Introduction to interfacial waves
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Lecture - 02
Coupled, linear, spring - mass systems

Recap. We were looking at two coupled mass systems connected through three springs, and we wrote down the equation of motion governing the two masses.

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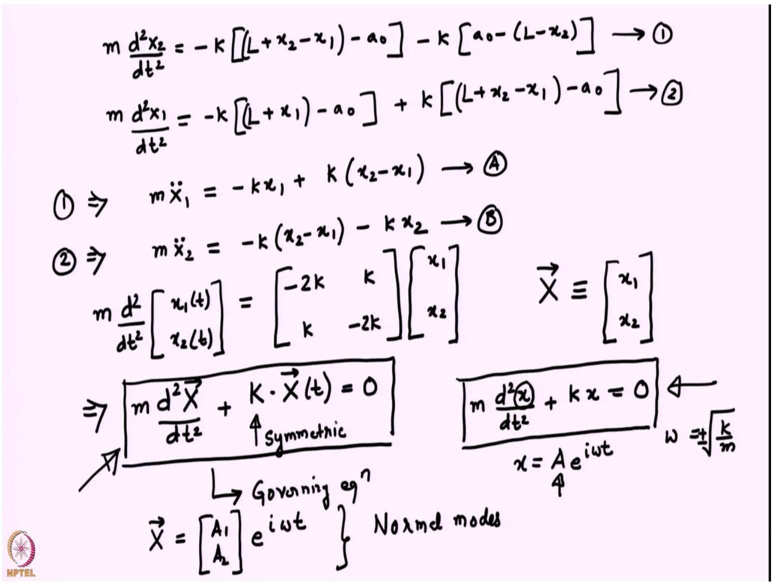

We found that we obtained two coupled linear ordinary differential equations. And we said that we are going to solve this using the method of normal modes. In this case, the method of normal modes looks like this. Once again we are looking for oscillatory solutions.

And an important thing to note is that in a normal mode motion, the entire system vibrates at the same frequency. Now, with this you can see that we have derived the equations and we have written the equations in matrix form.

If you compare this equation with the equation for a single degree of freedom, you recall that we had found that the frequency in the single degree of freedom case was omega is equal to plus minus square root k by m. So, we expect that the frequency will somehow k here in this two degree of freedom case is now a matrix.

So, we expect that the frequency of this system will somehow be related to some property of this matrix and the mass m. So, let us see what property is that. Also notice that this matrix is a symmetric matrix. It is a good idea to think about why it is a symmetric matrix.

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$$-m\omega^2 A_1 = -kA_1 + k(A_2 - A_1)$$

$$-m\omega^2 A_2 = -k(A_2 - A_1) - kA_2$$

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{Non-trivial } \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

For non-trivial sol^{ns}, det$|\quad| = 0$

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} = 0 \Rightarrow (2k - m\omega^2)^2 = k^2$$

$$\Rightarrow 2k - m\omega^2 = \pm k$$

$$\Rightarrow \omega^2 = \frac{k}{m} \text{ or } \frac{3k}{m}$$

So, now let us substitute our normal mode approximation into the governing equations A and B as expected it will lead to algebraic equations and we have to solve those algebraic equations. So, let us do that. So, our algebraic equations are minus m omega square A 1 is equal to, so this is minus here sorry.

And if you write it once again as a matrix equation, then we have now our column vector becomes A 1 A 2 is equal to 0 0. Now, we have to find non-trivial values of A 1, A 2 which means that we are not interested in A 1 A 2 both being 0. Because these are homogeneous set of equations for A 1 and A 2 0 0 is always a solution. We are not interested in this. So, we are interested in non-trivial A 1 A 2.

Now, recall from linear algebra that if you have a homogeneous set of equations, then for non-trivial solutions the determinant of the matrix needs to be 0. This will give you non-trivial solutions, but then those non-trivial solutions will not be unique. This non uniqueness will have important physical consequences as we will see shortly. So, for non-trivial solutions determinant of a matrix has to be 0, that matrix is this matrix.

Now, you can immediately notice that this matrix is just our original matrix or minus of that. And you can see that this quantity minus m omega square is basically the eigen value. So, it is clear that the eigen values of the matrix that we had seen earlier will be related to the frequency of oscillation. Because this is a 2 by 2 matrix, we are in general going to get two eigen values. They may or may not be distinct. Let us find those eigen values by setting the determinant equal to 0, so is equal to 0.

And this is easy to solve. And this basically leads to. So, we get two eigen values or two frequencies of oscillation in which the system can oscillate it is two because there are 2 degrees of freedom and the frequencies are distinct from each other. One is k by m; another is 3k by m.

Now, how do we find out? Now, because we have converted this into an eigen value problem. Now the frequencies are related to the eigen values. What information is contained in the corresponding eigen vectors? The eigen vectors corresponding to k by m and 3k by m. Let us find those eigen vectors and then answer this question.

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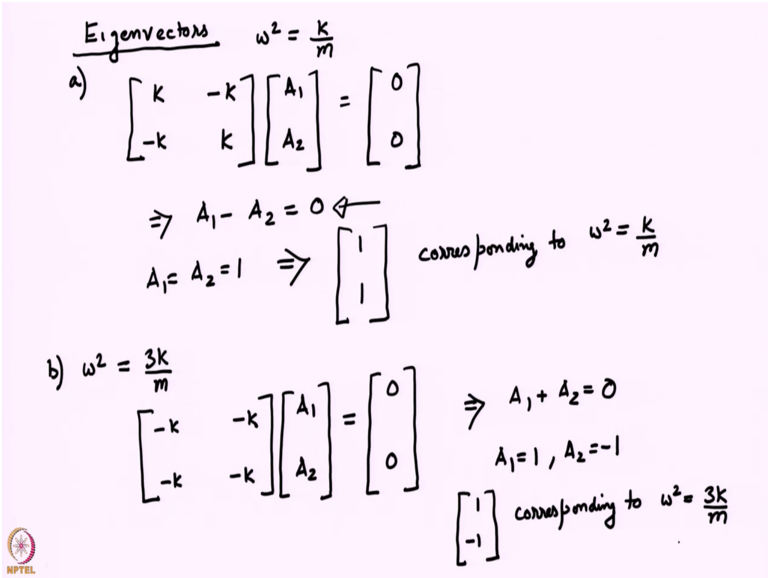

So, eigen vectors, now let us find the eigen vector for omega square is equal to k by m. If you go back to your eigen value problem and substitute this value of omega square, it leads you to the matrix equation k minus k minus k K into A 1 A 2 is equal to 0 0.

As expected this will not produce two linearly independent equations, but only one. You can see that it will give you only one equation which is A 1 minus A 2 is equal to 0. So, both the equations are just this. And so we have freedom in choosing one of the two way unknowns A 1 or A 2 and this equation will determine the other unknown.

So, if we choose A 1 is equal to A 2 is equal to 1, then this equation is automatically satisfied, and we get our eigenvector 1 1. And this is corresponding to omega square is equal to k by m. The eigen value k by m or square root k by m. The eigen vector corresponding to k by m is 1 1.

You can immediately see that this choice is not unique; you could have chosen 2 2 or you could have chosen minus 2 minus 2. All of these choices will also satisfy this equation. This is the non-uniqueness that I was referring to earlier. And it tells you that the choice of eigen vectors is not unique. We will see the physical meaning of this slightly later.

Now, let us work out the so this is eigen vector a, this is eigen vector b corresponding to 3k by m. If you once again go back and substitute omega square is equal to 3k by m in the eigen value problem that we had seen earlier. You will get the following matrix equation. Once again this does not lead to two independent equations, but only one which is A 1 plus A 2 is equal to 0.

We can choose A 1 to be 1, in that case A 2 becomes minus 1. So, we have A 1 is equal to 1, A 2 is equal to minus 1. And so we have an eigen vector 1 minus 1 corresponding to omega square is equal to 3k by m. Here also you could have chosen 2, minus 2, or minus 2, 2 and all of which would also be an eigen vector. You can multiply this 1 1 by any positive or negative number. And what you get would still continue to be an eigenvector corresponding to the eigen value 3k by m. The same is true for k by m alright.

So, now we have found the eigenvectors. We have also determined the eigen values which are related to the frequency of oscillation. Now, let us solve the problem. We had written two equations A and B. Now, let us write the answer to this equation. The general solution to these set of equations in terms of those eigen vectors and eigen values that we have found.

Now, obviously, because this is a second order ordinary differential equation with two unknowns x 1 and x 2, so we will have to do something more complicated than what we did last time. So, let us generalize that.

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$$\text{Real}\left\{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\right. = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\sqrt{\frac{k}{m}}t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-i\sqrt{\frac{k}{m}}t}$$

$$+ c_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{i\sqrt{\frac{3k}{m}}t} + c_4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-i\sqrt{\frac{3k}{m}}t}$$

$$c_1 = \bar{c}_2, \quad c_3 = \bar{c}_4$$

$$x_1(t) = (c_1 + c_2)\cos\left(\sqrt{\frac{k}{m}}t\right) + i(c_1 - c_2)\sin\left(\sqrt{\frac{k}{m}}t\right) + (c_3 + c_4)\cos\left(\sqrt{\frac{3k}{m}}t\right) + i(c_3 - c_4)\sin\left(\sqrt{\frac{3k}{m}}t\right)$$

$$x_2(t) = (c_1 + c_2)\cos\left(\sqrt{\frac{k}{m}}t\right) + i(c_1 - c_2)\sin\left(\sqrt{\frac{k}{m}}t\right) - (c_3 + c_4)\cos\left(\sqrt{\frac{3k}{m}}t\right) - i(c_3 - c_4)\sin\left(\sqrt{\frac{3k}{m}}t\right)$$

So, the general solution now is written as a linear combination of the eigen vectors. So, 1 1 was our eigenvector. So, corresponding to the first frequency omega square is equal to k by m, we will again have two omegas plus square root k by m and minus square root k by m. So, like before we write it as a linear combination of plus and minus, but now we have one more frequency.

So, we will have to write more quantities. Once again plus square root 3k by m and minus square root 3k by m. This is the most general solution to this set of coupled linear ordinary differential equations. What are C 1, C 2, C 3 and C 4? These are constants of integration which have to be determined from initial conditions. Like before you can see that this and this are just the complex conjugates of each other.

So, because the eigenvector is the same here and there, and because this quantity has to be real so C 1 and C 2 must be complex conjugates of each other. We will find that. Similarly, this and this are complex conjugates of each other. Once again the eigen vectors are the same in both the terms and so C 3 and C 4 will have to be complex conjugates of each other in order to keep x 1 and x 2 real.

This will come out automatically. So, we will have to determine C 1 and C 2 in terms of initial conditions. Before we do that, let us write and shift gradually to real notation. Once again like before we use e to the power i theta is equal to cos theta plus i sin theta. Using that and now I am leaving my matrix notation and writing out an expression for x 1 alone. You can write that by getting x 1 on the left hand side, and doing the matrix multiplication with C 1 and just writing the first row.

So, the first row would be x 1 of t would be C 1 plus C 2 cos square root k by m t plus i times C 1 minus C 2 sin square root k by m t plus C 3 plus C 4 cos square root 3k by m t plus i C 3 minus C 4 sin square root 3k by m t. Similarly, you can write an expression for x 2. The first part will be exactly the same; the second part will be different because the second column the second row in the eigenvector is different. So, we will have C 1 plus C 2 cos square root k by m t ok.

So, that is our expressions. These are still not completely in the in real notation because there is an i sitting in these expressions. But as it will turn out C C 1 plus C 2 will turn out to be a real number, C 3 plus C 4 will also turn out to be a real number, i times C 1 minus C 2 will turn out to be a real number, and i times C 3 minus C 4 will also be a real number making the expression for x 1 and x 2 completely real.

So, now let us do the same thing that we did earlier. Let us say that; let us say that we have initial conditions. Now, how many initial conditions do we have? We will have four initial conditions because now there are two oscillating masses. So, we have to specify the initial position and the initial velocity for every mass.

This is consistent with what we have done so far. In the sense that, there are four unknown constants C 1, C 2, C 3, C 4, and we have to determine them using the four initial conditions that can be specified. So, let us do that.

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$$\vec{X}(0) = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix} \qquad \dot{\vec{X}}(0) = \begin{bmatrix} V_1^{(0)} \\ V_2^{(0)} \end{bmatrix}$$

$$x_1(0) = (C_1 + C_2) + (C_3 + C_4) = x_1^{(0)}$$

$$x_2(0) = (C_1 + C_2) - (C_3 + C_4) = x_2^{(0)}$$

$$\dot{x}_1(0) = i\sqrt{\frac{k}{m}}\,(C_1 - C_2) + i\sqrt{\frac{3k}{m}}\,(C_3 - C_4) = V_1^{(0)}$$

$$\dot{x}_2(0) = i\sqrt{\frac{k}{m}}\,(C_1 - C_2) - i\sqrt{\frac{3k}{m}}\,(C_3 - C_4) = V_2^{(0)}$$

$$C_1 - C_2 = -\frac{i}{2}\sqrt{\frac{m}{k}}\left[V_1^{(0)} + V_2^{(0)}\right] \qquad C_1 + C_2 = \frac{1}{2}\left[x_1^{(0)} + x_2^{(0)}\right]$$

$$C_3 - C_4 = -\frac{i}{2}\sqrt{\frac{m}{3k}}\left[V_1^{(0)} - V_2^{(0)}\right] \qquad C_3 + C_4 = \frac{1}{2}\left[x_1^{(0)} - x_2^{(0)}\right]$$

So, if we say that X of 0 is equal to x 1 0 some initial position of the two masses, and X dot 0 is equal to some V 1 some initial velocity of the two masses. So; obviously, x 1 0, x 2 0, V 1 0, and V 2 0 are real numbers. And we are now going to express our unknown constants C 1, C 2, C 3, C 4 in terms of these real numbers. So, we obtain x 1 0 from the previous equation by substituting t equal to 0, we obtain.

We are just substituting t equal to 0, sorry we are just substituting t equal to 0 in the expressions written at the bottom. And you can immediately see that the cosine terms will go to unity, the sine terms will vanish and that is what is allowing me to obtain these equations that I am writing. Similarly, we can take the derivative of the expressions that I just showed you, and substitute t equal to 0 in them.

If we do that, then we would obtain x 1 dot of 0 and this by definition is equal to V 1 0. Similarly, x 2 dot dot is a derivative with respect to time x 2 dot of 0 is the first term is the same, and this by definition again is V 2 0. So, those are linear equations for C 1, C 2, C 3, C 4. And if you if we solve them, we can determine C 1, C 2, C 3, C 4 in terms of x 1 0, x 2 0, V 1 0, V 2 0.

You can immediately see that we do not need their individual values the difference is just this. We had expected earlier that C 1 minus C 2 i times C 1 minus C 2 will be an imaginary quantity. And you can readily see that from what I am writing down similarly i times C 3 minus C 4 would be a purely real quantity and that you can see here.

So, similarly we can obtain expressions for C 1 plus C 2 and we expect this to be purely real; C 3 plus C 4 is also purely real, and so that completes our exercise.

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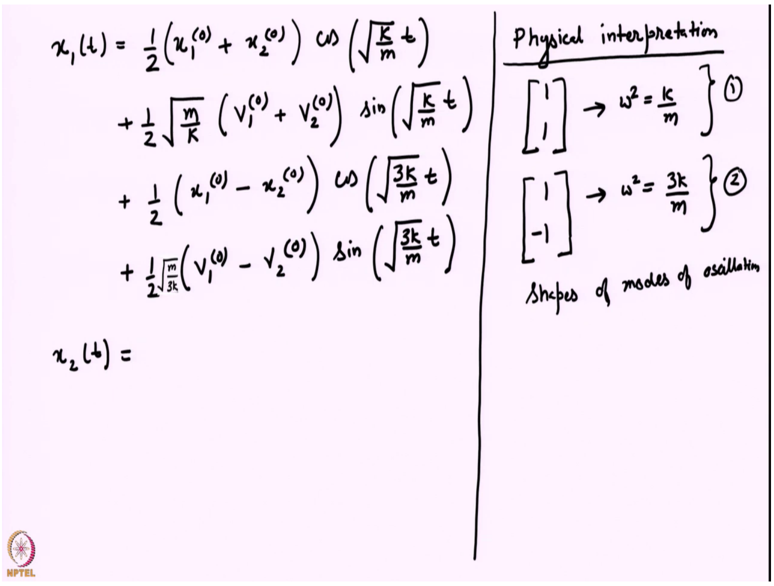

Now, writing substituting this and writing the final answer, we obtain, we are now shifting completely to real notation. There are no complex quantities in our final expressions like before. So, this is the expression for x 1 of t. You can write a similar expression for x 2 of t. I am leaving it to you to write that. It is very easy.

Now, let us come to the physical interpretation of this mathematics. We have seen that there are two eigenvectors for this system. Those can be chosen to be 1 1 which has a frequency omega square is equal to k by m. Another eigen vector which can be chosen to be 1 minus 1, and this corresponds to the frequency omega square is equal to 3k by m.

Now, these eigen vectors are what are known as the shapes of oscillation or the shapes of modes of oscillation. Let us try to understand why is it called a shape. Let us look at the first mode. So, I will call this the first mode, and I will call this the second mode. We have seen that there are two independent normal modes here.

How do we get the system to vibrate in normal mode 1? This is an oscillatory solution. So, we expect that if we set up the right kind of initial conditions, our system would vibrate only in normal mode 1. Now, it is immediately clear that in by looking at the eigenvector, it is immediately clear what should be done in order to set the system in normal mode 1. We go back to the diagram of our system that we had drawn for clarity.

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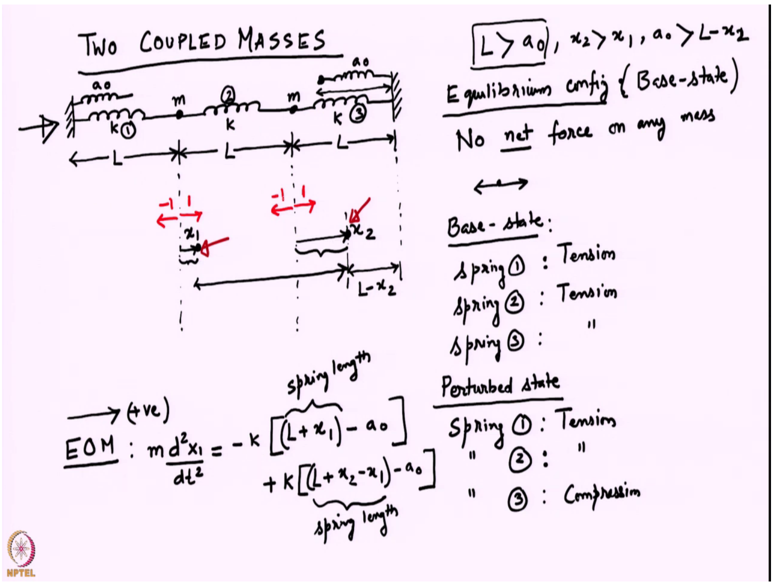

1 1 is the eigenvector. So, this is telling us that we have to displace mass; we have to displace mass 1 in the positive direction by a unit amount. You also have to displace the second mass again in the positive direction by a unit amount. If we do that, then we will set up an

oscillatory mode of motion. And in this mode, the system will have frequency k by m. It is also clear that it is this is not the only way of doing this.

If you did this in the other direction, so we recall that 1 1 was an eigen vector, but minus 1 minus 1 was also an eigen vector. So, if you give a unit displacement in either directions, then you would have the system oscillating in mode 1. So, you take the two masses and either displace them both to the left or both to the right.

It is from this thing it is also clear that if you displace both the masses by an equal amount, then the length of the spring in between does not change beyond what it is in the base state. In the base state, the length of the spring is l as I have indicated in the diagram.

So, you can immediately see that the middle string does not participate in this mode of motion. It is not getting extended beyond its base state length in this mode. Consequently, the two masses behave as if they are moving independent of each other because the connecting spring does not get extended beyond its base state length. Thus it is not surprising that the frequency of motion turns out to be k by m. Because these are these two masses are behaving as if they are uncoupled to each other.

Similarly, you can go back and think about what is the displacement 1 minus 1. So, this indicates that you take one mass 1 displace it to the right by a unit amount, take mass 2 displace it to the left because the second quantity is minus 1. So, you can either do 1 minus 1 or you can do minus 1 1 because if 1 minus 1 is an eigen mode, then minus 1 1 is also an eigen mode.

In other words, you have to displace them by equal and opposite amounts. If you do that, then you will set up the second mode of oscillation. I leave it to you to think why is the frequency of this mode of oscillation more than the frequency of the first mode.