Introduction to interfacial waves
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Lecture - 17
Perturbative solution to the non-linear pendulum

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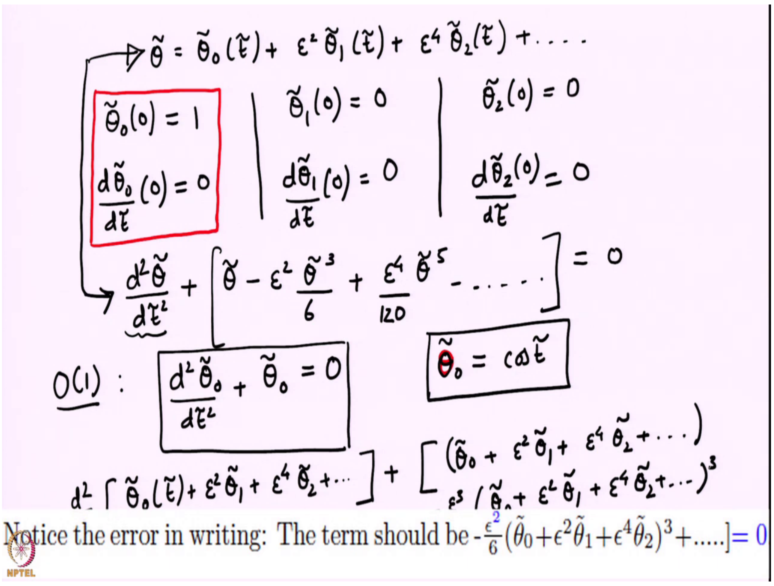

We were solving the non-linear pendulum using a regular perturbation expansion. We had seen that we have non-dimensionalized the equation and then we had hypothesized that we would have a expansion like that; note that the expansion starts at epsilon square and not epsilon.

Now, at order 1 we had this problem. So, at order 1 we had found that the equation is governed by a linear equation and we had solved that to obtain this a linear homogeneous equation. Let us obtain the higher order non-linear corrections. So, for that it is useful to substitute this series. So, this term is really simple.

So, we have d square by dt square of plus theta tilde is just this expansion minus epsilon cube by 6 whole cube. And, I am not writing the epsilon 4 term because we are not going to go up to epsilon 4. We are we just want to find out the first correction and you will see that there is a very interesting feature which will come out.

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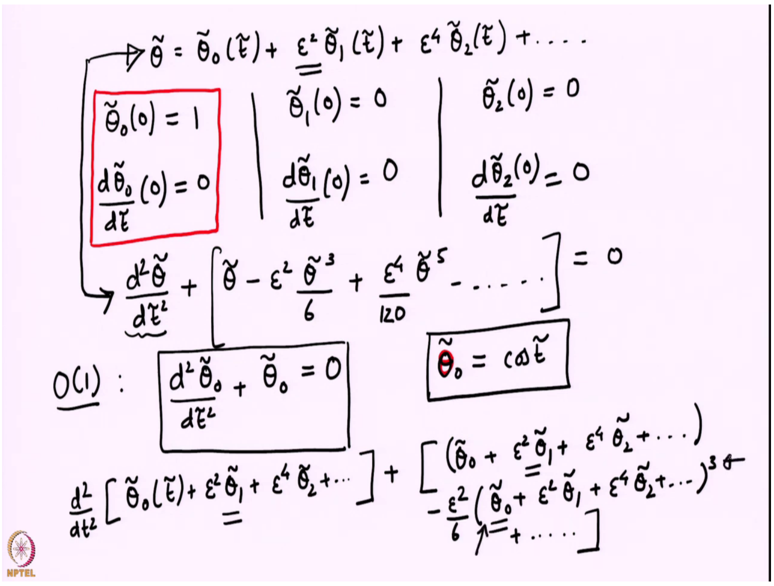

So, I will just put a dot dot dot here. Now, if I have to gather terms which are of the order epsilon square after order 1; order 1 is like epsilon to the power 0, then after order 1 we have epsilon square because that is our first correction. See if I have to gather epsilon order epsilon square terms.

Then you can see that this term only contributes through d square theta 1 by dt square as is expected. On the right hand side you will have this term which is order epsilon square and then you will also have and then on the on this term you with also be a term which will be the cube of this term with a coefficient which is order epsilon square.

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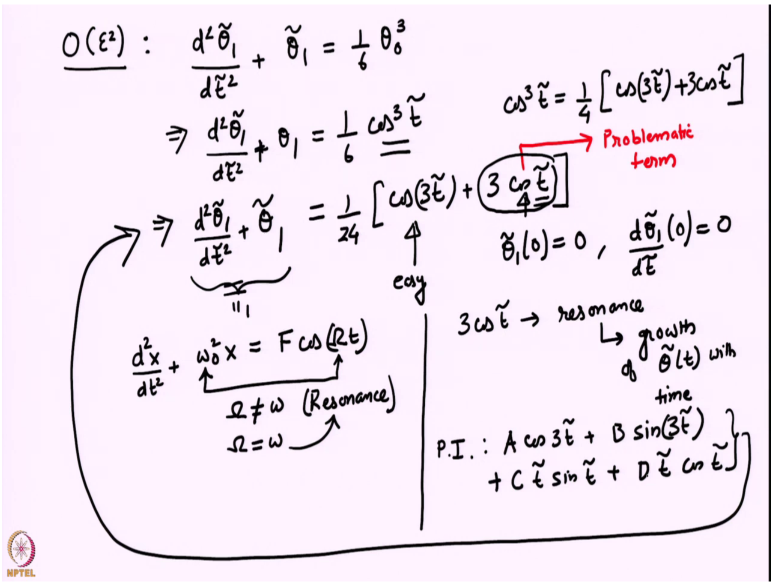

So, if we just collect all the order epsilon square terms and write down the equation, it turns out to be order epsilon square it turns out to be; so, the left hand side again remains the same like before except that now it is for theta 1 and not theta naught and the right hand side is just 1 by 6 theta naught cube. I am not writing the epsilon square on both sides. So, you can see that.

So, we had a theta naught cube because there is a cube here, alright. So, now, how do I have does one solve this equation? This is easy to solve because we already know what is theta naught, theta naught was just cos cube t or cos cube t tilde. So, this is an ordinary differential equation once again it is linear, note that it is linear the effect of non-linear this is this equation will correct this solution at this order theta 1 will actually bring in a non-linear correction to the linear solution.

But, notice that the effect of non-linearity is being felt through this term which is an inhomogeneous term on the right hand side. So, as you all know this can be written as a sum of complementary function plus particular integral. In order to calculate the particular integral it is useful to convert this using the cos 3 theta formula. We know that we write it here cos cube t tilde is equal to 1 by 4 cos 3t tilde plus cos 3 cos t tilde.

This is just from the formula of cos 3 theta is equal to 4 cos cube theta minus 3 cos theta. So, if I replace it here, then this becomes 1 by 24 of cos 3 tilde plus 3 cos t tilde. So, that is our equation that we will have to solve at order epsilon square in order to determine the first non-linear correction which is theta 1 tilde.

Say may like before we have initial conditions which basically says the theta 1 tilde of 0 is 0 d theta 1 tilde by d t tilde of 0 is again 0. Now, before we solve this we all know how to solve this we have to basically take a guess for particular integrals which are proportional to cos 3t. Now, this term is easy to compute as far as particular integrals is concerned.

However, this is a problematic term this is a problematic term. Why is that a problematic term? You can see that the frequency of the term. This is this you can see that this is like the equation of a forced harmonic oscillator. The left hand side is as if it represents a linear oscillator and the right hand side is like there is a forcing on it, ok. So, this is an equation of this form plus omega 0 square X is equal to some F cos capital omega t.

Now, we can solve this equation using the same technique, complementary function plus particular integral. Now, the particular integral is fine as long as capital omega and small omega are not equal to each other; small omega naught represents the natural frequency of the oscillator, capital omega represents the forcing frequency. You are forcing the oscillator at a harmonically at a certain frequency. So, capital omega represents that frequency.

When capital omega is not equal to small omega then we can guess the particular integral. However, when capital omega is equal to small omega we start getting resonance and that when this condition is fulfilled we start getting resonance and then the form resonance implies growth because it is feeding energy to the oscillator at the same frequency and so, you will get a t sin t t cos t kind of terms in the particular integral.

Now, why is that a problem? As you can see that a term like t sin t will oscillate, but it is amplitude will grow linearly or quadratically with t; if it is t sin t it will to grow linearly with t with t square sin t then it will go quadratically. Now, recall that this is the solution to a simple pendulum.

It is a non-linear thing, but we know that no matter at what angle we leave the pendulum. If we leave it with 0 velocity at any angle let us say we leave it at 90 degrees. 90 degree we will soon see is a large angle for linear theory to be good. But, whatever it is if we leave it at 90 degree the pendulum can only go on the other side at 90 degrees if it is a conservative system and keep oscillating like this forever. This is what our mathematical model will predict. Now, we have taken that mathematical model and we have tried to solve it by a regular perturbation method and we are finding that the solution may possibly contain terms like t sin t or t cos t.

The reason is because this term has a frequency which is 1, the capital omega here is 1 and the natural frequency of this part is also 1. So, this term is going to act as a resonantly forcing term on the oscillator and so, the solution to this equation will have terms like t sin t t cos t. Now, apriori even before writing the solution we anticipate that that is going to be a problem because that is going to cause the energy of my pendulum to grow with time.

We know that this is an energy conserving system and so, I do not expect my energy to grow with time as time becomes larger and larger. Let us see we this is a very important thing that we will see and we will have we will soon argue that there is a problem with this way of solving the equation perturbatively. So, we will have to do something special about this equation and we will discuss that shortly, but let us for the moment write down the solution to this problem.

So, we have seen that the 3 cos t tilde term will cause resonance and this in turn will cause growth of the amplitude of growth of theta with time and this growth is an unbounded kind of a growth. So, it will keep growing forever ok. And, this filets our basic intuition that this is an energy conserving system.

So, now, with that in mind let us formally write down the solution to this system and see what does the structure look like. So, the we have to basically guess a lot of particular integrals. So, you can so, for particular integral we can take the general thing as a linear combination of because we have A cos 3 t on the right hand side the first term which is the easy term we will take a linear combination of A cos 3 t tilde plus B sin 3 t tilde are the same frequency a sin and a cos.

And, then because we have a cos t tilde term which is the; which has the same frequency as the natural frequency of the system on the left, the particular integral for will be of the form t cos t or t sin t. So, I will write C t tilde sin t tilde plus D t tilde cos t tilde. Now, our task is to substitute this into the equation and because cos 3t tilde sin 3t tilde sin 3t tilde and t sin t tilde and t cos t tilde are all independent terms will have to equate the coefficients of each term to 0.

If we substitute it carry out the algebra, so, we are substituting this form into my governing equation for theta 1 t tilde. One just needs to do a little bit of algebra one has to take this expression, this is this expression with four terms A, B, C, D differentiate it twice substitute it and collect all coefficients of collect all terms with cos 3t tilde all terms with sin 3t tilde all terms with t sin t tilde and all terms with t cos t tilde. If you do that you will get the following equation.

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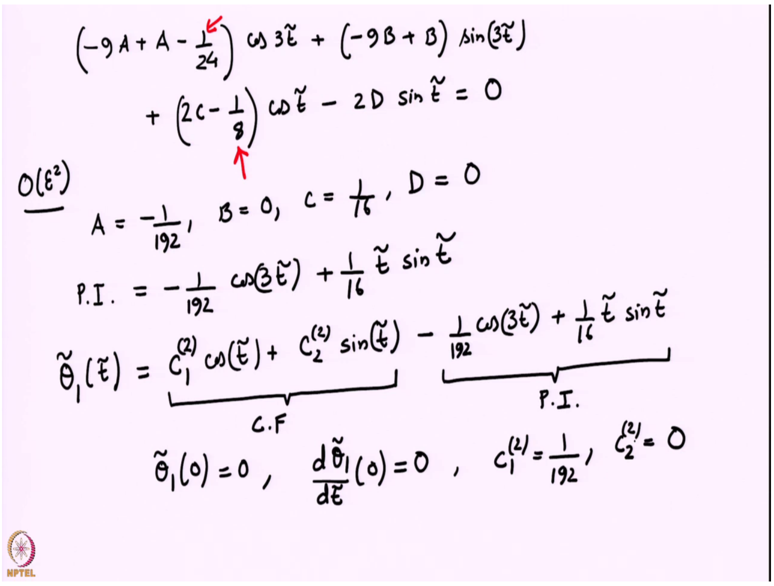

So, you will get. So, I am shifting everything to the left hand side, notice that I have also shifted these terms to the left hand side the entire in homogeneous terms the entire in homogeneous term also I have shifted to the left hand side ok. So, that is how I get that 1 by 24 and 3 by 24. So, 3 by 24 is 8 this term and this term from the right hand side.

So, now, as I said before you have to equate the coefficients of each of these terms to 0 and that will give you the values of A, B, C and D. So, you can readily see that A is equal to minus 1 by 192, B is equal to 0, C is equal to 1 by 16 and D is equal to 0. So, your particular integral has the form that it is minus 192 cos 3t tilde plus 1 by 16 t tilde sin t tilde.

As expected we still have a problematic term. The t cos t tilde term went away, but the t sin t tilde term is still there and it is the second part of the particular integral and this is going to cause some trouble for us. So, let us go further. So, this is only the particular integral. So, if I want to write down the most general solution for theta 1 of t tilde I have to add the complementary function also. The complementary function is just a linear combination of cos t tilde and sin t tilde.

So, the most general solution we will look like so, I am again following the same pattern. This is this calculation is at order epsilon square. So, there will be a subscript 2 for every constant of integration. So, this is my complementary function. So, this is. Now, we have to satisfy the initial conditions. The initial conditions in this case are very simple initial conditions we have theta 1 of 0 is 0 and d theta 1 of 0 is also 0.

If you will do that then you will find that C 1 is equal to 0 we were solving for the non-linear pendulum using a regular perturbation method and we had found the general solution up to some order and then we had fitted the constants of integration using the initial conditions that was provided. Using this we had found that C 1 of 2 is 192 and C 2 of 2 is 0. So, with this information let us now write the solution of the non-linear pendulum up to order epsilon square.

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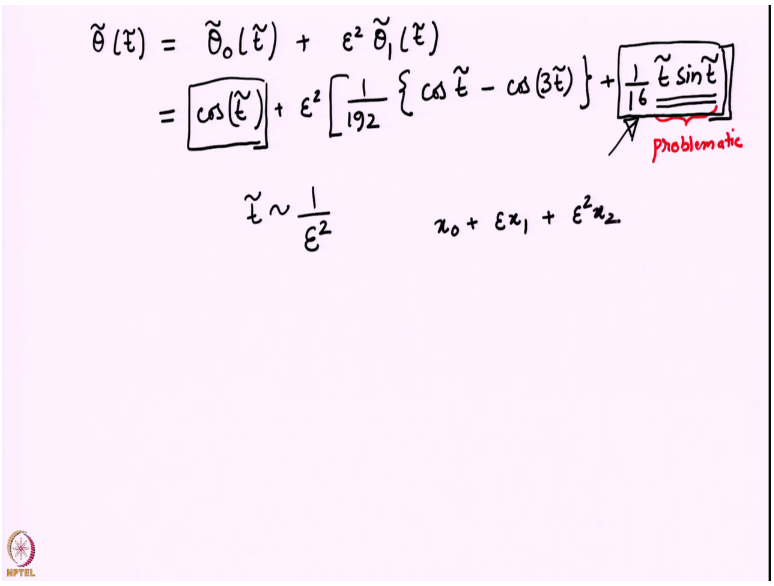

So, we have seen that theta tilde of t tilde is theta 0 plus epsilon square theta 1. Theta 0 we have found already this cos t tilde and theta 1 we have just determined to be 1 by 192. So, this is the solution up to order epsilon square. So, this is a the second term is a non-linear correction. As I remarked before you can immediately see that all the other terms are oscillatory and remain bounded, but this term is a problematic term.

In particular, you can see that if I truncate the expansion up to order epsilon square and then I take this expression and then I make t large and larger. So, if I look at larger and larger values of time you can clearly see that this term is going to cause a growth. In particular, you will see that when the time is approximately 1 by epsilon square; epsilon is a small number. So, 1 by epsilon square is a large number. So, this is a large time.

So, when time reaches of the order 1 by epsilon square then this term this t into epsilon square becomes an order of magnitude one number. So, consequently this term becomes as large as the first term. This is a problem. The reason is because in any perturbative solution when we write a solution like this and so on. In this case this is not there.

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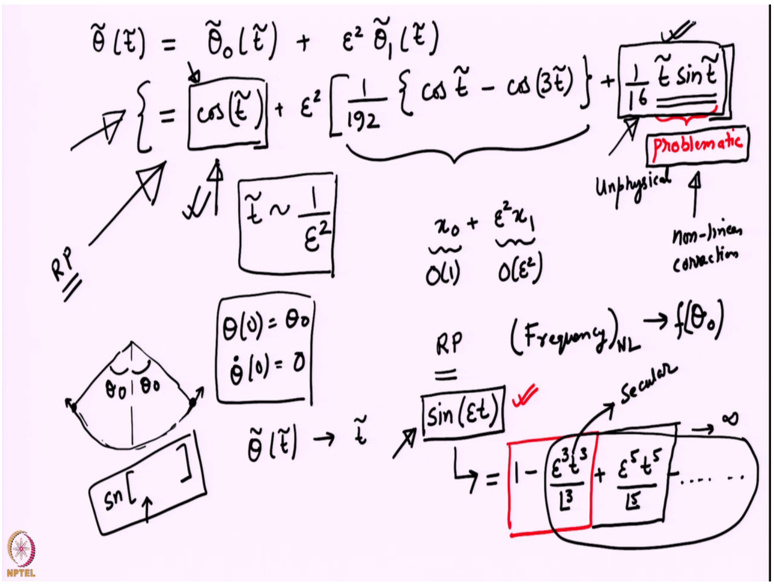

So, I will straightaway write square x 1 let us say, then it is inherently assumed that this term is always smaller than this term. This term if it is order 1, then this term is order epsilon square and this should be true at all times. In this example that we are seeing this is not true at all times. If I take my time to be a finite number, then I can arrange epsilon to be sufficiently small so that the second term always remains small compare to the first term.

However, if I allow time to become large and very large then at some time the second term is going to become as large as the first term and so, the ordering of the perturbation series which says that every successive term should be smaller, then all the terms on it is left gets disturbed. So, this is clearly a problem.

In addition, you can also see that this is a very unphysical kind of a solution. Recall that we are looking at a pendulum whether we are under the linear approximation or under the non-linear approximation, the pendulum and if we give it an initial condition where we are looking at the oscillatory solutions then the pendulum just oscillates.

So, if we leave it at an angle theta naught independent of whether theta naught is small or large the pendulum can only go on the other side up to the same theta naught. This is in particular for theta of 0 is equal to theta naught and theta dot of 0 is equal to 0. We are solving this under this initial condition. So, for you in this initial condition we know that if we leave the pendulum at theta naught, it will go up and climb all the way to the other theta naught.

Look at what this pendulum is doing or other what this approximate solution is doing. This is an oscillatory solution. This entire term is also an oscillatory solution, but the last term which I have written in red here is a problematic term because it will cause my angle theta to slowly increase. So, this is a pendulum which actually will climb to bigger and bigger angles on either sides.

You can clearly see that this is an unphysical solution and we have to understand why does the regular perturbation. So, this came out from a regular perturbation. So, for me why does the regular perturbation cause this problem? We will understand that, but let us look at a plot of this theta tilde of t tilde as a function of t tilde.

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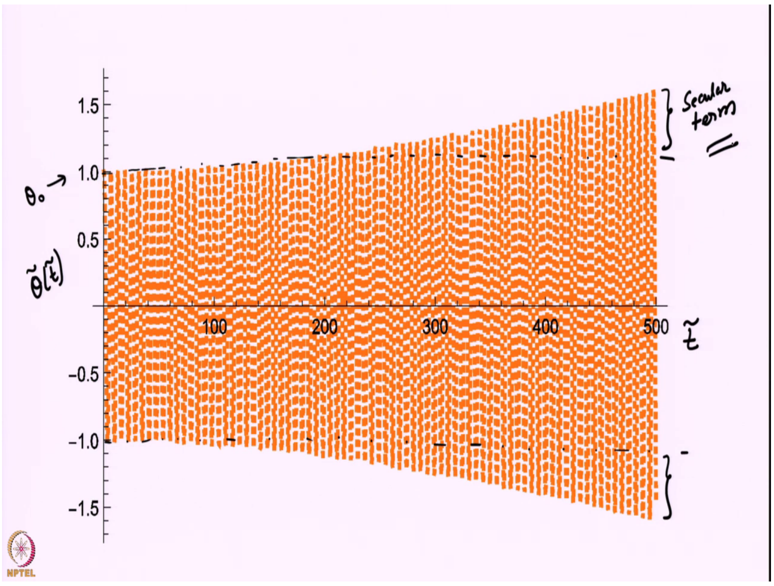

So, this is I have plotted many oscillations. So, this is theta tilde of t tilde the full expression that I have provided as a function of t tilde. You cannot see the individual oscillations because there are many of these and I have taken a rather long time window in order to show that the amplitude of motion is not bounded at 1 or minus 1 recall that this is a scaled angle. So, this in unscaled variables this would actually be the initial angle theta naught. So, this pendulum is actually going beyond theta naught on either sides.

So, this part this growth and this growth is being caused by the secular term. So, clearly as we can expect this is not an acceptable solution at large times. So, can we do something to improve this and in particular can we understand why is the regular perturbation producing this. So, again let us go back to a regular perturbation exercise and ask ourselves what is causing this trouble.

So, here the most important thing to remember is the fact that this is a non-linear oscillator. We are this problematic term comes only in the non-linear correction. There was no problem in the linear part of the perturbative solution, this is the linear part ok. So, the problematic term came in the non-linear correction you can carry this non-linear correction to even higher orders and you will see that you will keep getting more and more worse such secular terms. You will get t square sin t, t square cos t and so on.

So, it will keep getting growing worse and worse and so, we need to understand what why is this happening. So, recall that I have told you before that the non-linear pendulum the time period of the non-linear pendulum or the frequency of the non-linear pendulum. So, the frequency of the non-linear pendulum depends on theta naught is a function of theta naught.

In fact, we had found that it is a complicated function of theta naught given by an elliptic integral. We had calculated an approximation to that elliptic integral and we had found a correction the first non-linear correction which was found earlier. Now, you can see that in order to understand this that why this is happening, you can see I will take a simple example.

So, suppose you have a periodic function sine and the frequency of the periodic function depends on the small parameter epsilon. So, I will write sin epsilon t. Now, this for epsilon going to 0 or any finite epsilon is a perfectly well behaved function. If you plot it for a given epsilon then this will just oscillate with a certain frequency which is related to epsilon and it will always stay between plus 1 and minus 1.

Let us now represent this thing by a series expansion for small epsilon. So, we will use the Taylor series approximation and that gives me 1 minus epsilon cube t cube by factorial 3 plus epsilon 5 t 5 by factorial 5 minus dot dot dot so on. Now, suppose we obtain a so, this is a perturbative approximation. So, I replace this original function with its perturbative approximation. So, let us say I say that sin epsilon t is 1 minus epsilon cube t cube by 3.

Now, you can clearly see that this term is like a secular term and this is going to cause growth. This is going to cause growth and it is going to grow unbounded in time. This behavior will not be rectified if I include more terms. I will keep getting at every order these kind of secular terms. The interesting part is the sum of all such. So, you can clearly see if you could sum this series all the way up to infinity, then it would produce a function.

So, all of these each of which diverge in time would produce a function which is this which does not diverge in time. So, truncation at any finite order is going to cause the problem, ok. So, this should tell you what is happening. Our non-linear pendulum has a complicated function that is related to an elliptic integral and that function is a periodic function. It is not a sin or a cosine it is related to the elliptic Sn or elliptic Cn and that function has its argument epsilon or some function of epsilon rather. So, let me write it like this.

So, it is Sn into some function of epsilon into time and so, the formula that the regular perturbation is producing is basically nothing, but an expansion of that complicated function, a series expansion of that complicated function about epsilon equal to 0. I encourage you to go back and try expanding the exact solution that we had found for the non-linear pendulum subject to the same initial conditions using the elliptic Sn and put it in a package like Mathematica or Matlab and, Matlab or Mathematica will tell you what is the series expansion.

You will find that the series expansion of Sn, the exact solution about epsilon equal to 0 looks very similar to this that series expansion also contains these kind of secular terms at every order. So, the point is this that you have a non-linear you have in a non-linear oscillator there is a frequency term which depends on epsilon. If you expand that in a series, then you are going to get secular terms.

Those secular terms are going to add up if you can add all the secular terms of all the way up to infinity, then they are going to add up and produce a function which will not diverge, but if you chop the expansion anywhere then you will be left with a finite number of such secular terms and the some will typically always diverge. So, this is the basic logic behind this.

Now, the question arises is there a way to correct this behavior? Yes, answer is yes, there is a way to correct this behavior and that will lead us to the Lindstedt – Poincare method. In the next video, I will show you that how we can modify this expansion procedure and by allowing the frequency to be explicitly dependent on the perturbation parameter epsilon we will be able to bypass this the secular terms and produce a finite number of terms none of which diverges at large time. We will do this in the next video.