Introduction to interfacial waves
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Lecture - 19
Method of multiple scales

We were looking at the non-linear pendulum, where we use the Lindstedt Poincare Technique to eliminate the secular terms. And, obtain an expression which was bounded at all times, we determine the first non-linear correction. And, we found an approximation to the periodic solution of the non-linear pendulum.

We also saw that as angles as large as 90 degree, the two term expansion that we had found in the Lindstedt-Poincare method was a very good approximation to the exact solution, which was a more complicated formula expressed in terms of elliptic sn and sine inverse functions.

Now, we will come to another more general technique, which is called the method of multiple scales. Up till now we have been looking at the effect of non-linearity, and, we have looked at the non-linear pendulum and what it does, what does, how does perturbation help us in obtaining approximations to the solution to the non-linear pendulum.

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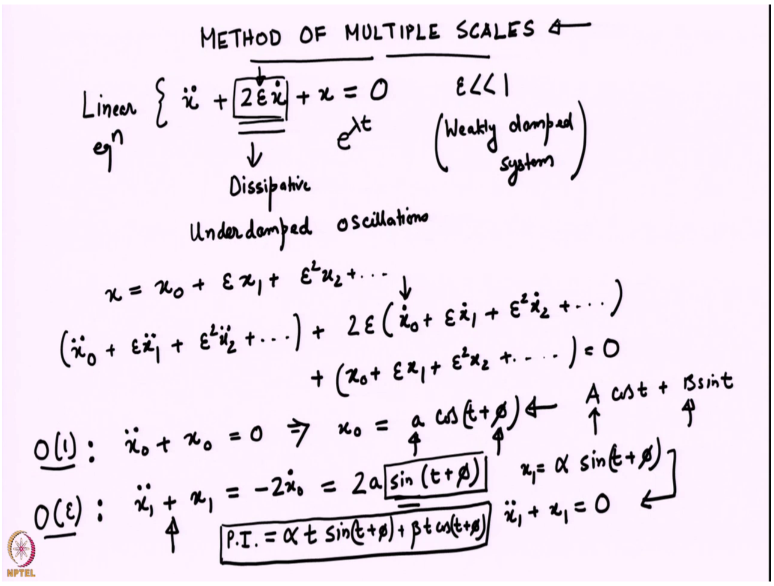

Now, I will show you that very similar problems arise even in linear equations. So, let us look at a problem, which is a slight modification to our simple harmonic oscillator problem, we are just going to put in a damping term. So, you can imagine that, this is earlier we had a mass connected to a spring, now we have a mass connected to a spring and a dashpot. So, this is the dissipative term.

You can also assume that this has already been non-dimensionalized. So, that introduces a small parameter in the problem. In this case the small parameter is a measure of the strength of damping. So, epsilon is less than 1 and so, it indicates that damping is weak. So, this is a weakly damped system. This also implies; so we are typically going to get under damped oscillations.

Recall that without when epsilon is 0, this is just a simple harmonic oscillator that we have already solved. By the addition of this term we have added damping. So, we cannot get periodic solutions, because this is like a friction term. And, so, our oscillator is going to damp out with time, because epsilon is much less we are going to typically get under damped oscillations.

Let us now try to, so notice that this is a still a linear equation. We can of course, solve it with this is a linear constant coefficient equation. So, we do know how to solve it exactly. Let us try to solve this perturbatively and you will find that there is an interesting feature, which also comes out of this equation, which is similar to what was seen in the non-linear pendulum.

So, once again because the equation has a term of the order epsilon so, I will say that the answer of the problem of the unperturbed problem, which is just a undamped oscillator, undamped harmonic oscillator, we will also have to be perturbed by order epsilon. So, x is equal to x naught plus epsilon x 1 plus epsilon square x 2 plus dot dot dot dot.

If, we substitute it into the governing equation, then we will obtain x 0 double dot, plus epsilon x 1 double dot, plus epsilon square x 2 equal to 0. Once again like before, we put together all the terms at various orders. So, at the lowest order, we just get the unperturbed oscillator. The solution to this is; the general solution to this is some a cos t plus phi.

I have just chosen to write it like this, note that you can also write it as A cos t plus B sin t. Where A and B are constants of integration; here A and phi are constants of integration. You can relate small a and phi to capital A and capital B.

Now, I am not going to solve an initial value problem here. So, I do not specify the initial conditions. One can do that, but the main point that we want to demonstrate here can be demonstrated without having to solve the initial, without having to account for the initial conditions.

So, now let us collect order epsilon terms and we will have. So, this left hand side is exactly the same operator, but now for x 1 and now it will be inhomogeneous. So, I will have a order epsilon term from here. So, that is minus 2 x naught dot. And, that is my only order epsilon term for x naught, for x 1 I have already shifted it to the left hand side. And, if I substitute what I know about x naught from here, then this term just becomes plus 2 a sin t plus phi.

Now, you can already see an interesting feature which you have encountered before. This formula sin t plus phi is a solution to that homogeneous equation, if, you take sin t plus 5. So, if you take x 1 is equal to some alpha sin t plus 5 and substitute in the equation x 1 double dot plus x 1, you will see that it exactly satisfies the equation. So, this is an exact solution to that equation.

Now, what does this imply? Whenever, we have a right hand side which happens to be a solution to the homogeneous part of the left hand side or in other words is a multiple of the complementary function, then the particular integral has to be multiplied by some power of t. In this case, it is t sin t plus phi. So, the particular integral in this case we will take the form.

So, the we will have to determine the exact coefficient, but so, the particular integral, in this case has to be written as a linear combination of alpha times t sin t plus phi plus beta t cos t plus phi. And, we have to go back to this the inhomogeneous equation and substitute it and find out, what is the value of alpha and beta, which satisfies the inhomogeneous equation.

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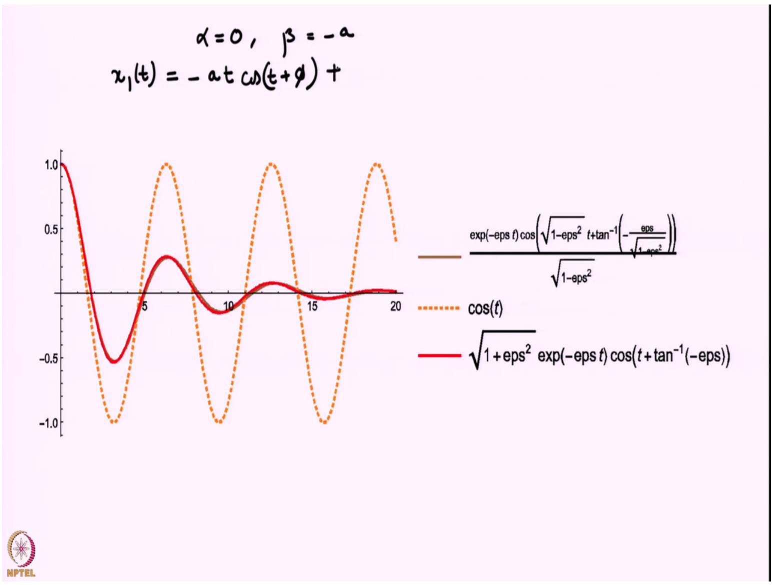

Alpha is equal to 0, beta is equal to minus a. So, the solution for x 1 of t is just minus a t cos t plus phi. I am not writing the most general solution, if I have to write the most general solution, then I will have to add a solution to the homogeneous part, which in this case is some c 1 times cos t plus c 2 times sin t.

That part is necessary only if I am accounting for the initial conditions, because the initial conditions will determine the unknown constants of integration. So, let me just not write the complementary function, let me let just write x 1 in terms of the particular integral, and this is the particular integral.

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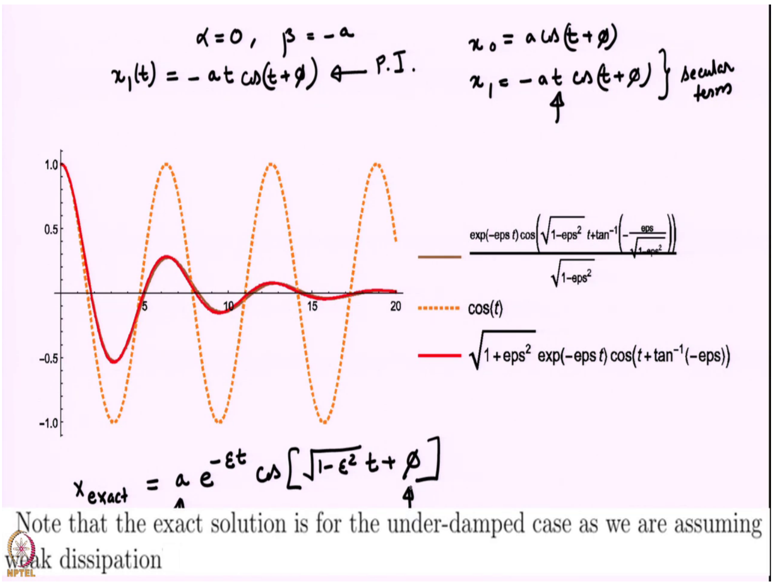

The basic point that I want to emphasize here is that your x naught was an oscillatory function and x 1 has turned out to be a function, which has an oscillatory part, but also a part which is going to grow with time.

We have seen this behavior before, this is a secular term. This is arising in exactly the same manner, we at the first order of correction, we had a resonant forcing term. And, the resonant forcing term forced the left hand side and caused a secular term to be produced.

Note that we have found this now in a linear equation. Earlier we had encountered this while we had solved for the non-linear pendulum. However, this feature is turning up or showing up even in a linear equation. Why is this so? The reason actually remains the same.

Recall that by adding damping we have added a twice epsilon x dot. What is the exact solution to this equation? Because, this is a constant coefficient solution, you can go back and substitute e to the power lambda t, this will convert this equation into an algebraic equation for lambda.

You will determine a complex conjugate pair and then write the answer in terms of a linear combination of e to the power lambda 1 t plus e to the power lambda 2 t. Lambda will have a real part and an imaginary part. And, if you express the final answer in terms of real things, then you can show this is very easy to show. That the exact solution to this problem is of the form a e to the power minus epsilon t cos square root 1 minus epsilon square t plus phi.

Once again I have two constants of integration in my final answer. An amplitude a and a phase angle phi, you could have we could have written this in terms of a cos and a sin in which case there would be an a 1 and an a 2, the two forms are exactly equivalent.

Now, you can see that why is there a secular term in my regular perturbative expansion. This is a linear pendulum, but by this is a linear oscillator, but by adding a dissipative term, the exact solution is telling me that this is going to; this is going to have an oscillatory part, which oscillates at a certain frequency and a part which causes the amplitude to decay with time.

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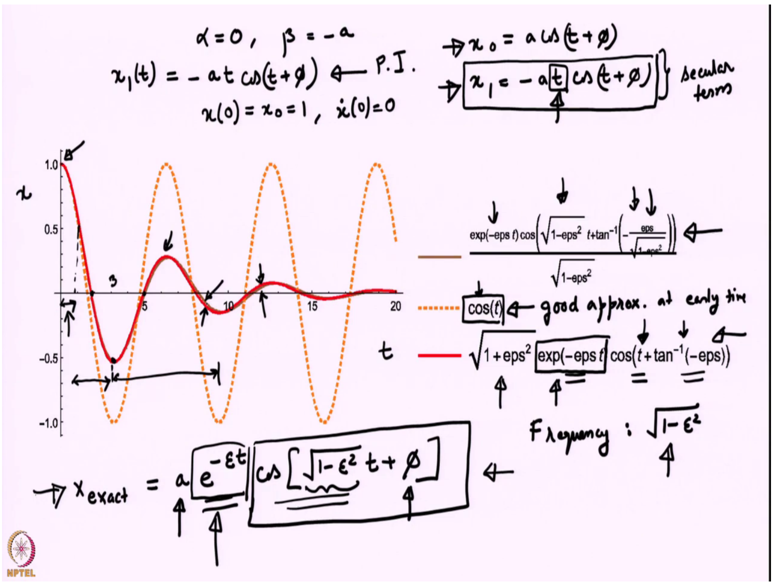

If you look at the oscillatory part which is this cosine function, you can see that the frequency of the oscillatory part; the frequency of the oscillatory part which is 1 minus epsilon square depends on epsilon, it is not independent of epsilon. So, as you change epsilon the frequency will keep changing.

Or in other words this is not exactly a time periodic system because the amplitude does not return back to it is value, but it is an oscillatory system. And, the 0 crossings, actually the gap between the 0 crossings actually depend on epsilon.

This is the same feature that we had encountered earlier in the non-linear pendulum also; there it was an exactly periodic solution. So, the amplitude would return if it started from 1, it would again return to 1. Here, because of dissipation if you start at 1. So, we start at 1, we do not return to 1 after one oscillation.

However, we one can still define a time period here, the gap between things so, the gap between here and here. And, so, this is the frequency associated with the oscillatory behavior and the frequency depends on epsilon.

Whenever the frequency depends on epsilon as I told you before you have a function whose, you have an oscillatory function whose frequency depends on epsilon, then the regular perturbation will typically lead to secular terms. And, you can get rid of the secular terms if you have a way of summing all these secular terms up to infinity.

Now, the regular perturbation cannot do better than this. So, if you take a two term expansion using the regular perturbation, then provided t is fixed you can make it a good approximation by taking your small parameter to 0, by making a small parameter smaller and smaller. However, given a fixed number of terms in the expression, we cannot take time to be bigger and bigger. So, this is a plot of x versus t ok.

So, the two term expansion, which would be a sum of x 0 and x 1 would clearly show growth in time, because of this part. And, this is a very unphysical behavior, because our system does not admit any growth, it does not there is no energy source. All the energy was injected into the system at time t equal to 0, either through an initial displacement or initial velocity or both.

Once that initial energy packet has been injected, the system can only lose energy through dissipation ok. So, we expect an oscillatory behavior, but the amplitude of oscillation is going to come down and it is going to come down exponentially as our exact solution tells us. This is in the under damped limit.

So, clearly the two term expansion produces an unphysical behavior and we need to do better than this. Can we do an improvement just like we had improved the non-linear pendulum solution? So, for that we will introduce a new technique that technique is called the method of multiple scales. It is more general than the Lindstedt-Poincare technique.

Now, here the in order to understand the basic idea of the technique, we I have plotted here the exact solution to this problem, which is basically just this. Then I have plotted the linearized solution to this problem which is just this. And, I plotted another approximation wherein I have an exponential damp part, but I do not have the frequency correction in the cosine part.

As, I told you the exact expression has an exponential damping and it is frequency of the cosine part also depends on epsilon. So, in this third approximation that I am indicating in red color, there is an exponential part, but the frequency of the cosine part is still the same as that of the linear part. There is that square root 1 minus epsilon square is not there.

Now, the reason why I am plotting this 3 is to help you understand, that we can think of the exact expression, we can approximate the exact expression which is this. So, in here it is plotted in brown, as having different parts to it as time progresses.

So, at very early times so, say, let us say we look at only within this time window. We can clearly see that the linearized approximation is good enough, I do not need to worry about damping. I do not need to worry about the fact that my oscillatory part, has a frequency which is dependent on epsilon. This part does not have any damping, the cos t does not have any damping, it also does not have any frequency correction beside t. So, the frequency is just 1.

If, we go to larger times so, let us say we went up to this time. So, up to here we were. So, this is a good approximation so, a good approximation at early time. Now, if I go further in time. So, let us say I go up to here. You can clearly see that this is no longer a good approximation.

The orange line is showing an amplitude which grows up to minus 1 whereas, the exact answer in brown has decayed. So, by the time you reach time which is about here about 3, you can see that you can start seeing an exponential decay. So, at very early times it is just cos t, at slightly longer times you start seeing the exponential decay.

If you go to even longer times, then you will see that this brown curve and this red curve show slight deviation from each other. Now, what is the difference between this brown curve and the red curve? As, I told you earlier the brown curve and the red curve are exactly the same, they both have the exponential decay factor, but the brown curve which is the exact solution, has the frequency correction inside cosine. The red curve does not have the frequency correction, it is cos of just t.

Do not worry about this extra factors that you are seeing here tan inverse of minus eps and there is something else here, all that has been put in so that all the three curves satisfy the same initial conditions. In this case I have chosen the initial condition to be x of 0 is equal to sum x naught, which has been chosen to be 1 and then x dot of 0 is 0.

Because the 3 formulas are different, so their constants of integration in general turn out to be different. But, if we adjust the constant of integration such that all the 3 formula satisfy the same initial conditions, then those will turn out to be the constant of integration.

So, in this case this will be the one constant and that will be the other constant. In this case this will be the one constant and the amplitude is just 1. So, you can clearly see that at early times cos t is a good approximation, at even longer times you start seeing the fact that there is an exponential decay. So, the decay of amplitude starts becoming apparent.

If, you go to even longer times on this time scale, you can see that the brown curve and the red curve show a slight difference. That difference can only be because the brown curve and the red curve have a cosine, which are at slightly different frequencies. One is at cos of square root 1 minus epsilon square and the other is at cos of t. So, the frequency difference also starts showing up ok.

So, what I am trying to say is that this problem can be broken up into pieces. So, at early times we only see an undamped oscillator, which is oscillating with unit frequency. At slightly longer times we see the decaying amplitude of oscillation of the oscillator. At even longer times we start feeling the fact that, the frequency of oscillation of the oscillator is slightly different from the exact problem.

So, this suggests that the problem can be broken up into parts, by treating each physical process. So, at early times it is just an undamped oscillator. So, that is an approximation; that is the first level of approximation. At slightly longer times you will feel the effect of damping. So, it is not just an oscillator, it is a damped oscillator. Then at even longer times it is a damped oscillator which is oscillating at a frequency, which has a epsilon correction.

So, that is the basic idea of the method of multiple scales. On different time scales, different processes will different physical features of the problem will show up. And, the basic idea is that you can treat each of those time scales as being independent of each other. So, let us solve this problem using the method of multiple scales.

We will separate time scales, we will introduce multiple times in the problem; this will convert our ordinary differential equation into an partial differential equation. But that will not complicate the mathematics, it is still an it the resultant partial differential equations can still be solved analytically. So, let us understand the basic idea behind the method.

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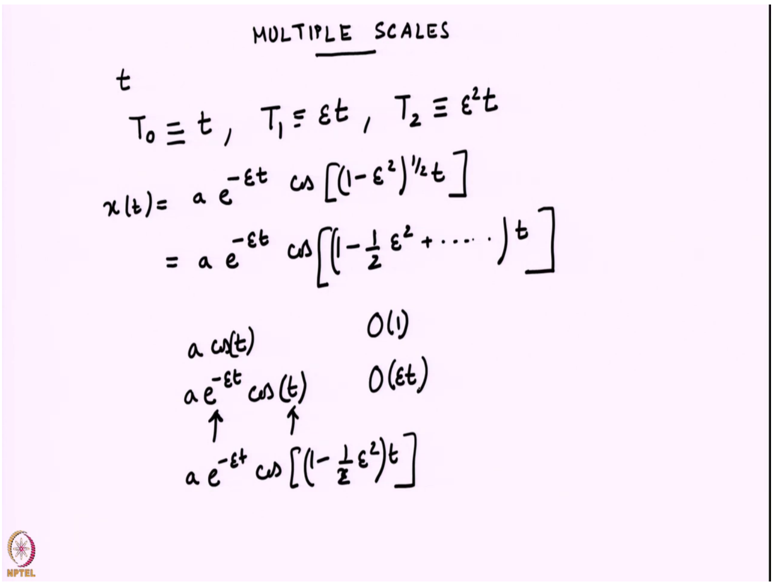

So, the basic idea behind this method, so, multiple scales is to correct the behavior of the regular perturbation technique, the regular perturbation technique had secular terms. And, so, what we will do is up to now our independent variable in the problem is time. Now, we will say that there are multiple time scales in the problem.

So, one time scale is t, another time scale so, these are all definitions epsilon T, another time scale is epsilon square t. Where do we get the these time scales from? Let us look at the exact problem.

So, the exact solution we have seen is a e to the power minus epsilon t cos 1 minus epsilon square to the power half into t. If, I keep the e to the power epsilon t intact and if I just express this using an expansion.

Then you can see, that at very early times this is just a cos t roughly. At slightly later times this is approximately a minus epsilon t into cos t. So, the frequency still remains unity, but we start seeing damping. At even later times, we start seeing damping and we also start seeing the fact that, the oscillations are happening at a slightly different of frequency. This effect is always there you know, but it starts it takes some time to show up ok.

So, that is the basic idea that at order 1 time it is just a damp undamped oscillator at order epsilon t. So, when time becomes slightly larger. So, then we start seeing a damped oscillator and at a even larger time, we start seeing the effect which is due to the fact that this is oscillating at a slightly different frequency so, this effect ok. So, this is where those time scales are coming from.

You can see that we will call these T 0, T 1, T 2 as so, this is early time.

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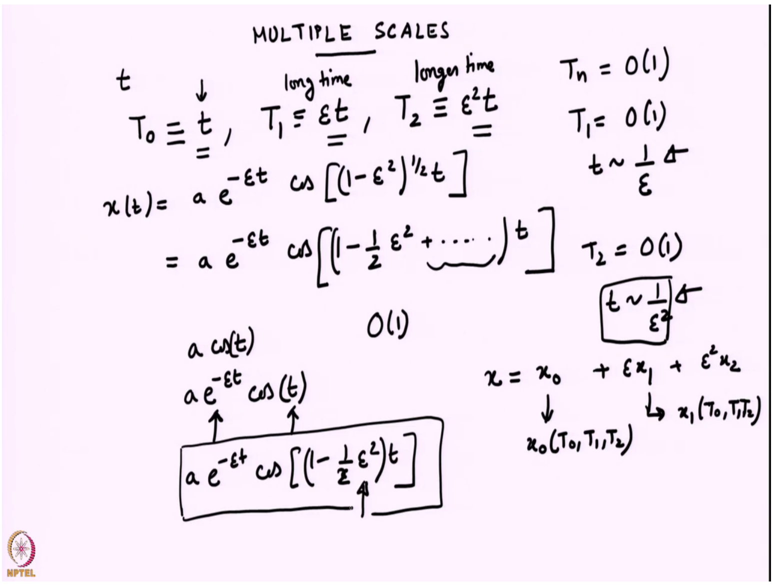

So, when t becomes so, all the T ns will be order 1, what does that mean? That means, that when T 0 is order 1 we are at early time, when T 1 is at order 1; that means, that small t is very large only then can so, when t 1 is order 1, then it implies t must be of the size 1 by epsilon since epsilon is a small number. So, 1 by epsilon is a large number ok. So, this is at long time.

When T 2 is of the order 1, then time is of the order 1 by epsilon square. This is even bigger than this. If epsilon is a small number then 1 by epsilon is a big number, but 1 by epsilon square is an even bigger number. So, this is an even longer time. So, this is a longer time.

So, we will find. So, what we will do is we will convert this problem, from small t to a problem where the independent variables are derivatives with respect to or were the independent variables are T 0, T 1 and T 2. And, we will have derivatives with respect to T 0, T 1 and T 2.

We will our dependent variable x will now be expressible as x 0 plus epsilon x 1 plus epsilon square x 2, but each of these quantities themselves will become functions of T 0, T 1 and T 2.

We can keep going further, we need not stop here, but we are going to do this process only up to order epsilon square, as you can see this part is not 0. So, there will be higher and higher corrections to the frequency and the method of multiple scales is going to precisely give me that approximation, if I solve it properly.

So, we are going to learn how to do this using the method of multiple scales, for this particular problem. And, we will see that the method of multiple scales will correct the behavior of the regular perturbation approach.

In particular it will give us a systematic way of keeping or eliminating secular terms up to any given order. So if you; if we say that we do not want secular terms up to order T 2, then it will give us a way of getting an expression, which has no secular terms up to order T 2.

This is a very important technique the method of multiple scales and is frequently used in the analysis of interfacial waves. Later when we do interfacial waves we will look at examples.