Introduction to Interfacial Waves
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Lecture - 21
Multiple scale analysis for damped-harmonic oscillator

We will looking at the solution to the damped-harmonic oscillator using the method of multiple scales. We have chosen three time scales T 0, T 1, T 2 and then we had expanded and expressed everything in terms of three independent variables; T 0, T 1, and T 2.

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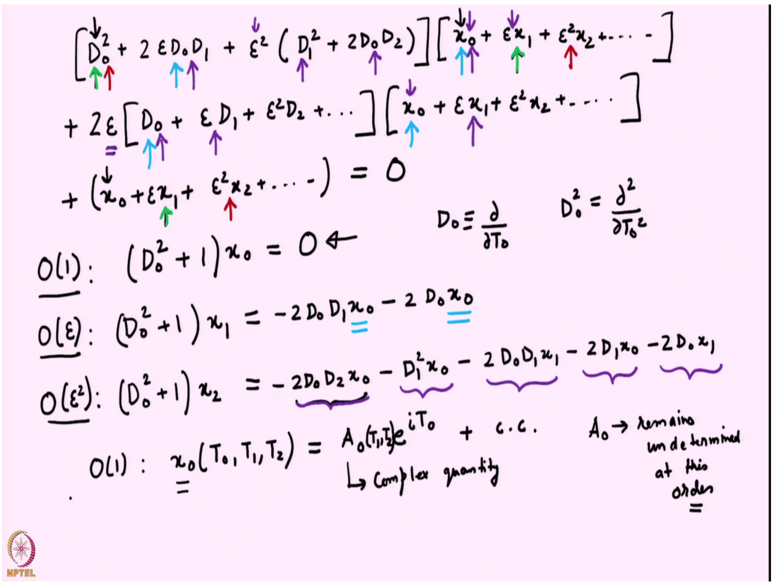

In particular we had found, that at the lowest order of description our system just behaves as a undamped oscillator with unit frequency. At the next order, where we expect some effect of damping to show up.

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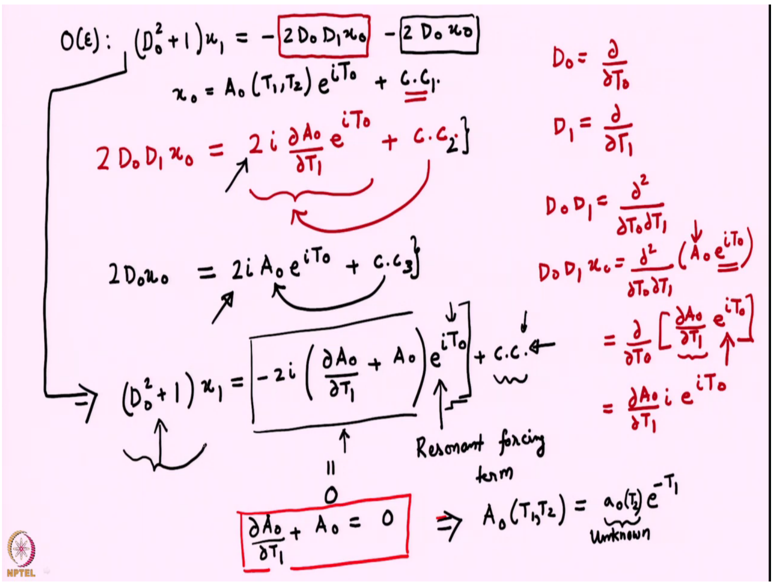

We had found, that we have an inhomogeneous ordinary differential equation, the right hand side has a term, which is proportional to e to the power i T naught, that is a resonant forcing term. And in order to; in order to prevent appearance of secular terms we need to eliminate that term. That in turn gave us an equation for the amplitude A naught. The A naught was actually a function of T 1 and T 2 and was appearing had appeared earlier as a coefficient of the lowest order solution.

So, now by eliminating the resonant forcing term, we have an equation governing A naught, if we solve this equation this is easy to solve, this is just a first order ODE this is actually a PDE, but we can again solve it as an ODE. So, you can see, that this equation just has the solution A naught which is a function of T 1 and T 2 as we had indicated earlier is equal to sum small a naught into e to the power minus T 1.

Once again this a small a naught is not a constant, but it is a function of the other unknown T 2 of the second independent variable T 2. So now, we know the T 1 dependence of capital A naught, the T 2 dependence is still unknown. So, this part is still unknown.

And so, now, we have found out the T 1 dependency of A naught, the T 2 is still unknown and that helps us in simplifying the equation governing x 1. So, this is the equation governing x 1. The right hand side is now completely 0, because we have said the only term which appeared was e to the power i T naught and e to the power minus i T naught.

By setting the coefficient of e to the power i T naught to 0, we have also set the coefficient of e to the power minus i T naught to 0, because the coefficient is just the complex conjugate of the expression that we have just set to 0. So, if this expression in the red box is 0, it is complex conjugate is also 0 automatically, alright.

So, now, this tells us that the equation governing x 1 is also a homogeneous equation, because the right hand side is now completely 0.

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$$(D_0^2+1)x_1 = 0 \quad \left[\text{After elimination of resonant forcing on R.H.S.}\right]$$

$$\underline{\underline{x_1(T_0,T_1,T_2)}} = \underbrace{A_1(T_1,T_2)}_{\text{complex}} e^{iT_0} + \underline{\underline{c.c.}}$$

$$\underline{O(\varepsilon^2)} : \quad (D_0^2+1)x_2 = \underset{1}{-2D_0D_2x_0} \underset{2}{- D_1^2x_0} \underset{3}{- 2D_0D_1x_1} \underset{4}{- 2D_1x_0} \underset{5}{- 2D_0x_1}$$

$$1: \; -2D_0D_2x_0$$

So, I can write the solution to x 1 as so, the equation after the simplification governing x 1 is just this. So, this is after elimination of resonant forcing on R. H. S; on the right hand side.

So, once we have eliminated that that gave us the T 1 dependency of the variable capital A naught ok. And, this is in turn leading us to this equation for x 1, this turns out to be the same equation which govern x naught also.

Again this is easy to solve. So, x 1 which is a function of T naught T 1 and T 2 can be once again written as some coefficient like before at the lowest order I was using the symbol A naught, now I will use the symbol A 1 because this is order epsilon calculation.

A 1 is A function of T 1 and T 2 into e to the power i T naught plus the complex conjugate of this part. Like before like A naught A 1 is also a complex function of T 1, T 2, you give it 2 real numbers it may give you a complex number. So, this is in general a complex quantity. And, because this is complex we have to add it is complex conjugate part to give us a real x 1, ok.

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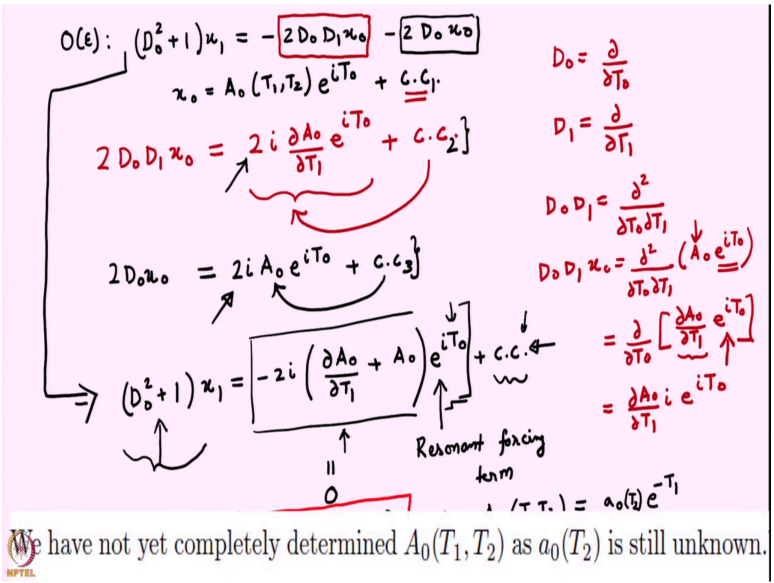

So, now we have determined x 1, we have determined A 1, A 1 is not completely determined because in the expression that we have for A 1 there is A small A naught sitting here. So, you can see that there is a small a naught sitting here this is a function of T 2 and we do not know what function of T 2 is this. So, for this we need to go to higher orders.

So, let us proceed now to the third order of calculation. So, order epsilon square. Here the left hand side is the same now operating on x 2. And, we had written earlier that there were a few terms in the right hand side. We had written that earlier that was minus twice D 0, D 2 operating on x naught minus D 1 square x naught.

As I said before, you can see that the right hand side is getting lengthier and lengthier. At the lowest order we had 0 in the right hand side, at the next order, we had we had 2 terms on the right hand side. The term indicated in the red bracket box and the term indicated in black box. Both of them produced a resonant forcing term we eliminated it and so, our right hand side was once again 0.

Now, at the or at order epsilon square I have 5 terms. And, so, you can see that the amount of algebra that we have to do keeps becoming more and more as we go to higher orders ok. So, let us do this. So, we now have to express. So, you can see that all the terms on the right hand side are known, because we know now what is x 0 and what is x 1, we only have to take the derivatives.

So, let us do them 1 by 1. So, I will call this 1 term 2, term 3, 4, 5. So, 1 is minus twice D 0, D 2 of x naught. So, for reference let me write let me raise this complex thing and let me write the solution for x naught also. So, that we have both the expressions for x naught and x 1 at 1 place.

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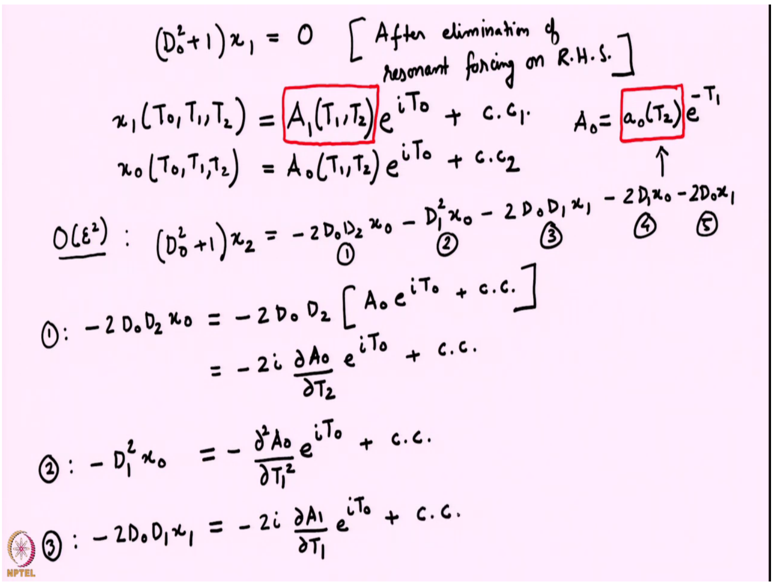

So, we had found that x naught plus complex conjugate, again if to eliminate confusion I will put a c c 1 and c c 2 it will indicate that this c c 1 is a complex conjugate of the term on it is left. Whereas, the c c 2 is a complex conjugate of the term on it is left ok. And, we had also found A naught; so I am just summarizing what we have found so, far. We had also found that A naught is small a naught, this is unknown yet e to the power minus T 1.

So, up to now this is an unknown and that is an unknown ok, so, because small a naught is unknown. So, you can also think that capital A naught is also an unknown, but it is partially known because we know that A naught depends on T 1 as e to the power minus T 1 ok. However, the T 2 dependence is not known. So, that is why I am not putting a naught; capital A naught in a red box, it is partially known and then partially unknown ok.

So, now let us calculate this term. This is equal to minus twice D 0, D 2 of A naught e to the power i T naught plus c c. Again, I am not going to write c c 1, c c 2, it should be understood that whenever I write a c c it indicates that it is the complex conjugate of everything that appears on the left of the c c. So, every c c is not the same. So, if I have multiple c c's in my in what I am writing, every c c represents a different expression that should be remembered, ok.

And, this is just minus twice i del A naught by del T 2 into e to the power i T naught. We know A naught from this expression, we will do that derivative later, but let me write it; let me keep it in this form right now. The second term is minus D 1 square x naught, this is equal to minus del square A naught. So, by del T 1 square into e to the power i T naught plus again the complex conjugate of this term.

The third term is twice D naught, D 1, x 1. We need all of these terms before we can work on the equation at order epsilon square, because each of these terms can potentially produce a particular integral. First we will have to see whether it produces a resonant forcing term or not.

If so, we will have to eliminate the resonant forcing term, then what is left we will have to work out the particular integrals of those terms. And, then we will have to write the general solution as a linear combination of the homogeneous solution plus the particular integral.

So, this is the general structure that we will follow. So, minus twice D 0, D 1 of x 1 is just minus twice i del A 1 by del T 1 e to the power i T naught plus c c, x naught.

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④ : $-2D_1 x_0 = -2\frac{\partial A_0}{\partial T_1} e^{iT_0} + c.c.$

⑤ : $-2D_0 x_1 = -2i A_1 e^{iT_0} + c.c.$

$$(D_0^2+1)x_2 = -\left[2i\frac{\partial A_0}{\partial T_2} + \frac{\partial^2 A_0}{\partial T_1^2} + 2i\frac{\partial A_1}{\partial T_1} + 2\frac{\partial A_0}{\partial T_1} + 2iA_1\right]e^{iT_0} + c.c.$$

$$A_0 = a_0(T_2)e^{-T_1}, \quad \frac{\partial A_0}{\partial T_2} = \frac{\partial a_0}{\partial T_2}e^{-T_1}, \quad \frac{\partial^2 A_0}{\partial T_1^2} = a_0(T_2)e^{-T_1}$$

$$\frac{\partial A_0}{\partial T_1} = -a_0(T_2)e^{-T_1}$$

$$(D_0^2+1)x_2 = -\left[2i\frac{\partial a_0}{\partial T_2}e^{-T_1} + a_0e^{-T_1} + 2i\frac{\partial A_1}{\partial T_1} - 2a_0e^{-T_1} + 2iA_1\right]e^{iT_0} + c.c.$$

$$\boxed{2iA_1 + 2i\frac{\partial A_1}{\partial T_1} = a_0e^{-T_1} - 2i\frac{\partial a_0}{\partial T_2}e^{-T_1}}$$

By setting the coefficient of e^{iT_0} on the R.H.S. to 0.

And that is minus 2 del A naught by del T 1 e to the power i T naught plus it is complex conjugate. And, the 5th term is just minus D 2 D naught x 1 is 2 D naught x 1. And, this is equal to minus 2 i A 1 e to the power i T naught plus it is complex conjugate ok. So, we have now calculated all the 5 terms, let us put them together.

So, the left hand side is just putting on x 2 and then it is 1 plus 2 plus 3 plus 4 plus 5. So, then that is minus I will put a minus common. So, then we have twice i del A naught by del T 2, that is my first term.

Then, my second term is a second derivative plus del square A naught by del T 1 square. Note that all the terms have e to the power i T naught; all the 5 terms have a e to the power i T naught. So, I am going to pull that out and keep it common, you can also see that it will produce all of these terms will produce a resonant forcing term.

The third term is twice i del A 1 by del T 1, you can check that and then the 4th and the 5th term. So, we can take a 2 also common, there is in the second term there is no 2, so we cannot take 2 common ok, fine. And, then the 5th term is twice i A 1 the power i T naught plus complex conjugate of everything on the left. So, the complex conjugate of this whole thing in square brackets multiplied by e to the power i T naught, ok.

So, now we can simplify this a little bit we have seen earlier, that A naught is small a naught which was a function of T 2 an unknown function as yet and then a known T 1 dependence. So, wherever we have derivatives with respect to A naught we can work out those derivatives. So, I have a derivative with respect to T naught A naught here a derivative here and a derivative there.

So, we have so, del A naught. So, let me write it on the side del A naught comma del T 2 is just del a naught by del T 2 e to the power minus T 1. And, then del square A naught by del T 1 square, it will take the derivative of e to the power minus T 1 twice, so I will get back the same expression. And, then we also have del A naught by del T 1, which is just minus of this.

So, if I plug in this on the right hand side, I have a minus this is twice i del a naught by del T 2 e to the power minus T 1 plus a naught e to the power minus T 1, that term remains intact. Then, the 4th term gives me and the 5th term is just twice i A 1 plus the complex conjugate. If, we remember that we have to take the complex conjugate we do not have to keep writing it all the time.

So, now you can see that all the terms; so, there are 5 terms and you can see that all the terms have a pre factor which is this, which is this or a post factor in this case, e to the power i T naught. As I told you earlier e to the power T i T naught is a solution to the homogeneous equation. If, I said the left hand side is equal to 0 any alpha into e to the power i T naught any constant into e to the power i T naught is a solution to the homogeneous equation.

Once again, we will have to set the entire coefficient of e to the power i T naught to 0. So, everything in square bracket here goes to 0. So, say everything in square bracket goes to 0 and so, we obtain; so, I am just going to write twice i A 1 plus twice i del A 1 by del T 1. So, I am writing this term and this term. And, I am shifting others to the right hand side.

So, this and this can be combined and it just gives me minus a 0 e to the power minus T 1. If I shift it to the right hand side, it just becomes plus a 0 e to the power minus T 1. And, then we have minus twice i del a naught by del T 2.

Please remember what are we trying to do here. We have obtained this equation by setting the coefficient of e to the power i T 0 on the right hand side to 0. So, this minus that is there is not relevant for our calculation. We do not have to worry about it, because I am taking the entire square bracket and setting it equal to 0. See, even if I had a minus it does not matter, because I am going to set it equal to 0.

So, I took the entire square bracket without the minus set it equal to 0. The 2 terms that I have encircled they are the same. So, I can add them and it gives me a minus a 0 e to the power minus T 1; I shifted to the right hand side. And, then this is the first term here; the first term here this is also shifted to the right hand side. So, it comes here. So, I have missed e to the power minus T 1 here.

And, then the terms that I have indicated by tick marks. So, let me put them in color. So, this tick mark and this tick mark. These two terms stay on the left, because these terms depend on capital A 1. And, the terms on the right depend on small a naught; this is the structure in which I have written down things ok.

So, this now is telling me something interesting. Like before it seems to be giving us an equation governing A 1. Recall that A 1 was an unknown and order epsilon, A 1 was just some unknown function of T 1 and T 2, we have seen this before, ok.

So, A 1 in square brackets in the rectangular bracket at the top of the slide, you can see is an unknown function yet. And, now by eliminating resonant forcing at second order, order epsilon square, we seem to be getting an equation for A 1. However, on the right hand side is small a naught and we do not yet know what is small a naught ok. So, let us see how to proceed with this.

So, we have to solve this equation. Let us do it. It is easy easier if we shift the 2 i on the left hand side, both terms have a 2 i, if I shift it to the right hand side it becomes a half there is an i in the denominator, but I can multiply numerator and denominator by i and bring i to the numerator and it becomes minus i, because of i square in the denominator, ok.

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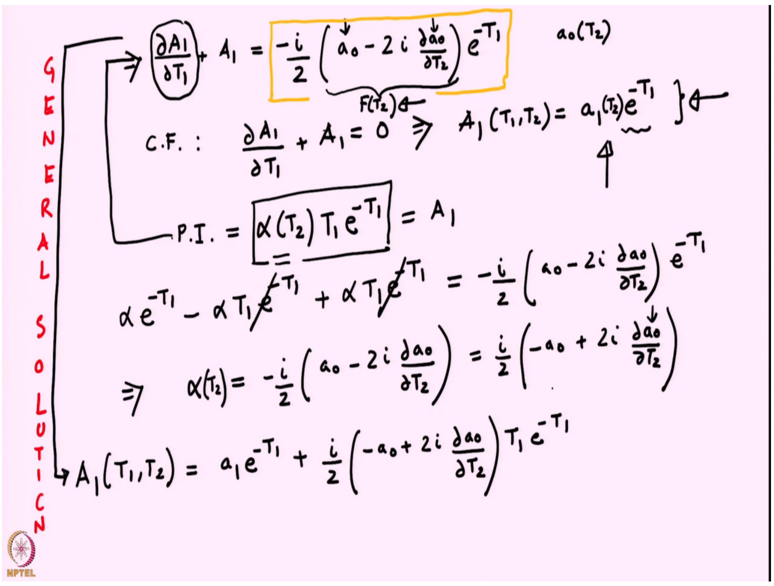

So, then the equation becomes del A 1 by del T 1 plus A 1 is equal to minus i, as I told you earlier and multiplied by half into whatever we had a naught minus twice i del a naught by del T 2 and I am taking the e to the power minus T 1 out, ok.

So, this is also again, this is actually a partial differential equation, but can be solved like an ordinary differential equation ok. There will be a C F and there will be a P I. So, the C F is just taking the left hand side and setting it equal to 0. So, C F is just del A 1 by del T 1 plus A 1 is equal to 0.

And so, this implies A 1 which recall is a function of T 1 and T 2 is some constant, it is not really going to be a constant into e to the power minus T 1 and because A 1 is a function of T 1 and T 2 and we are integrating in T 1. So, this small a 1 becomes a function of T 2. So, this small a 1 is the like the small a 0, which we had earlier and which is appearing in the right hand side of this equation.

So, A 1, A 0 small a 1 small a 0 are still unknowns, but we have figured out the T 1 dependence of A 1, from the homogeneous part. So, this is the complementary function. The particular integral is easy. You can see that this entire thing is just a function of T 2. There is no T 1 here why because small a naught is a function of T 2. We have said that earlier that small a naught is a function of T 2 alone.

So, this part that I have underlined here is a function of T 2, because it involves A naught and derivatives of A naught with respect to T 2. So, this is just a function of T 2 it is an unknown function, but it is a function of T 2. So, the entire term on the right hand side; so this entire term on the right hand side has e to the power minus T 1 with a coefficient, which is just a function of T 2.

So, I can find the particular integral for this very easily. I say that the P I is going to be alpha which is a function of T 2, T 1 e to the power minus T 1. Why did I take T 1 e to the power minus T 1? Because some constant or some function of T 2 into e to the power minus T 1 is a solution to the homogeneous part.

So, this is like a resonant forcing term, but you will see that this will not produce a secular term, but we will still have to eliminate it. So, I have taken T e to the power, because this right hand side is e to the power minus T 1 is the solution of the left hand side ok.

So, we have to take T e to the power minus T 1. So, that is why alpha T 2 into T 1 e to the power minus T 1, there has to be a function of T 2 here because the right hand side has a function of T 2, where small a naught is a function of T 2. So, I hope this part is clear.

So, now, we have to set as we have to in order to determine P I, we have only need to find out what is alpha ok. So, we have to go back and substitute into this equation. This is the usual way in which we find particular integrals and we will just get alpha into I take the derivative of T 1 first. So, it is just e to the power minus T 1 plus alpha so alpha never alpha is just a function of T 2. So, it does not participate in the differentiation. And, then this and there should be a minus sign here e to the power minus T 1, that is the first term.

Del A 1 by del T 1 plus A 1, A 1 we have said is alpha T 1 e to the power minus T 1, so, alpha T 1 e to the power minus T 1 ok. And, this is equal to minus i by 2 a naught minus twice I; I am just writing the right hand side of that equation. This term and this term cancel each other and this tells me that alpha is just minus i by 2 a naught minus twice i del a naught by del T 2.

It may appear a little bit tedious it may appear more complicated than what we had done in the Lindstedt Poincare method, but this is a much more powerful technique ok. And, we will see use of it when we do interfacial waves, ok. So, this and then I can push the minus sign inside if you want minus a naught plus. So, alpha as we had expected is just a function of T 2. This a naught is a function of T 2 del a naught by del T is also a function of T 2 so, that is alpha.

So, we now know the complementary function, the complementary function was this we also know the particular integral. The particular integral was alpha times T 1 times e to the power minus T 1 and we now know alpha. So, we can write the most general solution of this equation. This so, this is the equation whose general solution we are trying to write. So, let me write.

Say A 1 which is a function of T 1 comma T 2 is small a 1 e to the power minus T 1 ok, that is my complementary function. And, then a particular integral which is i by 2, I am just writing the value of alpha into T 1 e to the power minus T 1 ok. So, this is the expression for A 1. Now, what do we do next?

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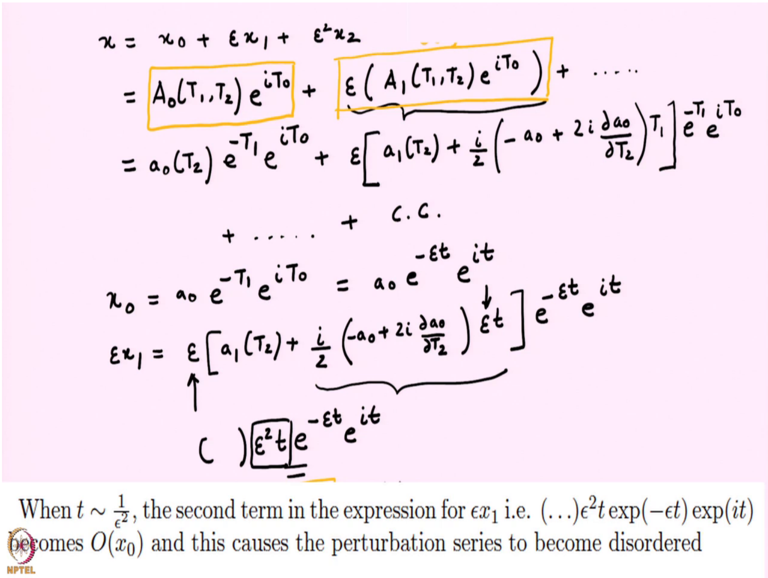

So, let us summarize what we have obtained so, far ok. So, far we have found that x is equal to x naught plus epsilon x 1 plus epsilon square x 2, we have not yet solved for x 2, but we have solved for x naught and x 1; x naught turned out to be A naught T 1, T 2 e to the power i T naught. I am not writing the complex conjugate I will write it at the end. Epsilon times x 1 was A 1 which is also a function of T 1, T 2 e to the power i T naught plus we have not solved for x 2 yet. So, let us put a dot dot.

We then found out that A naught is given by small a naught which is a function of T 2. Then, it is e to the power minus T 1 into e to the power i T naught. Similarly, epsilon times a 1 we have found a small a 1 which is some unknown function of T 2 plus I by 2 minus a naught plus twice i, this is the alpha that we had found earlier.

And, then we had a common e to the power minus T 1 and then this gets multiplied by e to the power i T naught. Plus we have higher order quantities and then we have to add the complex conjugate of everything to the left. So, this is what we have found so far.

Now, if you look at x naught then x naught is of the form a naught e to the power minus T 1 e to the power i T naught. If, I go back to my small t variable, then this will turn out to be e to the power minus epsilon t. So, there is the damping that you can see that we had anticipated ok. And, then this would be i t this is fine.

Epsilon x 1 the second term, the second term is epsilon x 1. What did we find this to be this is epsilon a 1 of T 2 plus i by 2 into what is there in the bracket. T 1 is epsilon t into e to the power minus epsilon t into e to the power i t. This is what we would found, if we went back to small t variables.

Now, look at the term inside the square bracket, look at the term; look at the second term. Now, what is here is a naught plus twice i del a naught by del T 2. You can see that this epsilon will multiply that epsilon and this will be a term of the form epsilon square t into e to the power minus epsilon t into e to the power i t with some coefficient. We can prevent this by setting the coefficient of this term to 0.

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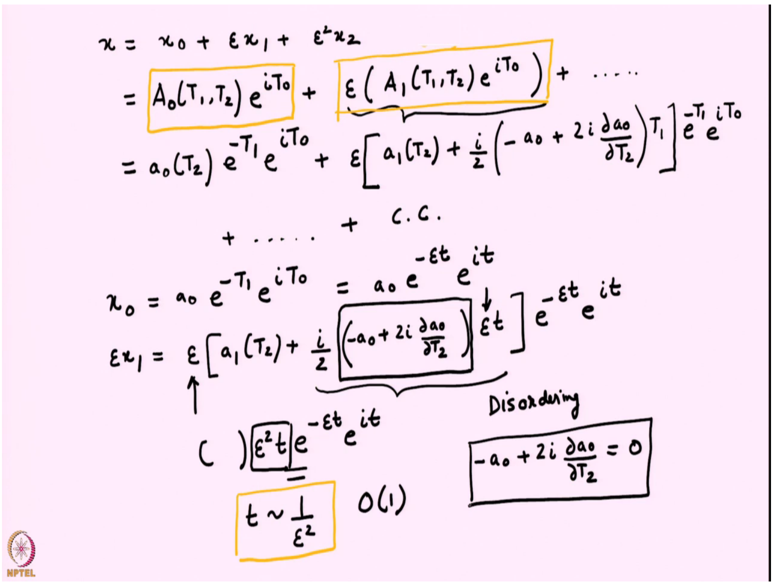

So, we can prevent this disordering, by ordering I mean every term in the perturbation series is smaller than all the terms on it is left. So, we can prevent this disordering by setting minus a 0 plus twice i del a 0 by del T 2 equal to 0. You can immediately see that this gives you the equation that we had sort for small a 0.

So, we are going to solve this equation and we will find that there are no more unknowns, there will only be an unknown constant when we integrate this equation. There will be no more unknown functions which appear here and that unknown constant can only be obtained by from initial conditions.

We will continue this in the next video.