Introduction to interfacial waves
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Lecture - 20
Method of multiple scales (contd..)

We were looking at the Damped Harmonic Oscillator and we had found that the solution to this equation to this linear equation for weak damping, using a regular perturbation series, produces similar secular terms as we had encountered in our solution to the non-linear pendulum, while using the regular perturbation method.

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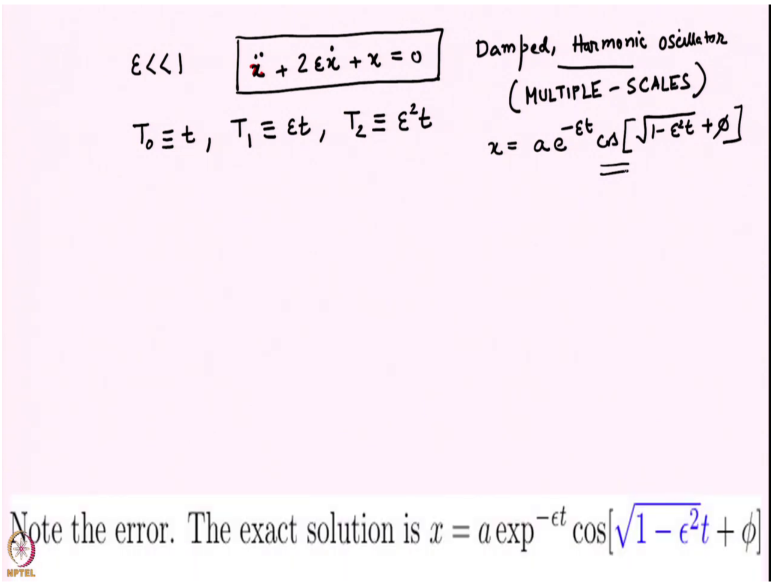

We had understood this that the exact solution of the damped harmonic oscillator contains an oscillatory term, which is cosine of t with a certain frequency and that frequency is a function of the small parameter epsilon. So, this was the origin of those secular terms. In particular we had mentioned that a generalization or we will introduce a more general technique, which is the method of multiple scales; using which we will be able to eliminate such secular terms up to any given order.

We are also motivated this technique by saying that, if you look at the exact solution of the damped harmonic oscillator, you can observe different processes showing up at different time scales. So, at early times you can just describe it as a harmonic oscillator with unit frequency and no damping.

At slightly longer time scales we had seen that the effect of damping starts showing up at even longer time scales, you can see that the frequency of the oscillator is slightly different from unity. So, in order to reflect that description, we said that we are going to convert from a single time to at least three different time scales ok.

T 0, T 1 and T 2 you can go higher up, but we will have to do more algebra in order to obtain solutions up to a given order in epsilon. So, we are going to now do this up to order epsilon square and we will see what we get as a consequence. So, let us now continue. So, as I had said we are going to convert this equation from a ordinary differential equation to a partial differential equation. The independent variables will become T 0, T 1 and T 2.

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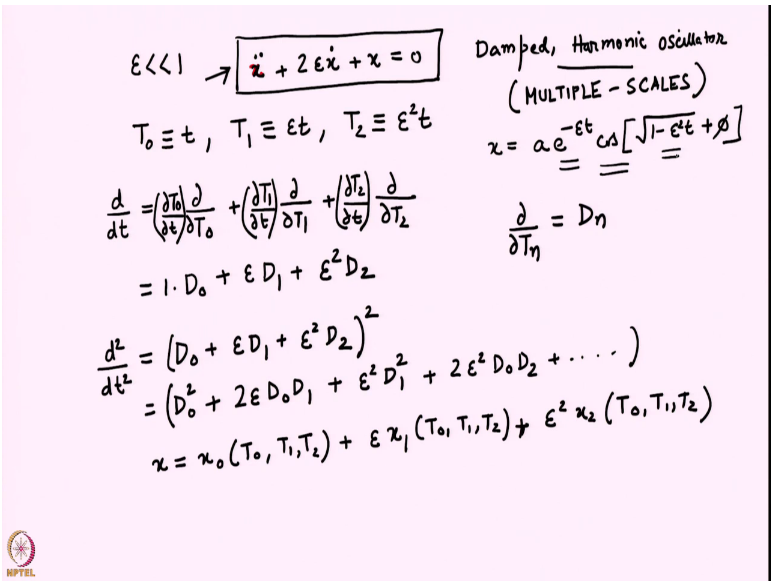

So, our derivative d by d t becomes del by del T naught into del T naught by del t. And, then this is just 1, I will replace this derivative del by del T n with the symbol D n. This is just a shorthand notation. So, del by del T naught which just becomes D naught, we have to just remember that all the d s are derivatives and the subscript indicates, which variable it is being derived with respect to.

So, this is epsilon D 1 plus epsilon square D 2. We want the second derivative in our equation that term the first term is second derivative with respect to time. So, we want this is an operator operating on itself. So, this is the square of this. And, if you open it up you get D 0 square plus twice epsilon D 0, D 1 plus epsilon square D 1 square plus twice epsilon square D 0, D 2 plus dot dot dot.

So, I am not writing any further, because these will be higher powers of epsilon, if, we now we also have to do an expansion for our equation. So, x is now x 0, which is a function of T 0, T 1 and T 2 plus epsilon times x 1, which is also a function of the same three variables plus epsilon square x 2 ok. So, let us find the solution. So, now, we will have to go back and substitute this expansion into the governing equation. So, let us do that.

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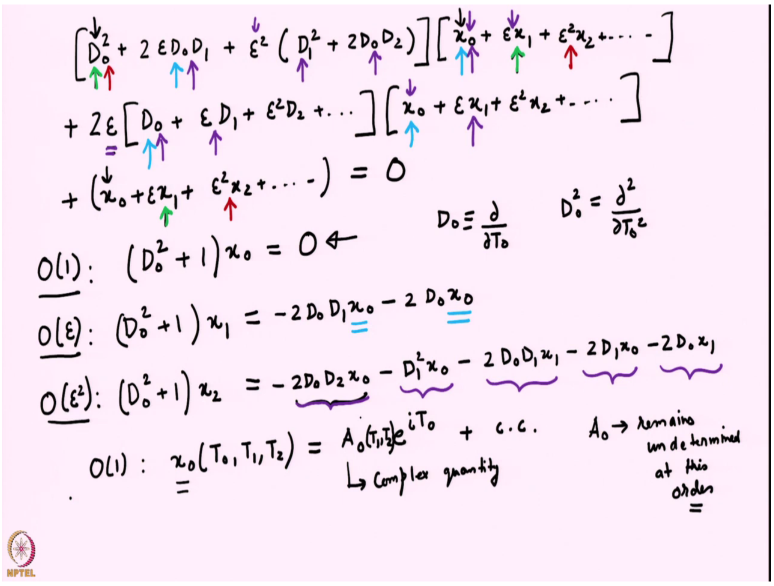

So, we have the operator in the first term and as a coefficient of epsilon square we had two terms D 1 square and twice D 0, D 2. And, this term this operator operates on x 0 plus epsilon x 1 plus epsilon square x 2, plus the second term is the damping term that has a first derivative.

So, the first derivative is just D 0 plus epsilon D 1 plus epsilon square D 2 operating on again the same thing and the last term is just x, which is and this is our equation. Now, like usual we have to collect terms at various orders. So, you can readily see that the term at order 1 is D 0 square.

So, that is this term operating on x naught, from the second term we do not get an order 1 contribution, because there is a prefactor epsilon overall. And, then we have this term which contributes an x naught. So, I can write it as D 0 square plus 1 x naught equal to 0. Remember that D 0, is defined as del by del T 0.

So, D 0 square is nothing, but del square by del T 0 square. Now, let us collect the terms at order epsilon. So, we will get so, we will get the structure of the left hand side will remain the same. So, we will obviously, have D 0 square plus 1 operating on x 1. So, that is D 0 square operating on x 1, that is an order epsilon term and then we have an additional x 1.

So, that is the so, let me put it this in color. So, this operating on this and then you have this. And, on the right hand side we will have so; we are collecting terms of the order epsilon. So, we will have minus 2 D 0, D 1 on x naught and minus 2 D 0 of x naught. How do we get those terms? So, we will have 2 epsilon D 0, D 1 operating on x naught, you can see that there is that is an order epsilon term.

We will also have D naught operating on x naught and that is an order epsilon term because of the epsilon prefactor in this term. We have already taken into account the order epsilon term from the last equation that I have indicated in green and that is on the left hand side of the equation.

So, this is my order epsilon equation. Note, the structure of the equation on the left hand side we always have D 0 square plus 1 operating on the variable, which reflects the order of the order at which we are operating. So, this is order epsilon to the power 1. So, that is why D 0 square plus 1 is operating on x 1.

At order 1 which was epsilon to the power 0, it was D 0 square plus 1 operating on x 0. And, on the right hand side at the lowest order it is always 0, at higher and higher orders we have more and more terms. But whatever appears on the right hand side; should not depend on a quantity which is at the same order.

So, you can see that all these terms operate on something which is already known. So, at order epsilon we already know the order one solution. We cannot go to order epsilon and solve the equations unless we already know the order 1 solution. So, on the right hand side should only be things which we know. And, on the left hand side should be things which we are going to find out at the given order. So, this is the structure.

Let us write one more order. And that is because so, we are really going up to order epsilon square, what we are going to do is we are not going to do a very long TDS calculation where we go all the way up to order epsilon square and find out all the corrections. We will just find the order epsilon square correction in the first term or in x 0. And, that you will see will contain the essential aspects of the calculation ok.

So, we are let us go to one more order, so, order epsilon square. So, we are going to order epsilon square, because as you can see the damping is going to show up at order epsilon as we will see, but the change in frequency. The fact that the frequency is not 1, but slightly different from 1, will appear only at order epsilon square. So, in order to see that we will really have to go all the way up to order epsilon square ok, so, with that in mind we are writing down terms up to order epsilon square.

So, again the same structure we will have D 0 square plus 1 now at or operating on x 2. So, that is coming from operating on that and then that. So, the left hand side is coming from the terms I have indicated in red. What do we have on the; so, on the right hand side we will have let me write down the terms and then I will explain.

So, the first term is twice D naught D 2 x 0. So, let me use another color. So, twice D naught D 2 x 0, will be coming from this term operating on that term. You can see that there is a prefactor epsilon square, then we have D 1 square x naught. So, that is going to come from this operating on this, again there is a coefficient x naught square, then we have twice D naught, D 1 x 1.

So, we have twice D naught D 1 operating on x 1. So, you can see the product of D naught D 1 contains an epsilon and then there is another epsilon. So, the whole thing will be epsilon square. So, then we have twice D 1 x naught. So, this is going to come from the product of this with that.

You can see that there is a epsilon sitting before. So, it will multiply and then we will get twice D 1 x naught. And, then at the end we have 2 D naught x 1. So, that will be once again this into that. So, that is the origin of various terms, I have listed them out one by one, so, this, this, this and that.

Note that all of them will have a minus sign, because they occur on the left hand side of the equation, but I am going to shift them to the right hand side to show that they are really in homogeneities, for the differential equation ok. So, now the procedure is exactly like before, it proceeds by solving the equations at every order, we have to start at the lowest order and then go step by step.

At every order we will have to find the complementary function and the particular integral. In particular we will have to be careful, that the particular integral should not contain any resonant forcing term. If, it contains resonant forcing terms we will have to set those to 0.

When we do that you will find that we actually obtain something called amplitude equations. And, the quantity which will appear as a constant at a given order will actually be a function of a slower variable or a longer time scale variable at the next order. And, the resonant elimination of the resonant forcing term will give us the equation which governs that quantity. So, let us learn how does this work?

So, at order 1 that is the simplest problem this is just a linear homogeneous equation x 0, which is a function of T 0, T 1, T 2 is a function of A naught e to the power i T naught plus complex conjugate; x naught is a real quantity. In this case x is a displacement and x is being written as a sum of x naught plus epsilon x 1 plus epsilon x x epsilon square x 2 and so on. All the epsilons are non dimensional. So, x 1, x 2, x 3 everything represents a displacement, so, it is a real quantity.

So, I have to if I am using complex notation I have to add it is complex conjugate to make it real like before. Why did I give a gap here? Because, I know that x naught now is not just a function of T naught, it is also a function of T 1 and T 2. We are integrating this equation as if it is an ordinary differential equation ok.

So, we are going to integrate it; however, when we integrate it, we have to remember that x naught is a function of T 1 and T 2. And, so, what I call as my constant of integration, actually here is not a constant, but is a function of T 1 comma T 2. Similarly, you will get a complex conjugate term, like before this is going to be a complex quantity, which means that if you give it a real T 1 and T 2 it will return a complex number to u.

So, a naught is a complex function of T 1 and T 2 ok. So, this is slightly more complicated than what we had encountered earlier. Earlier we were encountering complex constants, now we are encountering things which are functions of T 1, T 2 and are going to return complex numbers. Let us see, how to determine these unknown functions of T 1 and T 2.

So, it up to this order this is the only thing that we can write ok. If we, if y A naught is a function of T 1, T 2 is not clear to you I encourage you to differentiate this equation with respect to T naught and substitute back at the order 1 equation. And, convince yourself that this equation satisfies the differential equation that we have written at order 1 ok. So, now, with this so, our order 1 problem is completely determined, there is nothing more to be done at this order it is also clear.

That A naught remains undetermined at this order. Now, before we go further you can see that A naught is a function of the longer time scales T 1, T 2 are long and longer time scales. So, you can immediately see that on a short time scale. So, order 1 is the shortest time scale.

So, on the shortest time scale A naught is going to behave, so if you think of A naught as some function of T 1 and T 2. The dependence of a in the fact that A naught depends on time is not going to show up unless you go to times as large as T 1 and T 2. So, at very early times A naught is effectively a constant.

So, if we stop the solution to the problem at this order, it is just telling us, that this is a constant into e to the power i T naught. What does that mean? That means, that if you convert this into real notation you have to add the complex conjugate part, you will just get a constant into cos T, T naught is just small t.

So, a constant into cos T. This is consistent with what we have argued earlier, that at very early times it just behaves as a harmonic oscillator, no damping and with unit frequency. So, this is the description at this order. Suppose, I want a more detailed description, which is consistent with what I observe if I go to longer times. At slightly longer time I will start seeing the fact that this is a damped harmonic oscillator, the effect of damping will start to be seen. So, we have to proceed to the next order.

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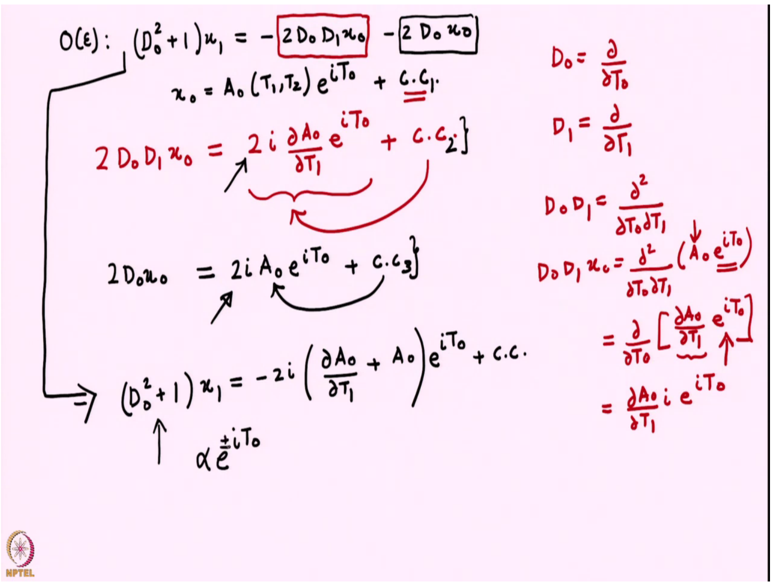

So, at order epsilon we have the left hand side remains the same operating now on x 1, but now it is an inhomogeneous equation. And, we had found that the right hand side is twice D naught, D 1 x naught and then there is one more term twice D naught x naught. We have just found that x naught is a naught which is a function of T 1, T 2 into e to the power i T naught plus complex conjugate.

So, what is so let me work out these terms on the right hand side this and that so, the red term is twice D naught D 1 x naught. And, so, you can see that this term is just twice i the D naught of x naught will bring will differentiate e to the power i T 0 and it will just pull out an i. And, then the D 1 will differentiate just A naught because e to the power i T naught is not a function of D 1 once again I remind you that D naught is del by del T naught and, D 1 is del by del T 1. So, D naught into D 1 is nothing, but del square by del T naught T 1.

And, if this operates let us say on x naught, then it is just del square by del T naught del T 1 operating on x naught, which is A naught e to the power i T naught. And, when you do this derivative suppose you do the T 1 derivative first then this becomes del by del T naught. So, when you do the T 1 derivative first it is only a naught which gets, whose derivative gets taken because this part does not depend on T 1.

So, it just becomes del a naught by del T 1 e to the power i T naught. And, then when you do the next derivative, which is with respect to T naught, then it is only this part which will get derived because this part is just a function of T 1 and T 2. So, this part will behave as if it is a constant and then you will have i e to the power i T naught. So, this is what I am writing here. So, it is twice i del A naught by del T 1 e to the power i T naught.

Now, that is not enough because I also had a complex conjugate. So, I need to take the complex conjugate also. I leave it to you to show that if you took the complex conjugate, then what you will obtain will actually be just the complex conjugate of this. So, this is just equivalent to saying that you can do the derivative first and the complex conjugation later or vice versa.

So, I will get a complex conjugate of this part. So, all the i s will become minus i s and remember that A naught is a in general a complex function. So, you have to put a bar on the top of the derivative del A naught by del T 1 also. In this way the one advantage of the complex notation is that we can only take 1 half of the part into account while doing our algebra.

However, when we multiply things we will have to be more careful ok. We will encounter that later ok. So, now, let us take the next term, the next term is twice D naught x naught. And, twice D naught x naught is just i. So, 2 out 2 i A naught e to the power i T naught and again like before it is c c.

Whenever, I am writing c c it only means that it is c c of this. So, this c c means c c of the term in red, this c c in black makes c c of the term in black. When I add them up I will have a c c of the term in red plus a c c of the term in black. So, I will write a single c c which means the complex conjugate of the whole expression the first term, plus the second term.

So, let us write the expression. So, I will have so, I am now just writing the equation it is D 0 square plus 1 into x 1 is equal to minus twice i, there is a minus because both the terms on the right hand side have a minus. And, then so, you can see that this term also has a twice i, this term also has a twice i; I am going to add them up.

So, I can put take them 2 i common and then I have a del A 0 by del T 1 plus A 0, e to the power i T 0 is common in both the terms so, I am writing it outside the bracket. And, as I said before the sum of this complex conjugate plus that complex conjugate, gives me another complex conjugate which is now the complex conjugate of this entire expression.

So, this complex conjugate is the complex conjugate of this entire expression. I hope it is not confusing to you that I am using c c c c. So, this c c is not the same. So, if you want to prevent confusion, then you can write c c 1, c c 2, c c 3 and this is the c c of the whole thing ok. So, just to ensure that this c c c c 1, c c 2, c c 3 is that is not the same as this c c ok.

So, now I need to solve the equation at order epsilon. So, I until now we have just written down the equation and we have figured out the explicit form of the right hand terms. Now, you can immediately see that although we are using complex notation, when you add the complex conjugate e to the power i T 0 is basically cos T 0.

So, this is going to give me a cosine term with unit frequency ok. So, it is cos T 0 ok. So, now, look at the solution to the homogeneous part. If you just had look at the left hand side and if you set it equal to 0. You will see the expression for the left hand side is the solution to the left hand side is sum alpha e to the power i T 0 ok.

So, some constant into e to the power i T 0 plus minus i T 0 is a solution. The plus is what comes here the minus will come in the complex conjugate ok. So, once again we have the familiar situation, where we have a right hand side term, which is a solution to the homogeneous equation ok. And, so, this term unless we eliminate this term this is going to cause secular terms.

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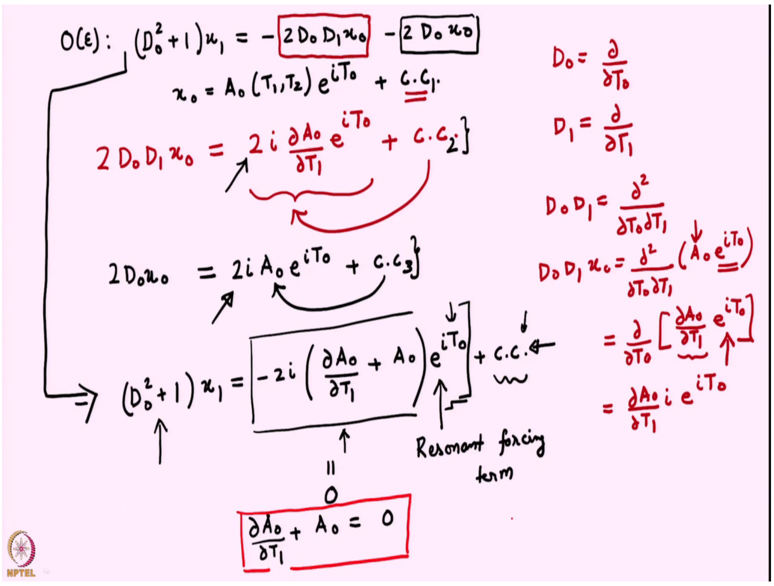

So, I am going to call this a resonant forcing term. It will give me a term of the form in expressed in real notation, it will give me a term of the form cos T naught. And, if you look at the left hand side cos T naught is a solution to the homogeneous equation for the left hand side ok. So, if you have to guess a particular integral you will have to take T naught cos T naught.

And, so, that will give me secular term which will grow in time, I do not want that because that was the problem we have encountered earlier when we were doing regular perturbation. Now, this method gives me a way of getting rid of such kind of secular terms. What do we do? We say that we want to set this entire term, this entire term to 0, you can pay attention that if you and set this entire term to 0.

Then you are also setting the complex conjugate to 0, because if you have some function a complex number and if you are setting it to 0. Then, it is equivalent to setting the complex conjugate also to 0. So, this is one advantage of the complex notation that if you just set this to 0. This part automatically the second part automatically gets taken care of and it is also set to 0.

So, what I am going to do is I am going to set the coefficient of e to the power i T 0 to 0, notice that it gives us an equation for A naught. This is exactly what it should be because as recall that in the earlier calculation in my order first order calculation. I had said that if you stop the calculation at the lowest order, then A naught is just a constant, A naught is not really a constant, we already know that a naught is a function of the longer variables T 1 and T 2.

So, at this order so, if you look at very short times you will find that a naught is just behaving like a constant, because you cannot see it is variation. You need to look at a longer time window to actually realize that A naught is varying and varying slowly as a function of time.

So, when you go to the next higher order that correction appears. Going to the next higher order is equivalent to asking what happens on longer time scales. So, on longer time scales A naught actually varies as a function of time. And, the process of elimination of resonant forcing term at the next order, tells us what is the equation for A naught. This is why I had mentioned that this is like an amplitude equation.

If, you think of A naught in at the lowest order, if you think of A naught as an amplitude of the oscillator. The e to the power i T naught term is an oscillatory term. And, A naught was it is amplitude. It is amplitude is at the lowest order, it is amplitude is just a constant, but if you go to a longer time window you actually realize that, it is amplitude is not a constant, it is slowly varying ok.

And, what is the equation, which govern that slow variation, that equation is determined by eliminating the resonant forcing term at the next order. So, the equation that we will find is just this del. So, I ignore the constants the minus 2 i is just a constant because I am going to set this term to 0. So, that constant is not relevant plus A naught is equal to 0.

So, this is my equation governing A naught. Note that A naught actually is a function of 2 variables T 1 and T 2, but this equation will not tell me the dependence of A naught on T 2. This will only tell me what is the dependence on T 1? We will still have yet another unknown amplitude, which is function now of T 2. And, one has to proceed once again to the next order in calculation to determine that unknown amplitude, we will continue in the next video.