Introduction to Interfacial Waves
Prof. Ratul Dasgupta
Department of Chemical Engineering
Indian Institute of Technology, Bombay

Lecture - 22
Duffing equation using multiple scales

We were looking at the damped harmonic oscillator using the method of multiple scales. We had expanded up to some order and we had found that at various orders; one has to eliminate the resonant forcing terms.

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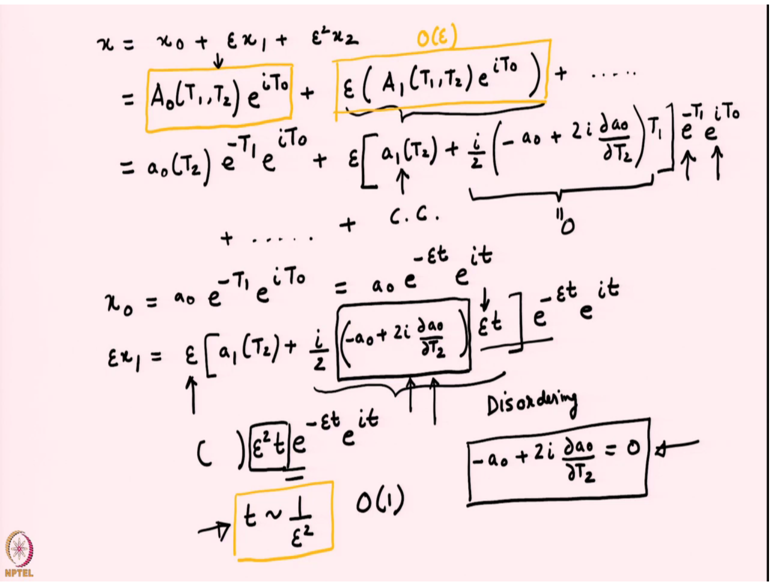

In particular at order epsilon, we had discussed that the coefficient of the term that I have indicated here, this term in the box. The coefficient of this term is something which will get disordered, this is an order epsilon term. And so, at large times time of the order 1 by epsilon square. This term becomes as large as the first term in the expansion.

So, to prevent that we would have to set the term in that rectangular box to 0. I had shown that this leads to an equation for a naught, recall that a naught is a function of T 2. So, we now have an equation governing a naught let us solve that equation and determine the functional dependence of a naught on T 2.

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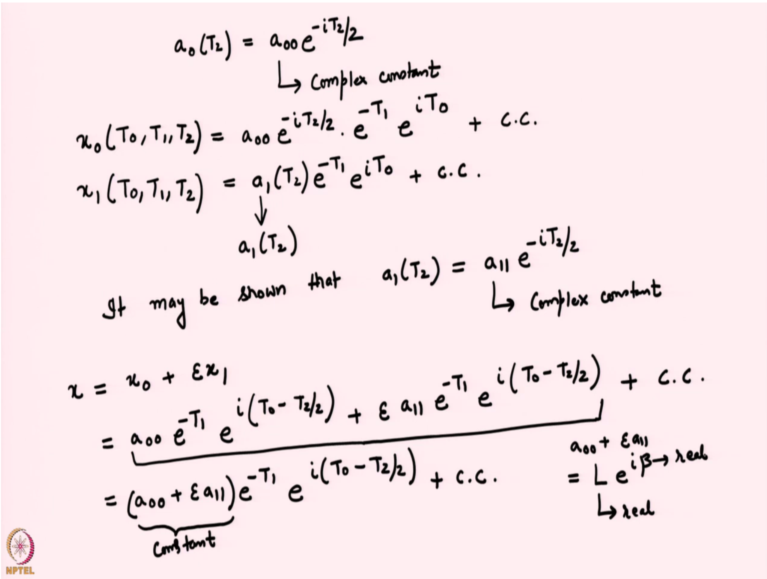

So, one can solve this equation easily and it leads to the solution a naught as a function of T 2 is now, some constant of integration. Now, the constant is no longer a function of any other variable, because a naught is just a function of T 2.

So, this is really a constant into e to the power minus i T 2 by 2. This is just the solution to the equation that I had indicated in the previous page at the bottom of the page. If you integrate that equation, you will find this solution. Now, this is the constant of integration and in general, it is a complex constant ok.

So, now let us put things together. So, now, we have found that x 0 which was a function of T naught T 1 and T 2 is now, a 0 0 e to the power minus i T 2 over 2 then, it is into e to the power minus T 1 into e to the power i T naught ok. This is essentially rewriting this expression in the yellow box; x naught is A naught e to the power i T naught and A naught in turn, we have found is equal to small a naught into.

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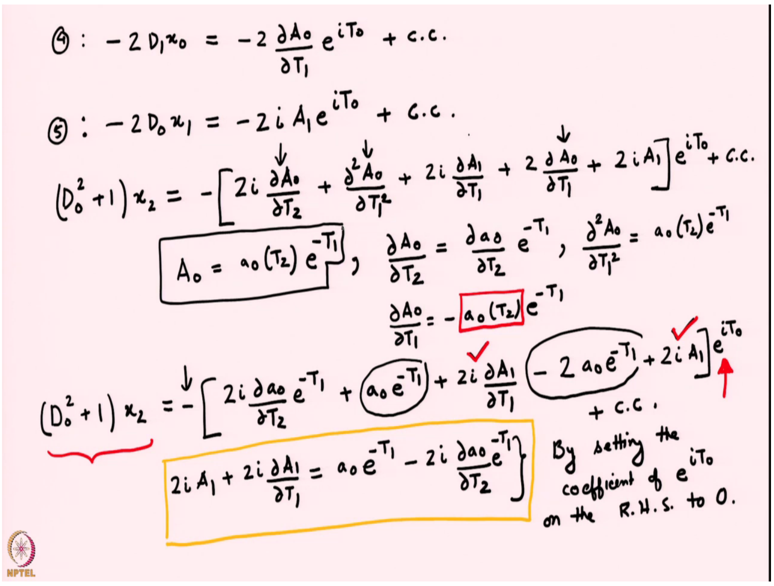

So, we have found earlier that A naught a small a naught into e to the power minus T 1 and using the equation for small a naught, we have now determined x naught completely. So, that is our x naught.

Of course, this has to have a complex conjugate added to it to give us a real x naught. Similarly, one can find x 1 and this is equal to a 1 and this is a function of T 2 e to the power minus T 1 e to the power i T naught plus complex conjugate.

Once again, we have found this part before. We had found this part earlier we had written that it is. So, this is basically just a 1 it is small a 1 into e to the power minus T 1 into e to the power i T naught. So, it is a 1 into e to the power minus T 1 into e to the power i T naught the second part is not there, because we have set it equal to 0, in order to determine small a naught. I hope this is clear.

So, we need to determine this a 1 as a function of T 2. It is clear that we will have to go to the next order in order to do this. I am just going to write down the answer for you, because the procedure is now straight forward.

Show, it may be shown that a 1 of T 2 is equal to some a 1 1. I am following the same notation as earlier into e to the power minus i T 2 over 2 where a 1 1 is once again a complex constant. This I will leave it to you to show yourself it is not difficult. Once you do this then, we can write down the answer up to order epsilon and we have found that this is a naught naught into. We can combine all the exponentials.

So, I can write. So, e to the power minus T 1 does not have i. So, I will write it separately and then, I will write i T naught minus T 2 over 2 and then, plus epsilon times a 1 and we have seen that a 1 is basically, a 1 1 and the same thing here, e to the power minus T 1 into e to the power i into T naught minus T 2 over 2 plus of course, the complex conjugate of the entire expression. So, there will be two parts to it.

Now, let us shift to real notation. Before we do that, let us you can see immediately that we will have a term which is like this. So, we can write this as a naught naught plus epsilon a 1 1 e to the power minus T 1 into e to the power i T naught minus T 2 over 2 plus complex conjugate.

Now, you can see that this is a constant. This is a complex constant, because a naught a double 0 is complex constant a double 1 is also complex constant and epsilon is real. So, the whole thing is a is another complex constant.

I can write that a double 0 plus epsilon a 1 1 is equal to some complex number, whose amplitude is L and whose phase is some beta with the understanding that L is now, real and beta is also a real number; both L and beta are real numbers.

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$$x(t) = \underset{\substack{\uparrow \\ \text{Real}}}{L}\, e^{-T_1} e^{i\left(T_0 - \frac{T_2}{2} + \underset{\substack{\uparrow \\ \text{Real}}}{\beta}\right)} + c.c.$$

$$= L e^{-T_1} \cos\left[T_0 - \frac{T_2}{2} + \beta\right]$$

Note that we are missing a 2 which multiplies L. This may be absorbed by redefining L

If we do that then, the expression that I had written earlier may be written as L e to the power minus T 1. I would also have a e to the power i beta, but I will absorb the beta a in the exponential which was already there with the i. So, I am going to just adding it up to this part and then, plus complex conjugate of whatever is on the left.

Remember that now, L and beta are real by definition and so, if we now shift to complex exponential to real notation, using e to the power i theta is equal to cos theta plus i sin theta. You can immediately see that this just becomes L e to the power minus T 1 cos of T naught minus T 2 by 2 plus beta and we do not have the sign part, because the sign cancels out.

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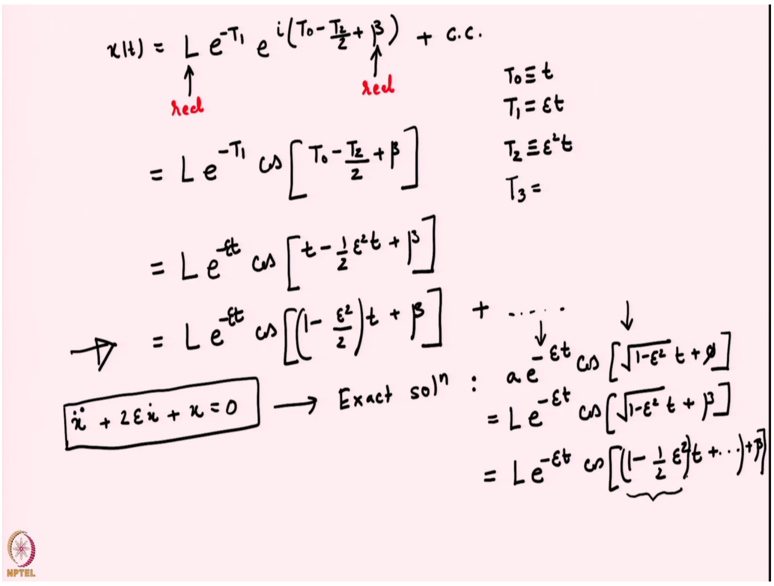

And so, this. If we now, go back to instead of writing three different scales T 1 T 2 T 0, we go back to our original scale then, this gives you e to the power minus t cos of this is t this is half epsilon square t plus beta. This is equal to L e to the power minus t cos of 1 minus epsilon square by 2 into t plus beta.

So, our final answer. Now, of course, there are higher order terms here. Now, notice that. So, I have missed epsilon here, T 1 is epsilon t. So, T 1 T naught is was defined as t, T 1 is epsilon t, T 2 is epsilon squared t.

Now, notice that this is not completely the solution up to order epsilon square. We would have to add; in general, I will have to add another part here, which would be epsilon square into x 2.

We have not done that step. You can go ahead and do this step and make it completely consistent, but the basic purpose of this exercise was just to demonstrate that what these multiple scales doing.

So, first thing that you notice is that the multiple scales is giving you an answer which will not give become disorder in time. It has also eliminated the process of doing multiple scales itself eliminates all the resonant forcing terms and consequently, there are no secular terms. Let us compare this solution with the exact solution that we had written at the starting of this example.

So, we had seen that this. So, we were studying the damped harmonic oscillator and this was the equation whose solution we have obtained perturbatively using the method of multiple scales ok and this equation we have seen earlier has the solution this is the exact solution this is a linear equation. So, it can be solved exactly very easily. It will be some a e to the power minus epsilon t cos square root 1 minus epsilon square into t plus phi.

If I replace the a by L a and phi are real numbers. So, I can use any other symbol and I am replacing phi by beta just. So, that we can compare with this expression. So, you can see that these are this is nothing, but we have recovered is nothing, but the expansion of 1 minus half epsilon square t plus dot dot dot plus t plus beta.

So, you can see that you can guess what will be the higher order corrections. The regular perturbation what it would have done is, it would have expanded this term also and this term also. And consequently, it would have taken the product of the two expansions. So, consequently, we would have found secular terms as long as we truncated the expansion up to any finite number of terms and took the limit t going becoming larger and larger.

Here what it does is it keeps the e to the power minus epsilon t outside. It recovers the e to the power minus epsilon t at the first order correction and then, it expands the frequency ok. So, it basically says that square root 1 minus epsilon square t is approximately 1 minus half epsilon square into t when epsilon is sufficiently small.

You can take this to higher orders of epsilon and you will recover more and more correction. So, you can go to more time scales. So, you can go to T 3 which will be epsilon cube into t. So, essentially if you go to even longer time scales you will discover that the frequency is not exactly 1 minus half epsilon square, but it is actually 1 minus half epsilon square plus some correction.

So, this term will have a small mismatch with the theoretical prediction ok. So, that is how the basic method works.

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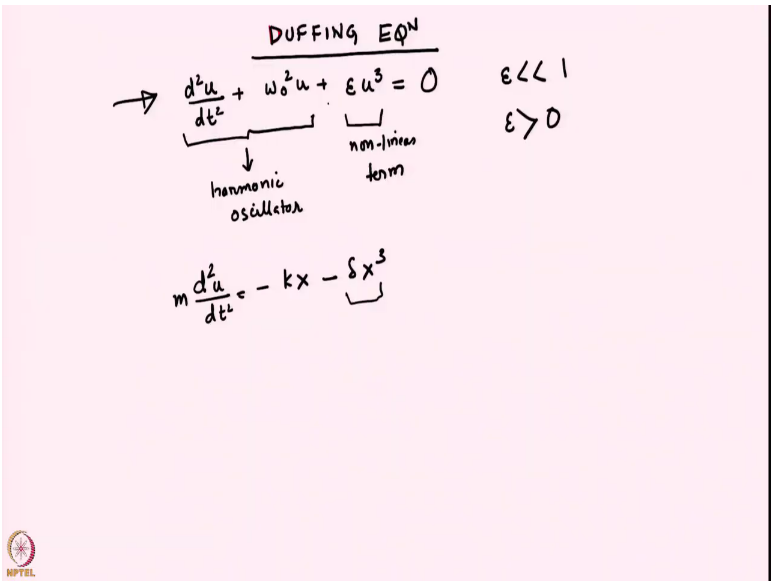

We will continue our discussion of the method of multiple scales particularly, because this is a very important technique, which is frequently used in analyzing interfacial waves.

So, we will go on now, we have just completed our discussion of using multiple scales to obtain an approximate solution to the damped harmonic equation. We now, go on to another equation. The damped harmonic equation was a linear equation. We now go on to a non-linear equation.

So, this equation that we are going to now do using the method of multiple scales will not have any damping. So, the energy. So, this equation will have something like an energy, which will be a conserved quantity.

However, this equation will be slightly different from the non-linear pendulum, in the sense that instead of having sin theta, which is a non-linear function of theta, it will have a just a cubic term in theta. So, this equation was one of the early equations, which was used for understanding an harmonic behavior of oscillators. So, the equation is like this.

So, you can immediately see that up to here this is a harmonic oscillator. Omega 0 square is just some square of some characteristic frequency, but this term actually is a non-linear term. Epsilon is small, but positive and so, you can see that this is a kind of a. So, you can think that this is the equation of motion of some mass, which is connected to a non-linear spring.

So, if it was a linear spring, it would just go as minus. So, this would be just this, but then I am adding some delta into some x cube term or rather this ok. And then, I am non-dimensionalizing it, which is leading me to the Duffing equation. Now, you can immediately see qualitatively what is going to happen.

If I did not have the second term; the second term you can see that if x is positive. If it was a linear spring, the force would be from right to left. It would try to the restoring force would try to pull it back to the equilibrium position here, to this restoring force tries to pull it back and it tries to pull it back harder than a linear spring.

So, we do expect periodic solutions. We also expect that, because the restoring force depends on epsilon. So, we do expect that the time period of motion will be dependent on epsilon ok, like the non-linear pendulum.

Note; however, that the first correction in the non-linear pendulum is a minus term approximate sin theta by Taylor series about theta equal to 0 then, it is theta minus theta cube by factorial 3. Here, it is a plus epsilon into u cube. Let us look at this equation. So, I will just erase this. So, now, before we do. So, this is we are going to do a multiple scale on this.

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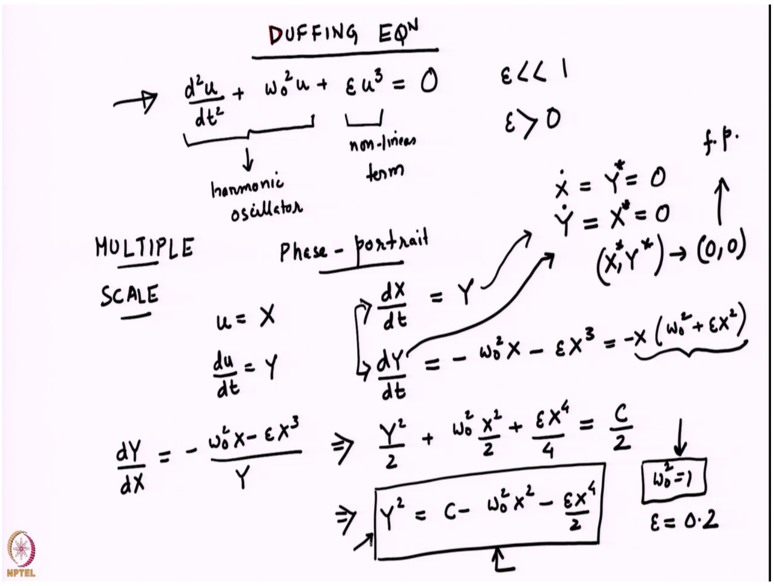

This can also be solved by the Lindstedt-Poincare technique. But this is just to show you that the multiple method of multiple scales is very general. It works it worked on the linear equation that I, that we discussed earlier and it is going to work on this equation as well.

So, let us draw the phase portrait of this oscillator before we attack it using the matrix method of multiple scales. So, as before we convert this. This is a second order ordinary differential equation. We convert this into two first order ordinary differential equations. They are expected to be coupled and non-linear.

So, we define u to be X and we define d u by d t to be Y. So, this immediately gives me d X by d t is d u by d t which is Y and d Y by d t which is basically the equation itself the Duffing equation itself ok d Y by d t. If I shift the omega square u and minus epsilon u cube on the left hand side then, I get minus omega 0 square into X minus epsilon X cube.

So, this is we have learnt to do this process before and I can write this as minus X into omega 0 square plus epsilon X square. Now, I can take the ratio of these two terms and if I take, if I divide the second term by the first term, then I get d Y by d X is equal to minus omega 0 square X minus epsilon X cube divided by Y. this equation can be easily integrated and this is telling me that Y square by 2 plus omega 0 square X square by 2 plus epsilon X 4 by 4 is equal to some constant of integration.

It is convenient if I choose the constant to be C by 2; so that I can cancel out a factor of 2 everywhere.

If I do this then, I get Y square is equal to C minus omega 0 square x square minus epsilon X 4 by 2. And so, like before we will have to plot this equations of Y as a function of X for various values of C. Now, for convenience what in I have plotted it in my next slide. I have chosen w 0 square is equal to 1. This is just for convenience, It does not need to be unity.

So, we will have two parts to it. One part will be the positive square root and one part will be the negative square root. This is very similar to what we did for the non-linear pendulum. So, let us look at the phase portrait of the Duffing equation.

Now, before we draw the phase portrait. Let us understand what are the fixed points of this equation. So, if you look at this equation; it is clear that X dot is equal to 0 when Y is equal to 0. Similarly, from this equation Y dot is equal to 0 when X is 0, this part. Omega 0 square and epsilon are positive and. So, the only value of X at which Y dot is equal to 0 is X equal to 0.

So, consequently X comma Y. So, let us call it Y star and X star. So, X star Y star being 0 comma 0 are the fixed points of the system. This is the only fixed point of the system. So, now, let us draw the phase portrait. So, we are going to draw this curve.

As I told earlier, we will choose omega 0 square is equal to 1 for convenience and we have chosen an epsilon which is equal to 0.2. So, I am going to plot various values of this curve, because this is a Y square here. So, there will be a plus square root and a minus square root. And for each value of C, I will get one phase space trajectory. Let us look at those trajectories.

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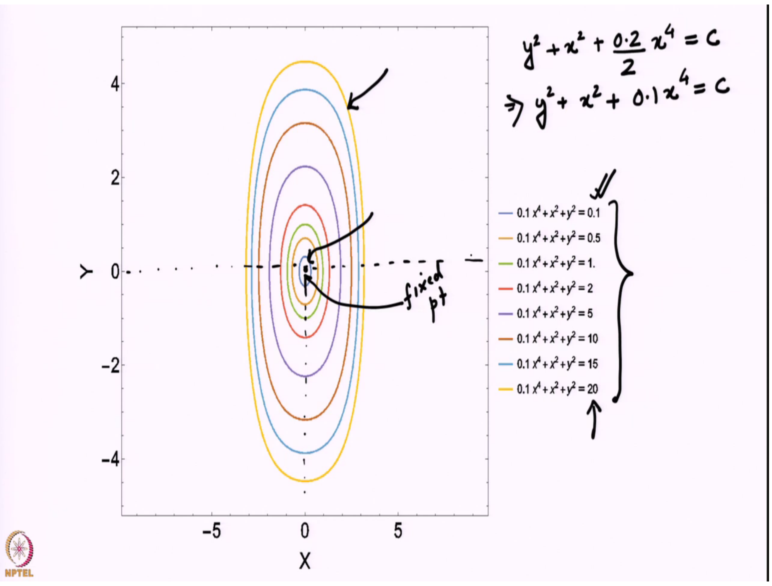

So, this is what the phase portrait looks like. So, this is the fixed point of the system. So, fixed point. This is corresponding to X is equal to 0 and Y equal to 0. Now, note; that I am plotting each curve as I said earlier for a different value of C. So, the curve recall are basically y square plus x square plus epsilon is 0.2 divided by 2 into x to the power 4 is equal to constant.

So, in other words I have y square plus x square plus 0.1 times x 4 is equal to C and I am choosing various values of C here, on the right hand side. So, the value of C ranges from 0.1 to about 20. You can see that as C gets smaller, the phase space trajectory also gets smaller. So, this is the smallest value of C for which I have plotted and that is the largest value of C for which I have plotted.

Once again you can split this curves into two parts. The upper part and the lower part, the upper part will correspond to the positive square root and then lower part will correspond to the negative square root.

So, we notice that there is one fixed point in the system and there are these periodic oscillatory solutions about the fixed point. One can go to higher and higher values of C. We call that C is related to the energy of the system. And so, higher and higher values of C just give me a larger and larger phase space trajectories.

So, now we are it is obvious from this figure that there are oscillatory periodic solutions to this equation. Let us find those solutions. Now, how do we find out these periodic solutions? Let us use once again the method of multiple scales. Once again, what are the time scales in the problem?

So, you can see that at very early times. We can we know that the period of the non-linear oscillator is different from the period of the linear oscillator. Now, at very early times, we are not going to see that period. We are not going to see the effect of that small difference in frequency, which is there between a non-linear oscillator and a linear oscillator ok.

So, as a first approximation. We will find that is just a harmonic oscillator with time period omega 0. Now, at the next order of approximation, we will find corrections to that. Let us work out those corrections.

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$$\frac{d^2u}{dt^2} + \omega_0^2 u + \epsilon u^3 = 0$$

$$T_0 \equiv t, \quad T_1 \equiv \epsilon t, \quad T_2 \equiv \epsilon^2 t$$

$$\frac{d}{dt} = \left(D_0 + \epsilon D_1 + \epsilon^2 D_2 + \dots\right)$$

$$\frac{d^2}{dt^2} = \left(D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2 D_1^2 + 2\epsilon^2 D_0 D_2\right)$$

$$u = u_0(T_0, T_1, T_2) + \epsilon u_1(T_0, T_1, T_2) + \epsilon^2 u_2(T_0, T_1, T_2) + \dots$$

$O(1)$: $\quad (D_0^2 + \omega_0^2) u_0 = 0$

$O(\epsilon)$: $\quad (D_0^2 + \omega_0^2) u_1 = -2 D_0 D_1 u_0 - u_0^3$

$O(\epsilon^2)$: $\quad (D_0^2 + \omega_0^2) u_2 = -2 D_0 D_1 u_1 - 2 D_0 D_2 u_0 - D_1^2 u_0 - 3 u_0^2 u_1$

So, we have. We are not going to use the Lindstedt-Poincare technique. We will use the method of multiple scales. So, we will define like before. We do not intend to solve the problem up to order epsilon square, but we have to go to order epsilon square in order to determine all the unknowns up to order epsilon. So, one has to do a little bit more algebra for here.

So, we have this and then, we have epsilon square t. So, like before d by d t is just D 0 plus epsilon D 1 epsilon square D 2. And d square by d t square. If I take this and then, I have to do an expansion which is x 0 sorry u 0 is equal to u is equal to u 0 which is a function of T 0 T 1 T 2 plus epsilon u 1 T 0 T 1 T 2 plus epsilon square u 2. Now, let us do that expansion.

So, we have to substitute all of this. So, the procedure is pretty standard now I am sure all of you are familiar with this. So, I will straightaway jump to collecting terms at various orders.

So, all this is obtained once you substitute this into this equation and this also into this equation and then, start collecting terms at various orders.

The left hand side is the same the right hand side. Now, you are familiar with this, one has to be careful while doing the right hand side. If you miss even one term, your answers will be incorrect. So, one has to be careful that you have not missed any term, nor have you included a term, which is not supposed to be there a t that given order.

Once again, I would like to reiterate that we are not going to solve the problem up to order epsilon square, but to determine one unknown at order epsilon, one has to go up to order epsilon square. So, that is why I am having to write up to order epsilon square.

The algebra is slightly lengthy here, but since we have already learnt how to do this and we are familiar with this from the damped harmonic oscillator, I am going to skip some steps.

It is easy for you, if you work out the algebra to do those steps. It is exactly similar to what we had done earlier. The only difference is here we are dealing with a non-linear equation. In the earlier example, we dealt with the linear equation ok.

So, these are our solutions at various orders. Now, let us solve these. So, let us write down the solution at order 1.

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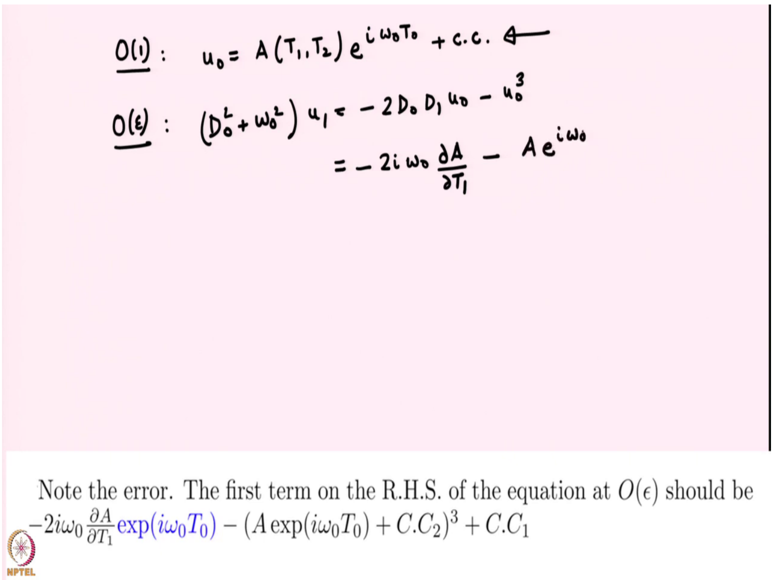

So, at order 1, the solution is the simplest. So, u 0 is just A, which is a function of like before T 1 T 2 into e to the power i omega 0 T 0 plus C C. Now, at order epsilon, we will have to plug this in. The left hand side remains the same and we had it we had written in the last slide that the right hand side consists of these terms, which depend on the previous order.

So, if I apply. Now, I know the solution at the previous order. And so, this just becomes twice i omega 0 del A by del t 1 minus A e to the power i omega 0 T 0.

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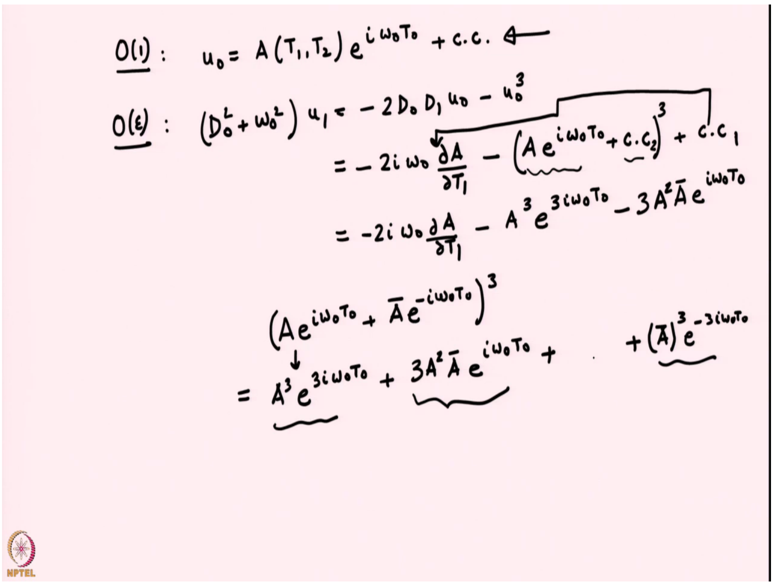

Plus it is complex conjugate whole cube. And there will be one more complex conjugate, which is the complex conjugate of this term. So, you can put C C 1, C C 2 if you want. So, this is C C 2 and this is C C 1. For example, now we will have to work out the right hand side. So, this term is obviously there then, we have A cube e to the power thrice i omega 0 T 0.

Now, there will be one more term, which will be the product of three times A square into B and that will give me. So, there is a minus 3 A square e to the power twice i omega 0 T 0 into B and B is just A bar e e to the power minus i omega 0 T 0 ok. So, I will have a A bar here, and e to the power i omega 0 T 0.

I hope this is clear. What we are doing is we are just taking this and adding its complex conjugate part and cubing the thing. So, the cube of this thing I have already written plus 3 A square B is what I have written. You can see that e to the power twice i to omega 0 T 0 minus.

So, this will give me e to the power i omega 0 T 0 plus i will have thrice A B square and that will have e with a negative exponent, but that will exactly be the complex conjugate of this part. And then, I will have one more term, which is A bar cube e to the power minus thrice i omega 0 T 0 and that is the complex conjugate of this part. So, I will just write the first two terms and then, put a C C at the end to indicate the other two terms.

We will continue this in the next video.