Introduction to interfacial waves

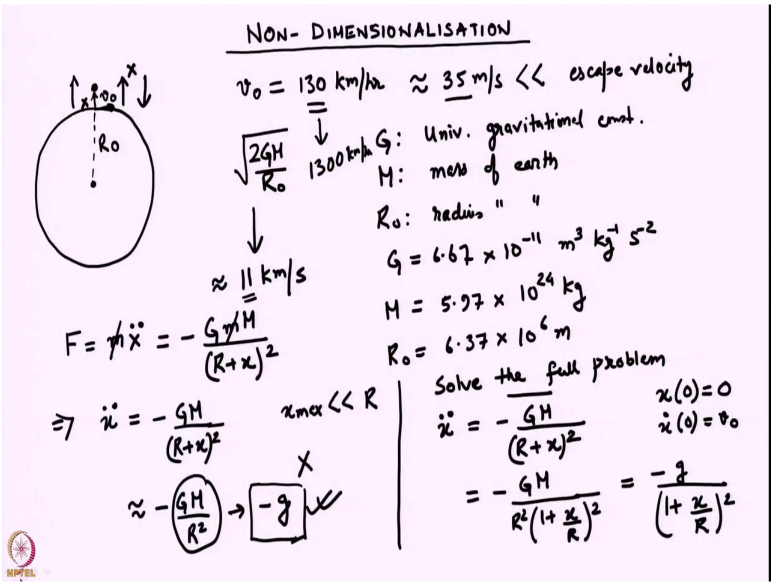
Prof. Ratul Dasgupta
Department of Chemical Engineering
Indian Institute of Technology, Bombay

Lecture - 16
Perturbative solution to the projectile equation

We had began our discussion on one aspect of perturbation method which was non dimensionalisation. For understanding this, we had taken a simple problem of a projectile being thrown at a speed v 0 from the surface of the earth and writing down the exact equation which govern the motion of the mass.

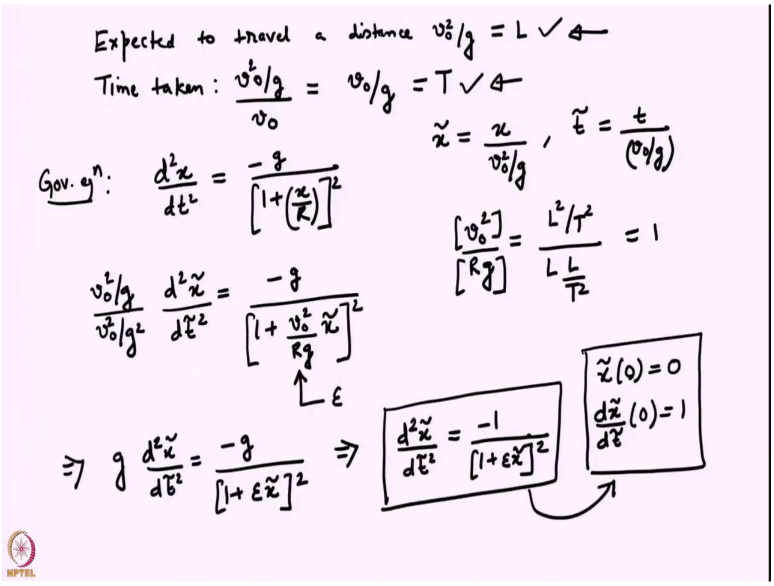
In particular, we had said that we are not going to make the assumption that the force is constant, but we also realized that if v 0 is much much smaller than the escape velocity, the v 0 the speed with which the projectile is launched is much smaller than the escape velocity, then assuming the force to be constant is a good approximation.

(Refer Slide Time: 00:53)

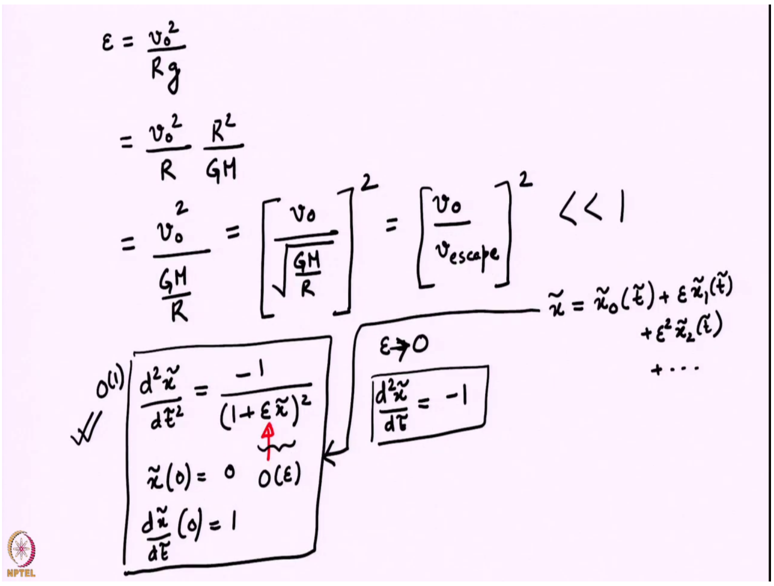


Now, with that in mind we had written down the equation the complete equation of motion.

(Refer Slide Time: 00:56)



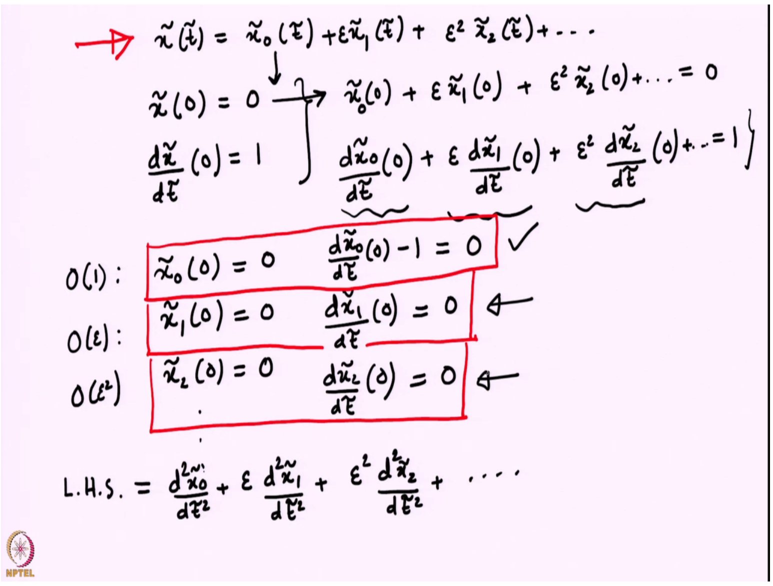
(Refer Slide Time: 00:59)



And then we had chosen appropriate scales for non-dimensionalization. This in turn introduced a small parameter indicated by the red arrow epsilon. Epsilon, we had seen earlier was the ratio of the launch speed to the escape speed the square of that. Now, we have this intuition that if the launch speed is much much smaller than the escape speed, then the square of that is going to be a very small number compared to 1. And so as a first approximation it is justified to assume that the acceleration is just a constant.

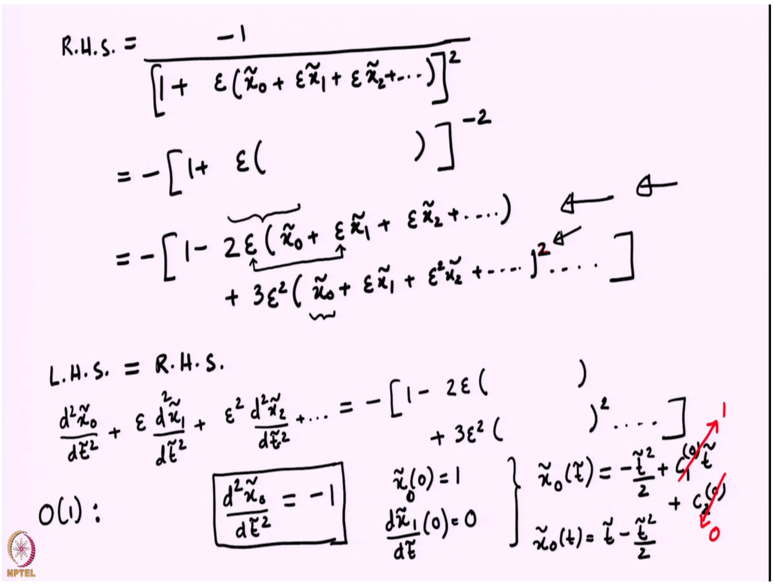
Now, in order to solve this problem perturbatively, we had assumed an expansion in x naught in the non-dimensionalize variable x which represented the vertical displacement of the mass at any instant of time. And then we expanded it in a power series in epsilon.

(Refer Slide Time: 01:45)



We then took the initial conditions and then we wrote them at every order. We found that the effect of initial conditions is only failed at the lowest order, all the other initial conditions essentially go to 0.

(Refer Slide Time: 02:01)

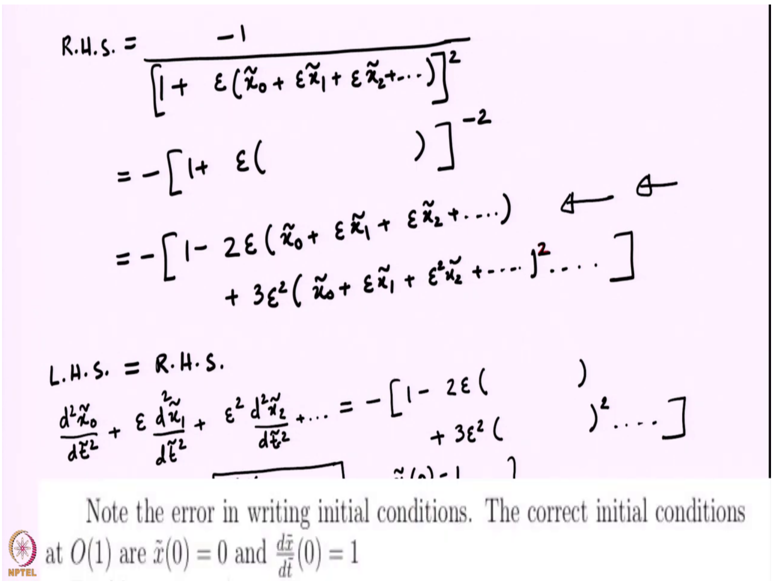


Now, our task is to solve this equation by equating the left hand side to the right hand side. So, we have already simplified the right hand side. There should be a square here. So, on the left hand side, we have seen the left hand side is very simple that is our left hand side plus dot dot dot is equal to this right hand side which we have written here. So, let us write this into whatever is written in the bracket inside into the square of what.

So, now, let us like before let us equate let us collect terms at every order. So, at order 1, we have is equal to you can see from this expression that the only quantity on the right hand side which is of order 1 is just minus 1, everything else has a coefficient which is epsilon dependent. So, this is our equation at the lowest order.

This is just a non dimensional way of saying that at the lowest order when epsilon is sufficiently small at the lowest order it is justified to assume that the force is a constant or the acceleration is just minus G. This is just a non dimensional statement of that. We have to solve this with the initial condition is 0. Now, you can see this equation is readily integrable.

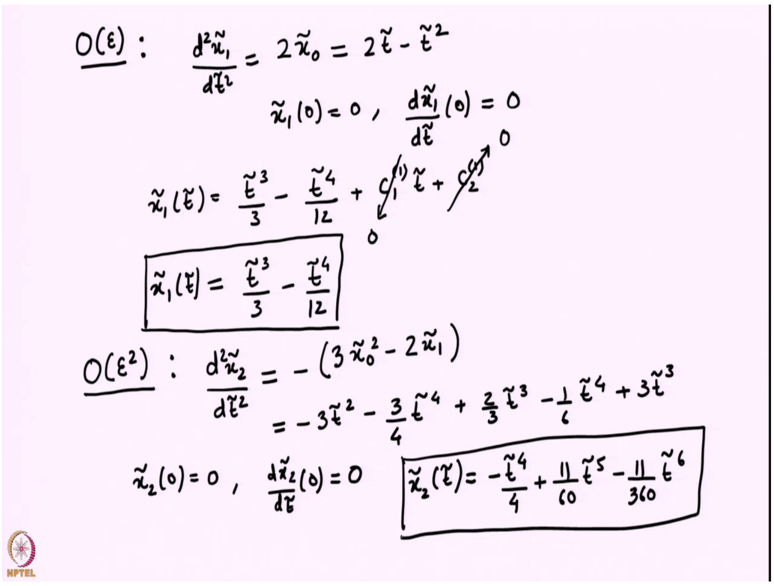
(Refer Slide Time: 03:56)



So, you can just integrate it. And if you integrate it, you have to integrate it twice. So, I am going to follow a pattern here. So, if you integrate a tire twice, you will find two constants of integration. So, I will follow up pattern in writing down the constants. So, because this is the 0th order constant. So, the constants will be of, so let me write down the expression and then I will explain.

So, there are two constants. So, C 1 and C 2, and the superscript at the top indicates that these are constants at the 0th order. So, now, if you replace this, you will basically find that C 1 0 is 1, and C 2 0 is 0. So, this leads us to the 0th order solution that x 0 tilde of t is equal to just t tilde minus t tilde square by 2, that is a simple conclusion alright.

(Refer Slide Time: 05:17)



So, now let us go to the next order. At the next order, we have to take into account order epsilon terms. So, at the next order, the left hand side is simple. If I look at the right hand side, then you can see that the right hand side we are looking for terms which have epsilon to the power 1. And you can see that the only term which will have epsilon to the power 1 is this term, the product of 2 into epsilon into x naught. So, I am not writing the epsilon. So, it is just 2 into x naught tilde.

You will see a pattern here that the left hand side remains the same at every order. It is just at order epsilon you will have d square by d t square of x 1; at order epsilon square, you will have d square by d t square of x 2, and so on. And it is the right hand side which becomes more and more complicated. Pay attention to the fact that at every order we are solving a linear equation still solving a linear, but in homogeneous equation because of the two x naught term.

So, let us write it. We know what is x naught tilde. So, this is just 2 t tilde minus t tilde square; x naught tilde we had written it earlier. This again is very easily integrated with the initial condition x 1 tilde of 0 is 0, and d x 1 tilde of d t tilde of 0 is again 0. We have derived these initial conditions earlier. So, we are now using this. We have already used this at the lowest order. Now, we are going to the next order which is order epsilon. And we are now using this. At the next order, we will be using that. And these are all simple initial conditions, everything is 0 alright.

So, now again this is easy to integrate. And if you integrate this, in general you can show that this can be written as again I am following the same pattern. There are two constants of integration C 1 and C 2, and the superscript indicates that this is an order epsilon constant. Now, if you substitute these initial conditions, you can easily see that both of them both of the constants of integration are 0. And so, you just recover x 1 tilde of t tilde is just this.

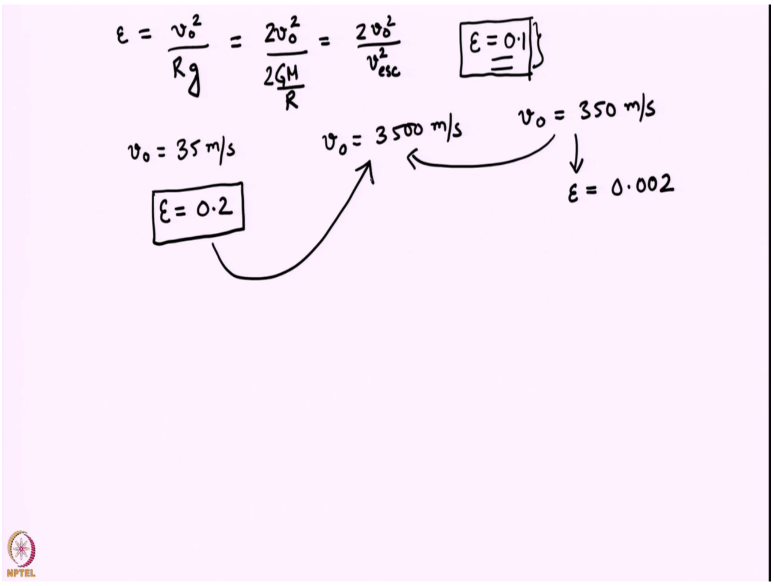
So, we have solved the problem at the lowest order in nonlinearity. Let us go one step further. The pattern is again the same. Now, we are going to order epsilon square. So, we have, so we have to look at this term. So, we are looking at order epsilon square. So, you can see that at order epsilon square, you will have a correction appearing from here. So, the product of these two will give you a order epsilon square term.

The square of this will also give you remember there is a square here. So, the x naught square will also have its coefficient and epsilon square. If you put those are the only contributions that order epsilon square and so you will have. And we already know x naught and x 1 tilde, so we can substitute.

And if you substitute this, then this just becomes after simplification I am just going to write down straightaway the expressions. So, this is non-linear effect at order epsilon square. And so you can simplify the right hand side a little bit more. If you do that and then if you carry on the integration, you will find like before we have to carry on the integration this integration is very easy to do. And once again you will have to substitute the initial conditions x 2 of 0 is equal to 0, dx 2 by dt of 0 is equal to 0.

Now, if you substitute if you do carry out this integration this the right hand side can be easily integrated. If you carry out this integration and then use the initial conditions to determine x 2 as a function of t tilde, it looks like this. So, this is x 1, this is x 2, and we had x 0 earlier in the previous slide. So, if you solve it, let us compare our numerical solution to this ordinary differential equation compared to the perturbative solution up to order epsilon square. Now, for that we have to choose a value of epsilon.

(Refer Slide Time: 11:07)



Now, we have seen earlier that epsilon was defined as v 0 square by R g or it was actually we had managed to write it as v 0 square divided by GM by R. Now, the escape velocity actually has a factor of 2 here. So, I would like to put a 2 here and a 2 there. If I do that, then this becomes 2 v 0 square by v escape square.

So, we had not written down this factor of 2 earlier when we were making estimates. This factor of 2 here will not make any difference because we were saying that epsilon is much, much less than 1. So, epsilon is much, much less than 1, then 2 times epsilon is also much, much less than 1 ok, so that was the basic argument.

But now because we are trying to estimate the value of epsilon by choosing a certain value of v naught, we need to put the factor of 2. So, that we get the exact value of epsilon if we launch the projectile at certain speeds. Now, I have told you before that v naught was, for a bowler it was around 35 meter per second or about 130 kilometers per hour. Now, this is a very very small speed compared to the escape velocity.

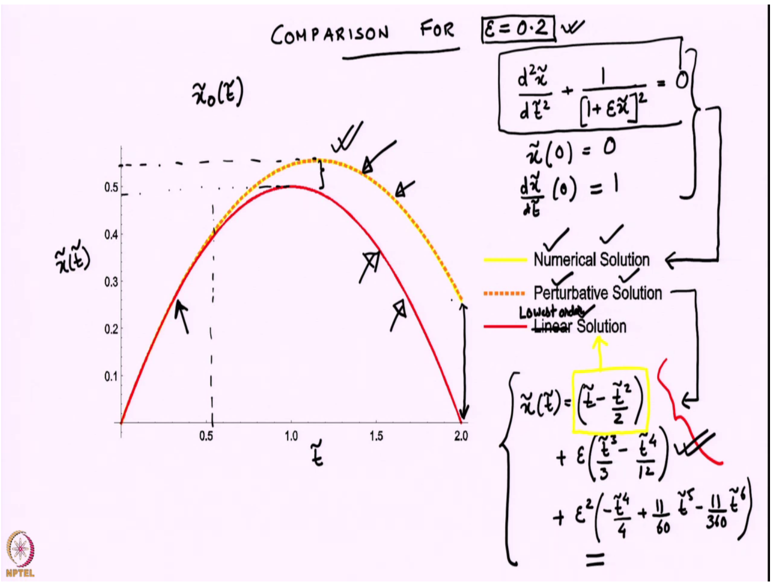
So, we already know that we really do not need to solve the right hand side. The right hand side can just be approximated as minus 1 at the lowest order. We really need to push up v by a much larger factor in order to get an epsilon which is small, but not extremely small ok. So, we would like to have epsilon typically about 0.1 or so. This as you can see by a quick calculation that unless we make v naught to be about 3500 meter per second or 3.5 kilometer per second we are not going to get an epsilon which is about 0.1.

If you use for example, v naught equal to 350 meter per second, this is close to the speed of sound in air, then you will get an epsilon which is about 0.002. If you push it by a factor of 10 here, so from 350 we are going to 3500, this is a very large speed. The speed of sound in water is about 1.4 kilometer per second; this is about 3.5 kilometer per second. So, that is about 3 times in the speed of sound ok.

So, we are really launching it at supersonic speeds. So, if we launch it at supersonic speeds, then epsilon climbs to about 0.1. And then it makes sense to ask that are there any corrections to the fact that the force, that the earth exerts on the object is not constant all along its motion, but actually reduces as it goes higher and higher.

So, we are going to use a value of epsilon which is about 0.2. So, this would correspond roughly to having throwing the ball of throwing the mass with speeds of that order 3500 meter per second. So, if you launch it at that speed, then you will get an epsilon which is about 0.2 and. So, let us compare the solution to the equation that we had just written for the case where epsilon is 0.2. We will have. So, this is what we are doing here.

(Refer Slide Time: 14:51)



So, this is we have chosen epsilon equal to 0.2. So, there are three plots here, numerical solution, perturbative solution and linear solution. I should probably replace the word linear with lowest order ok. So, this is the lowest order solution. So, the linear the numerical solution basically takes the full problem, and solves it numerically ok. So, you can see that the yellow curve here is the numerical solution.

The lowest order solution is in red. It is just the solution for x naught of t tilde. And the perturbative solution is this whole thing, x of t tilde is equal to x naught plus some epsilon into x 1 plus epsilon square into x 2. I have already written these expressions. You can clearly see that during the climb part of the journey when in the mass is going up and up and up, almost up to here the three solutions are indistinguishable.

However, the maximum height reached by the exact solution is quite more is substantially more than the, so this is the maximum height reached and this is the maximum height reached. You can also see that the numerical solution and the perturbative solution are lying on top of each other. So, at this value of epsilon, the perturbative solution is doing much better than the lowest order solution. The perturbative solution actually has term which is linear in epsilon, and then a quadratic term.

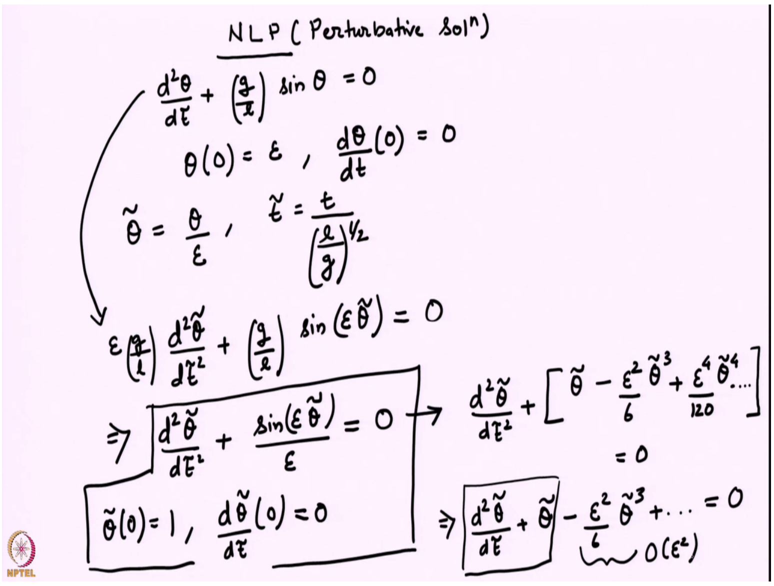
So, you can plot these things and you can play with this by switching of various terms and seeing that how good or bad is the approximation. So, you can see that even if you did not know how to solve this equation exactly, the perturbative solution actually gives you a way to evaluate answers which are quite accurate. So, just by climbing up to epsilon square for epsilon equal to 0.2, you can get an solution which within this timescale is as good as the numerical solution.

You can also see that this is going to make an error in if you want to evaluate the total time of flight, it is going to make a large error because it the linear solution predicts the that non-dimensional time t tilde equal to 2, it is; it has come back to where on the surface of earth ok, whereas, the non-linear solution is still up in the air ok. So, this should give you a flavour of how these problems are dealt with.

Now, I am going to go back and we are going to go back to our old problem which is the non-linear pendulum. And we are going to once again go back to the non-linear pendulum, we are going to say that suppose we did not know elliptic functions, can I solve the non-linear pendulum using such perturbative techniques?

And in the process, we will discover a method which is very useful. And later on in the course, when we do interfacial waves, we will use that method for understanding capillary gravity waves.

(Refer Slide Time: 18:12)



So, we come to the perturbative solution of the non-linear pendulum. Up to now we have solved the non-linear pendulum either through linearization or we have solved it exactly using elliptic functions. Now, we will find a systematic method of solving the non-linear pendulum up to the desired order of accuracy using a perturbative method. It will be seen that for angles which are approximately 35 or 40 degrees, the linear solution does not do a very good task; it is not a good approximation.

So, for such angles, the perturbative solution provides a much better way of understanding the solution rather than the exact solution. The exact solution in any case may not always be available plus it comprises of functions for which we may not necessarily have a very sound physical intuition. So, the perturbative solution in some sense is a simpler approximation, and it does a reasonably good task as we will shortly see.

So, we had seen that the equation to the non-linear pendulum. Now, because we have used epsilon as a small parameter, so here the small parameter is the initial angle with which I am leaving the pendulum. We have use theta naught, but because in our introduction to perturbation methods, we have always used epsilon as a small parameter. I am going to call my initial angle as epsilon.

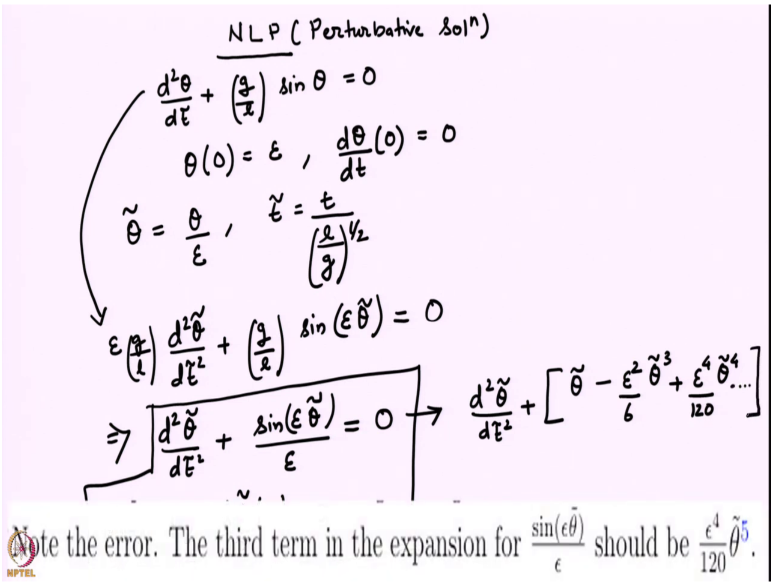
So, epsilon is just theta naught. We have also seen these are the initial conditions for which we have been solving all along. And we have seen that if we non-dimensionalize this, so some theta tilde and the timescale is just the linear time period or it is proportional we are neglecting the factor of 2 pi.

Now, if you do this, then you get if you substitute these scales and non-dimensionalize this equation, then you get. And now I can cancel out the g by l and write this as. The initial conditions become. So, now, I have to solve this set in a perturbative manner, because epsilon is a small quantity it makes sense to expand sin epsilon theta tilde in terms of epsilon.

Note that like before we are assuming that after non dimensionalisation, all our variables are of the order 1. So, for example, this d square, the first term in the equation d square theta tilde by dt t tilde square is an order 1 term, and we will soon see what is the size of the second term. So, sin theta or sin epsilon theta is just epsilon theta divided by epsilon.

So, we will just get a theta tilde plus rather minus, it is minus, and then we will have epsilon cube theta tilde cube divided by factorial 3 which is 6. So, if I divide that epsilon cube by the epsilon in the denominator, I get an epsilon square. Then we have theta tilde cube and then 6. And then we will have next epsilon to the power 4, theta tilde to the power 4, and the there is a factorial 5 I think which gives you 120 and so on.

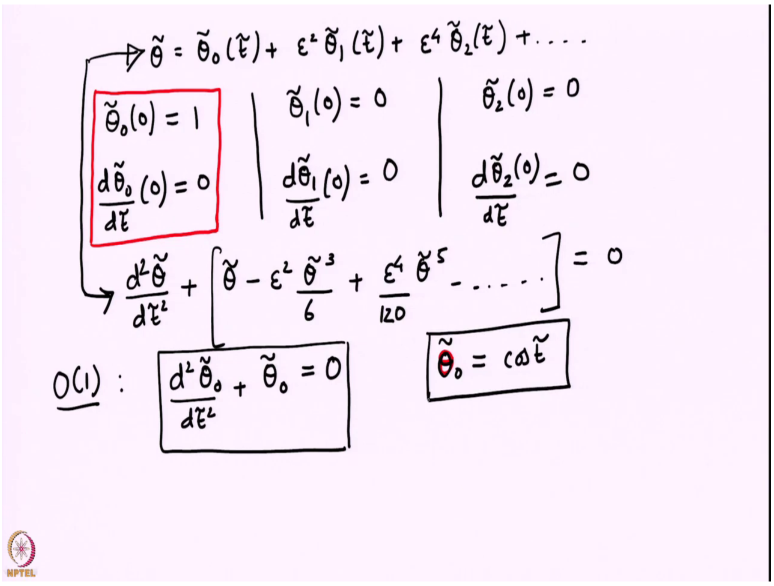
(Refer Slide Time: 22:29)



So, you can see that this whole term which I have put in square brackets is an order 1 term provided theta tilde is of the order 1. So, this is equal to 0. And we have to solve this in a perturbative manner subject to initial conditions. So, once again you can see that I can write this equation as all order 1 terms minus epsilon square by 6 theta tilde cube. Let me just stop here. These are higher order terms is equal to 0.

You can see that the linear pendulum which is represented by the first term is perturbed by a term which is order epsilon square. So, the unperturbed problem which in this case is the linear pendulum is perturbed by a quantity which is order epsilon square. So, when I write down an expression form a for theta tilde as an expansion in powers of epsilon, I should start with epsilon square and not epsilon.

(Refer Slide Time: 23:45)



We have theta tilde is equal to theta tilde cube t plus epsilon square t plus epsilon 4, theta 2. We are not really going to use the perturb, and plus there will be dot dot dot. If we do the same expansion, now if we substitute back into the differential equation and the initial conditions, we are going to get quantities at various orders. What happens to the initial conditions? Like the previous problem, anything which is nonzero in the initial condition gets collected at the lowest order.

So, you can immediately see that the initial condition on theta 0 is this. The initial condition on theta 1 an onwards for all higher values and so on, this is exactly like before. Now, if we plug in this initial condition this expansion into our C's into our differential equation, our differential equation was of the form, this was a form of a differential equation. So, we have to plug in this into this.

And you can immediately see that I can collect the order 1 term, the order 1 term is d square theta tilde 0 by d tilde square plus theta tilde 0 is equal to 0. And we have to solve this with that initial condition. This is easy to do. If you do it, then you can immediately see that theta tilde of 0 is a linear combination of sin t tilde and cos t tilde. There will be two constants of integration using the initial conditions you can eliminate, and you can express the constants in terms of the initial conditions.

If you do that, you will find that one of the constants is 0 the other one is just 1, and so the solution is cos t tilde. This is our solution to the linear pendulum. We will look at higher order corrections in the next video.