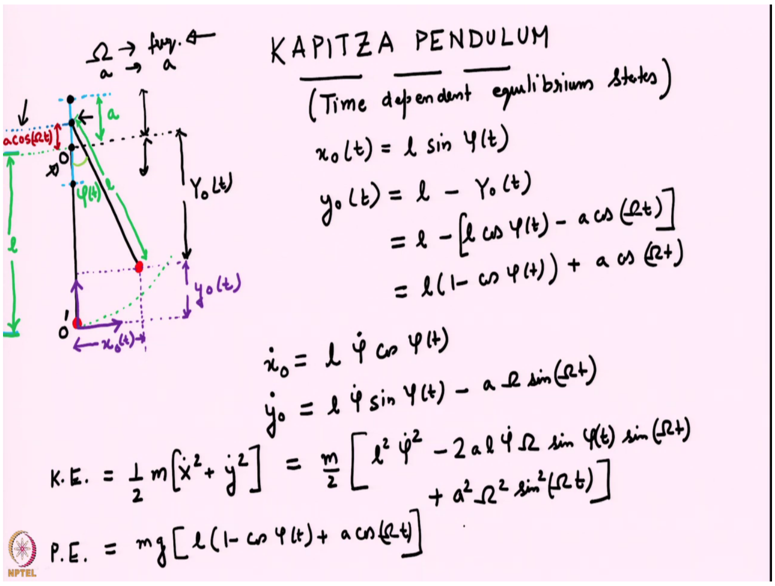

Until now we have worked on a number of problems of point mass systems connected through multiple springs or doing non-linear oscillations and things like that. But in each of these examples that we have studied so far, our systems in the equilibrium state were always in a state which was time independent or in other words the equilibrium state was not a function of time. There was no variable in the equilibrium state which would be a function of time.

We now come to a more complicated example of a pendulum which is called as a Kapitza Pendulum after the name of a very famous Russian scientist who first proposed this pendulum. And this is very similar to the simple pendulum that we have studied until now with an important difference that the point of support of the pendulum in this case. As you can see on the figure on the left.

So, the point of support of the pendulum let us call it point o that is indicated with the black dot there. So, the pendulum is of length l you can see it here and what we are doing is we are the point of support of the pendulum is moved vertically along the vertical direction in an oscillatory manner with a frequency omega and an amplitude a. So, frequency and small amplitude a.

So, as you can see the pendulum is going to, the point of support of the pendulum is going to do oscillations about the point o, at some time it will be above the point o at a at max up to plus a or at max up to minus a; its plus a above it or plus a below it. So, this is the point below it and this is the point above it.

Now what I have done is the we are modeling the pendulum the string of the pendulum as being inextensible. So, its length is l and that length is not a function of T, its just the point of support which is moved up and down. And so at I have drawn the pendulum also at some angle psi which is a function of T when the point of support is at this point. And this point from the point o is a cos omega T, I have written that in red.

Now, the variable the; what we want to write down if we want to write an equation of motion for this pendulum. This problem has some very interesting aspects which will later show up when we do interfacial waves in particular Faraday waves. When we do Faraday waves on time dependent basis states.

So, that is why I have said that this is an example where the equilibrium state is a time dependent state. Why is the equilibrium state time dependent? You can see that if the pendulum was not oscillating the lower most point is an equilibrium position. The force that is exerted by gravity is exactly balanced by the tension in the string.

If I start moving the pendulum up and down then provided the tension in the string adjusts to the local acceleration, you can think of this problem in the accelerating frame of reference. So, if you sit on the point which is moving up and down you will not see any motion of the pendulum, because the length of the string is inextensible; however, you will feel the pendulum will feel a non inertial force ok.

That is equivalent to saying the tension in the pendulum is fluctuating or rather oscillating as a function of time to adjust. So, effectively the pendulum feels as if the instantaneous value of the acceleration due to gravity has changed. And so the tension in the string adjust instantaneously to keep the pendulum in that position under vertical equilibrium.

So, that is why here we are talking about time dependent equilibrium states. You will see that the equation that governs the pendulum will be as if it is a regular pendulum, but the gravity the g sin theta term that appears in the pendulum as if gravity has become time dependent and it is becoming an oscillatory function of time ok. And we will learn how to analyze this equation in particular we will learn a technique called flock analysis which we are later going to use for understanding faraday waves on a fluid interface ok.

So, I have drawn a figure and I have noted down the various things here. So, let me call my origin with some other name. So, I will call it o prime to distinguish from the point of suspension of the pendulum.

And my origin is o and as you can see the pendulum at some instant of time is making an angle psi of T with the vertical. Now with respect to this origin o prime I would like to write down the equation of motion of the pendulum. Now this can be done in various ways. I am just going to write down. So, I will show you one way of doing it.

So, x 0 of t we know is from this figure is l sin psi t, y 0 small y 0 of t is the length of the pendulum which is a constant minus capital Y 0 of t. Capital Y 0 of t from the figure you can immediately see. So, this is length this is length l. So, capital Y 0 is l cos psi of t. That is the projection of the inclined string on the vertical and from that we will have to subtract this distance in red that I have indicated here. So, minus a cos omega t. The point of suspension is being moved with the frequency omega and the amplitude of the oscillatory motion is plus a and minus a. So, at most it goes to plus a above and at most in the reverse direction it goes to minus a below ok.

So, with that we have expressions for; so we can write this as l into 1 minus cos psi of t plus a cos omega t. Let me calculate the kinetic energy of motion. So, the kinetic energy of motion in the pendulum is given by x naught x dot square. So, for that I need x. Let me first write x, the derivative of x naught with respect to time and that is just l psi dot cos psi t.

Similarly, you can get y dot y 0 dot and that is l psi dot t minus a omega sin omega t. So, the kinetic energy at any instant of time of the mass with respect to the coordinate system which I have drawn whose origin is at o prime is x dot square plus y dot square. if you do that then you will get.

So, was cos square psi plus a sin square psi adds up to just unity and then you have two more terms. The expression for the potential energy is easy that is just m g into small y naught. Small y naught we have already got and so this is; so this is my expression for kinetic and potential energy.

I am going to use the Euler Lagrange equation of motion one does not have to use this. In case you are unfamiliar with the Euler Lagrange equations of motion and how they lead to the equation of motion of the pendulum, you can try analyzing this system in the non inertial frame of reference where you sit at the point of suspension which is moving up and down and so in that frame of reference you have to add a non-inertial force. One can do it either ways both approaches will lead to the same equation of motion of the pendulum.

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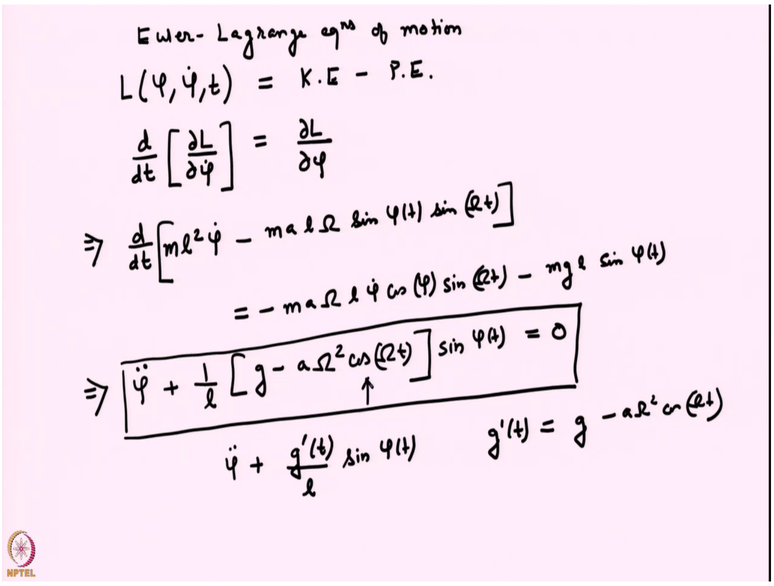

So, the Euler Lagrange equations of motion. So, the Lagrangian is a function of the variable psi the angular velocity and t and that is defined as kinetic energy minus the potential energy. So, the Euler Lagrange equation of motion is just del l by del psi dot. It is just like the approach where we treat position and velocity as two independent variables ok.

So, the way we did it when we did the phase portrait of the system; x and x dot were treated as two independent variables that led us to instead of a second order equation we got two first order equations. This is the same approach where psi and psi dot are treated as two independent variables. And so the Euler Lagrange equation here is just because this is a single degree of freedom system. So, there is only one variable psi. So, psi and psi dot and we have this equation.

Now if we do del psi by del psi dot, while doing this del l by del l by del psi dot we have to take derivatives, we have to treat as if psi and psi dot are independent variables. When we do that then we just get l square psi dot, there is a m minus m a omega sin omega t del l by del psi. Again psi and i dot have to be treated as independent variables. So, any variable where there is a psi in it only will get differentiated.

If you simplify this expression then it will give you the second order equation that we seek. So, I will use a double dot for differentiation with respect to time. So, psi double dot plus 1 by l g minus. Some terms will get cancelled out on both sides. If you cancel it out after differentiation you will see that some term on the left hand side and right hand side will get cancelled out, what is left if you collect it and put it together you will get this equation.

So, that is our equation which governs the motion of our pendulum whose point of suspension is being oscillated with the frequency capital omega with an amplitude small a. As I had said before this if you set for example, if you set the amplitude of motion to 0. So, it is not really oscillating, then you will recover the equation that we are all familiar with psi double dot plus g by l sin psi is equal to 0. So, this is the regular pendulum whose point of support is not oscillating. So, this just generalizes what we already know.

Notice that instead of g by l what actually appears. So, it is as if. So, you can think of this equation as if psi double dot plus some effective g prime which is a function of time divided by l into sin psi of t, where g prime of t is defined as the regular gravity minus a omega square cos omega t. So, it is as if the gravity is fluctuating or oscillating up and down in time.

This makes physical sense. Imagine you being inside an elevator and the elevator is moving up and down you will the instantaneous value of gravity that you will sense will become a function of an oscillating function of time, this is a very similar thing.

So, we are going to analyze this equation, in particular we are going to see we can we can quickly write down the fixed points of this system. Note that this is this is slightly different from the system that we have worked on until now.

There is an explicit time dependence in the coefficient of this equation through this term cos omega t. You will still see that it is possible to write down the fixed points of the system. In fact, the fixed points of the system are exactly the same as that of the pendulum whose point whose point of suspension is not moving.

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So, the lower most point and the uppermost point. So, when the pendulum is like this or when the pendulum is like that those two continue to be the fixed points of this system as well.

We will expand a little bit around the fixed point in a Taylor series and we will look at the oscillatory motion about the lower fixed point. This will lead us to an equation which will which is known as the Mathieu equation and we will analyze the solutions to that equation using flock analysis. We will encounter the Mathieu equation again later in this course when we study interfacial waves.