Introduction to interfacial waves

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Lecture - 15
Non – dimensionalisation

In the last problem, we had looked at solving for roots of a quadratic equation where the equation was slightly perturbed from a state where it was factorizable as two integers, two integer roots into a slightly perturbed state. There we I had stated that the epsilon which is typically a non-dimensional parameter and a small parameter for solving the problem perturbatively is introduced through non-dimensionalisation.

We did not discuss non-dimensionalisation for that problem. We now return to non-dimensionalisation, and what is the optimal way of doing non-dimensionalisation and introducing epsilon into our problem. The important thing to realize is that that the choice of typically for a problem for a mathematical model that we want to set up in this case we will look at a simple example.

Now, later we will have to do the same for our interfacial waves problem; typically there are more than one choice for length, time, length and time scales. This leads to some kind of ambiguity for the choice of for the correct choice of non-dimensionalisation for quantities like velocity and so on. Here we have to use our physical intuition in order to choose the scales correctly. Let us look at an example.

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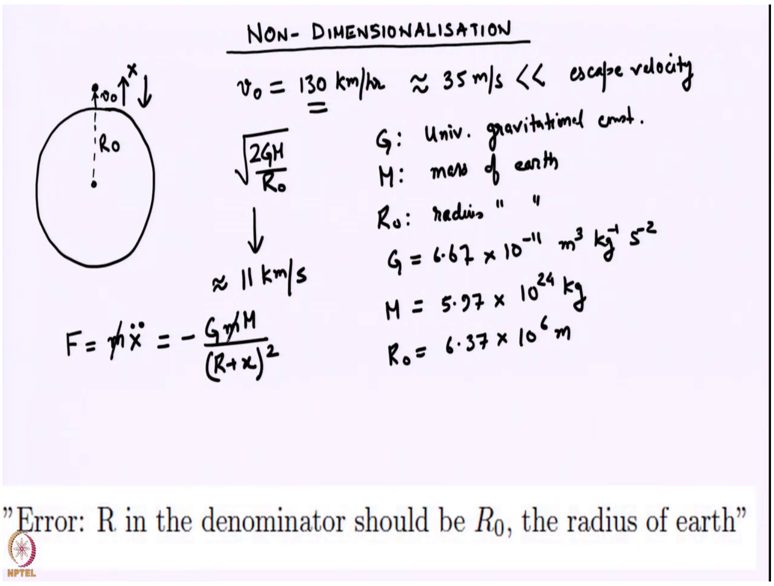

So, let us take a simple example, where let us say we are standing on the surface of earth, and we are throwing a ball upwards at a speed v 0. Now, of course, the distance to which the ball travels will depend on the speed v 0. We typically if we throw with our hands, we can we cannot throw the ball very high. When I say very high I mean the distance that it would travel would be very small compare to for example the radius of the earth.

Let us write down some numbers to form our estimates. So, typically v 0, I will take an upper limit on what, what is the fastest that a human hand can throw a ball. So, typically a fast bowler, for example, in cricket can throw typically at about 130 kilometers per hour. This is approximately it is slightly more than this, but approximately this is 35 meter per second. It is above 36 point something, but I have a rounded it off to 35.

So, this is, this is a representative estimate of how fast can the human hand throw. So, if we have a fast bowler in cricket throwing, the ball is expected to be thrown at a typical velocity of 35 meter per second. Now, you can see that this is much, much less than for example the escape velocity.

We all know how to estimate escape velocity. Recall that the escape velocity is given by the formula 2 G M by R 0. Here G is the universal gravitational constant appearing in the law of gravitation, M is the mass of earth, and R naught is the radius of earth. Now, we know that when you throw a ball upwards, the acceleration the force that the earth exerts on the ball via gravity is not really a constant.

It depends on the separation between the ball and the earth, and the center of the earth. Now, typically for the speeds at which we throw a balls the distance at which it travels is so small compare to the radius of the earth that as a first approximation, it is usually safe to ignore that distance and treat the force as if it is a constant that gives us a constant value of force consequently a constant value of acceleration which we know to be small g which is 9.8 meter per second square.

Now, suppose instead of throwing 130 at 130 kilometers per hour, we throw at a much higher speed you know 10 times or may be even 100 times of that. So, this is a hypothetical situation when a where we are launching the ball not my hand, but using some kind of a mechanism which can let us say a launch it at about instead of 130 kilometers per hour, we are launching it at let us say 10 times. Let us compare it with the escape velocity.

If you estimate this, so G is 6.67. So, these are the numbers in SI units. If you calculate, if you plug in it in here, you will get approximately 11 kilometer per second. So, you can see that this is even if I launch it at 1300 kilometers per hour that is much, much less than 11 kilometers per second. So, you can see that 11 into 3600 will be much more than that.

So, now compare a hypothetical situation where I am launching it at a high speed, the speed is high enough such that I do not want to ignore the fact that the as the ball goes upwards, the force that the earth exerts on the ball actually changes. So, I want to solve the exact equations without making an approximation that the acceleration or the force is a constant. We know how to do this. So, let us do this.

So, the exact force, so the exact expression would be, so let say this direction is X we are solving a one-dimensional problem. So, mass into acceleration because the force is always attractive it will be directed towards the center of the earth and we take all quantities in the readily outward direction to be positive, so upward direction is positive. So, the force is negative. And by Newton's law of gravitation the force is just this here small m is the mass of the body that I am launching.

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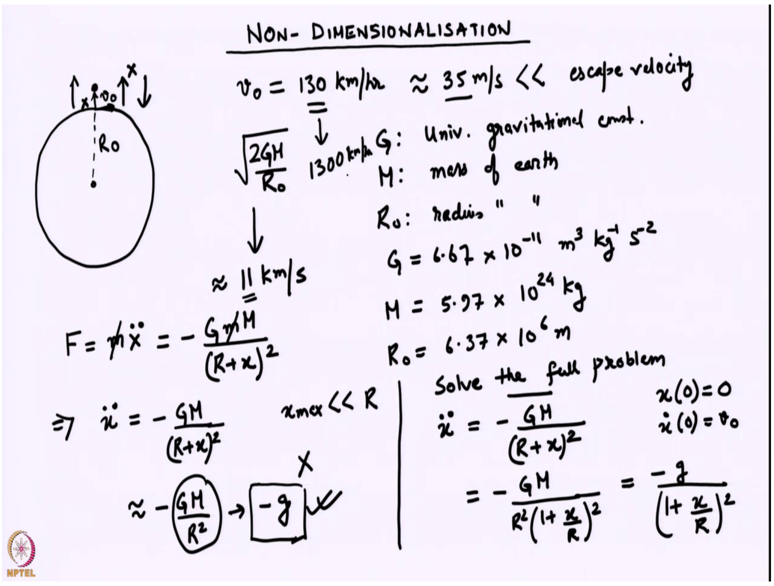

Now, this cancels out and so we get x double dot is equal to minus GM by R plus x square. Now, I can immediately see that for the typical velocity is which suppose we launch it at 35 meter per second or 130 kilometers per hour it would the distance that x max the distance then the ball would travel would be far less than R.

And so when we solve this equation as a first approximation, I can just ignore the x in the denominator and just replace this as minus GM by R square. This is essentially minus g, GM by R square is g which is 9.8 meter per second square. Now, suppose I do not want to do that approximation because I want to throw the ball at increasingly higher and higher speeds.

The speeds are sufficiently large that the ball goes distance, but it is still a much bigger distance than what it would if I launch it with 35, but they are still small compare to the radius of the earth, and the velocity is still much smaller than the escape velocity. So, let us see if we can systematically set up a way of non-dimensionalising this problem choosing the scales correctly.

So, how do I choose this? So, I want to solve the full problem. I do not want to make this approximation. This is just an exercise in non-dimensionalisation followed by solving this problem using perturbative techniques. So, I do not want to make this approximation to start with.

However, I also have an intuition that if I if my v 0 is sufficiently small, then at the lowest order of approximation this should be my this problem x double dot is equal to minus g should be my first approximation. So, with that in mind let us state the problem. So, x double dot double dot is d square x by dt square is equal to minus GM by R plus x square.

Now, I will launch it at time t equal to 0 at it was at 0 my origin of my coordinate system is here. So, this is x in the vertical direction. And I am launching it with some velocity v naught. Now, let us say that I am I want to write it like this. So, I pull an R out. So, it will come out as R square, and then I will have x by R whole square.

I am doing this because I want to write minus GM by R square as a single variable which is g small g. So, this is minus g by 1 plus x by R whole square. This is a very useful form because it tells us that if x max is less than R, then a first approximation is just x double dot is equal to minus g. So, the acceleration is a constant. Now, I will let us try to first non-dimensionalised this problem appropriately.

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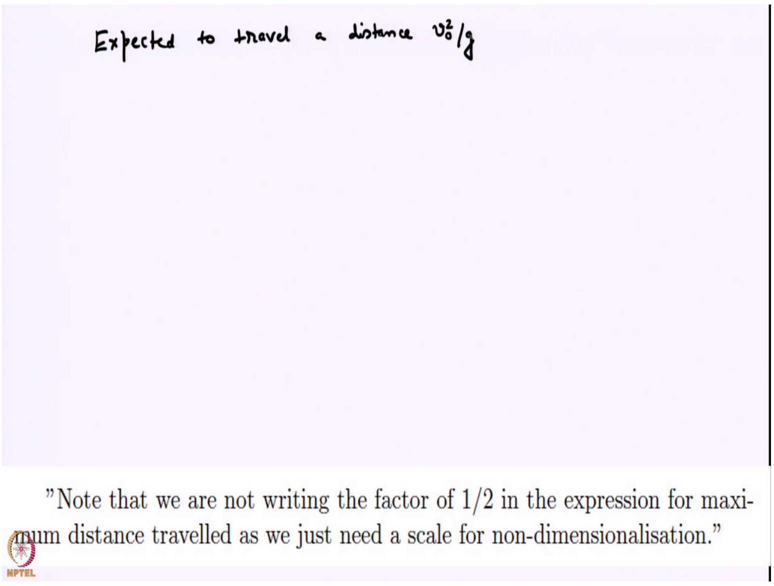

So, if I launch the my projectile with the speed v 0, I expect it to travel expected to travel a distance v 0 square by g. You can get this from a simple estimate assume that acceleration is constant. And if you launch something with speed v 0, then how much is the distance travelled in some time t. So, this is the maximum distance that it would travel before which after which it would start falling.

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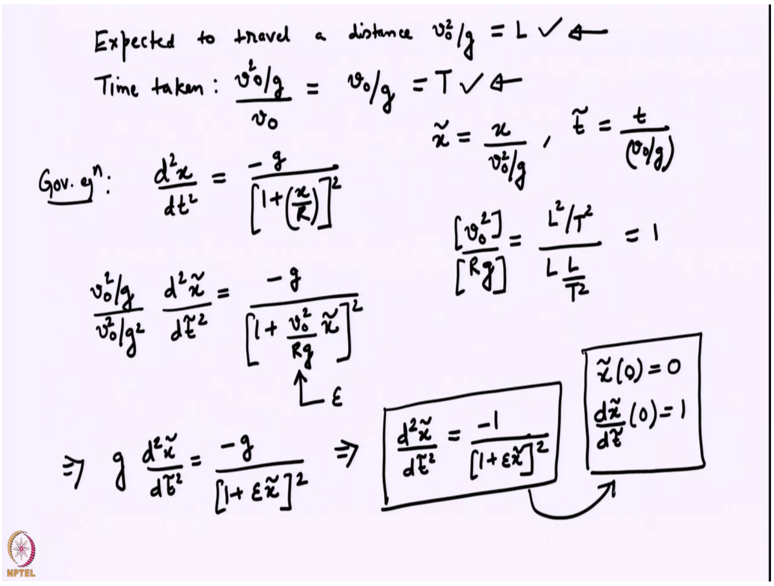

Now, it would travel this distance time taken would be if I know my distance and if I divided by the speed with which it travels, then I get a time, so that is v 0 by g. So, this is my choice of length scale, this is my choice of time scale. This is something v 0 by g is some quantity which has the units of time v 0 square by g is some quantity which has the units of length. We are going to non-dimensionalise by these scales my governing equation.

Note that the choice of distance and the choice of time is not unique. If I had gone back to my equation and if I had not done this step where I have replaced GM by R square, then I had more dimensional quantities like G capital M an R square, in particular, I could have non-dimensionalised by my x with the radius of the earth. I encourage you to try that, but first let us do it in the intuitively correct way.

So, I expect it to travel a distance L before it starts falling back, I expect it to travel that distance in this time which is given by v 0 by T. Now, let us scale my equations. So, I will write it now in terms of d square x by dt square is equal to minus g by 1 plus x by R whole square.

This is just the equation that we had written earlier. Now, I define a non-dimensional x which is an x with the tilde which is x the dimensional x divided by something with the units of with the dimensions of length I will choose that to be v 0 square by g – this scale. I also have to define non-dimensionalised time. I will define a new time variable t tilde, it is a non-dimensional variable. So, dimensional time divided by something with a dimensions of time which is this.

Now, using these scales, I have to non-dimensionalise my governing equation, so governing equation. Notice that this is a non-linear equation. If you solve the full problem, it is a non-linear equation in x may or may not be solvable analytically, but we are try to going we are going to try to solve this using a perturbative procedure. So, first let us non-dimensionalise it.

So, if you so I will have a v v 0 square by g in the numerator and I will have v 0 square by g square in the denominator, and then this will become d square x tilde by d t tilde square. And then I will have minus g 1 plus I can see that I will have a v 0 square by R g into x tilde whole square. You can see that this quantity v 0 square by R g is non-dimensional this is length square by time square length by time square. So, this is a non-dimensional number, I will call it epsilon.

What we have on the left hand side after cancellation, you can see is just g d square x tilde by dt tilde square, and then on the right hand side I have minus g by 1 plus epsilon x tilde whole square. I can cancel out the g. What about my initial conditions? My initial conditions is x tilde of 0 is anyway 0, dx tilde by dt tilde at time t equal to 0 would be 1. So, I have to solve this equation with those initial conditions. Let us estimate the value of epsilon.

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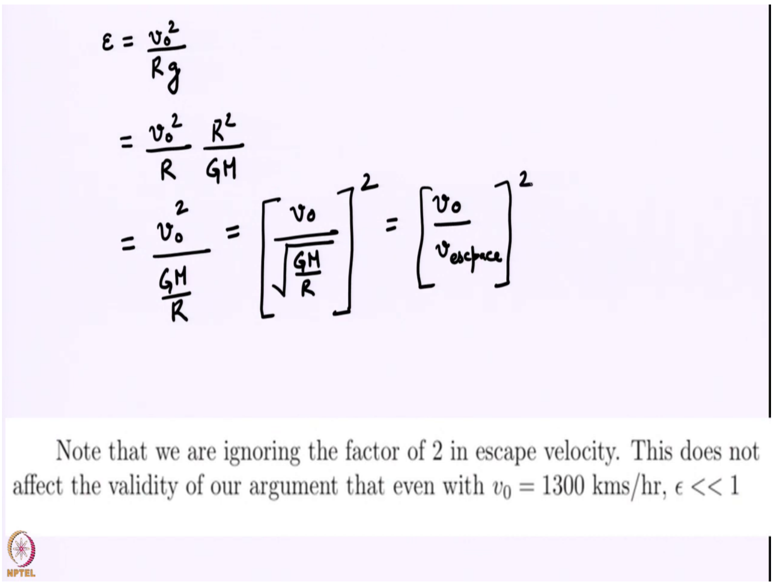

v 0 square by R g. And you can see that if I use the formula for small g that I had given earlier, then small g was so it is 1 by g. So, it will be R square by GM. And this is equal to v 0 square divided by GM by R. Now, this is equal to v 0 by square root GM by R whole square. What is square root GM by R? We have seen the 2 times square root GM by R is the escape velocity; square root of 2 times GM by R is the escape velocity. So, this epsilon is nothing but v escape whole square.

We have seen that even if we make v 0 as instead of 135 kilometers per hour we even if we make it 1350 – 10 times. If I make it instead of 130, I make it 1300 kilometers per hour, so it will travel 1300 kilometers in an hour. This travels 11 kilometer in a second. So, in 3600 seconds which is an hour, it will travel much much more than 1300. So, even if we make it 10 times, this ratio v 0 square, v 0 by v escape.

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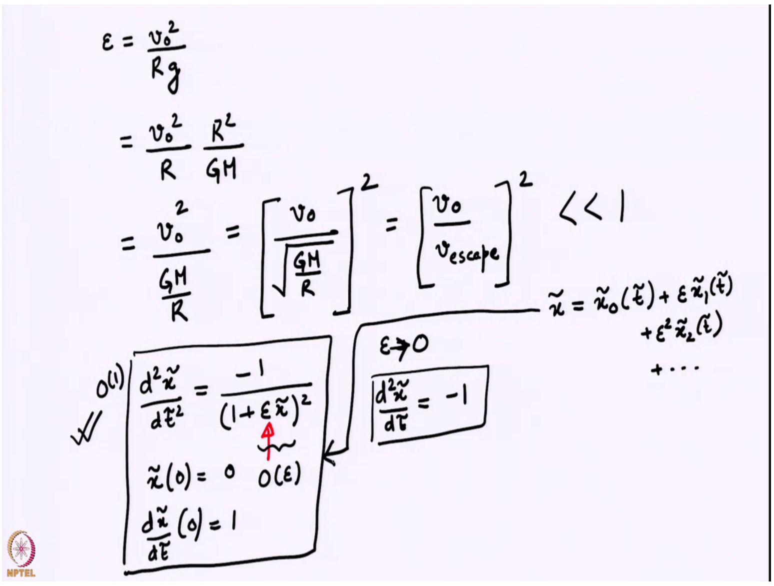

Let us write escape is much much less than 1. However, if we throw it at sufficiently large v 0, those corrections to the gravitational force might become important ok. So, let us find out a way to solve the full problem in a perturbative manner. Assuming that epsilon is less than 1 is much much less than 1 and so we can solve it perturbatively, but epsilon may not be so small that we can consider force to be a constant.

So, our scale problem looks like this. How do we know that this is the correct way of non-dimensionalising? First, check whether when epsilon goes to 0, do you recover the correct equation or not that you would expect on physical grounds. So, if you said epsilon equal to 0, you get the equation d square, your unperturbed problem is just this.

This is just equivalent to saying that if my epsilon is extremely small ok, so in the limit of epsilon go in rather you should think of it like this, in the limit of epsilon being extremely small, I just recover the fact that force is a constant and acceleration is just g. This is just the scale version of that.

However as if epsilon is small, but not very small, then I can use a perturbative procedure to on this equation. I encourage you to try this non-dimensionalisation by choosing other scales. So, for example, we have chosen the length, the length scale to be v 0 square by g. You could have chosen the length scale to be the radius of the earth, you could have chosen the time scale to be the time taken to cover the radius of the earth at a speed v naught.

Try choosing those scales and non-dimensionalising your problem, and see what happens to the problem in the limit of epsilon going to 0. You will typically find that it will reduce to either something which does not make mathematical sense or it violates your basic intuition that for sufficiently small epsilon taking acceleration to be constant is a valid approximation.

So, now this non-dimensionalisation has added a non-dimensional parameter to our problem which is epsilon. In this problem, it is the square of the speed with which we are launching our projectile to the escape velocity of the earth, provided we do not launch our projectile at the escape velocity, epsilon is usually going to be much lower than 1. Let us solve this problem perturbatively, and find out what does the perturb solution look like.

So, just like before we think of this as an order epsilon term because x tilde is an order one term. So, we are saying that x tilde is of order one after non-dimensionalisation if we have non-dimensionalised correctly and so this whole thing is an order epsilon term. So, and the left hand side is again order 1 because there is no epsilon anywhere.

So, let us see that we are going to set up a perturbation approximation to the solution which will be x 0 tilde which will be your function of t tilde plus epsilon x 1 tilde plus epsilon square x 2 tilde plus dot dot dot. We are going to substitute this into our equation and initial conditions, and then work out the solutions to this problem at every order of epsilon.

Once we get that, then we will have to solve the differential equations at every order. And important think to notice is that this procedure will reduce a non-linear differential equation into a series of linear, but in homogeneous differential equations. It will be easier to solve those equations because they are linear.

However, the right hand side of the equation in homogeneity will become more and more complicated as we go to higher orders. Let us do this exercise and understand how it works. We are going to first let us work on the initial conditions first. So, our initial conditions were so let me write down the expansion again.

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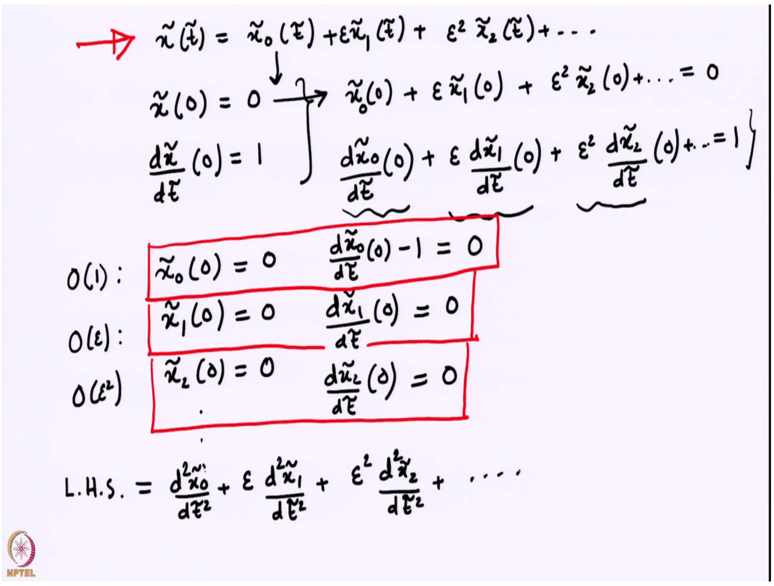

So, x of t tilde is x 0 tilde. This is my base state solution, I expect the base state solution to give me the lowest order approximation as epsilon goes to 0, the lowest order is just force is equal to constant. So, it should just tell me the d square x 0 by dt square is equal to minus 1 plus. Now, my initial condition is this.

So, what does it translate to? So, we will have to use this to obtain initial conditions for every order. So, how do we do this? So, we say x 0 plus epsilon x 1 of 0 square is equal to 0. This is obtained by substituting the above into this equation. We also obtain this is x 0 at 0 equal to 1. I can satisfy the first equation by choosing, I am equating the coefficient of every power of epsilon to be 0. So, I will have and so on.

I have to for the second one I have to be a little bit more careful. I, there is an order one term here, this is an order epsilon, this is an order epsilon square. So, if I collect all the order one terms which are basically coefficients of epsilon to the power 0, then I get. So, so this is all let me write it at the same horizontal level.

So, this is order 1 – this and this. Then at order epsilon, we will have. Then at order epsilon square, we will have. And like before if I shift the one to the left hand side here, then I have 0 on the right hand side. So, I have to set all the coefficients to 0 in order to satisfy.

So, this, is my initial condition. These are my initial conditions at order 1. This is my initial condition at order epsilon. This is my initial condition at order epsilon square. So, you can see that we have 0 0 initial conditions at every order except 1, except at the lowest order. So, this will particularly simplify the algebra ok.

So, now what we have to do is, we have to go back and substitute this form into our governing differential equations. Let us do that. So, if I substitute, then the left hand side becomes the left hand side of my equation becomes d square x naught tilde by dt tilde square plus epsilon d square x 1 tilde plus dot dot dot dot.

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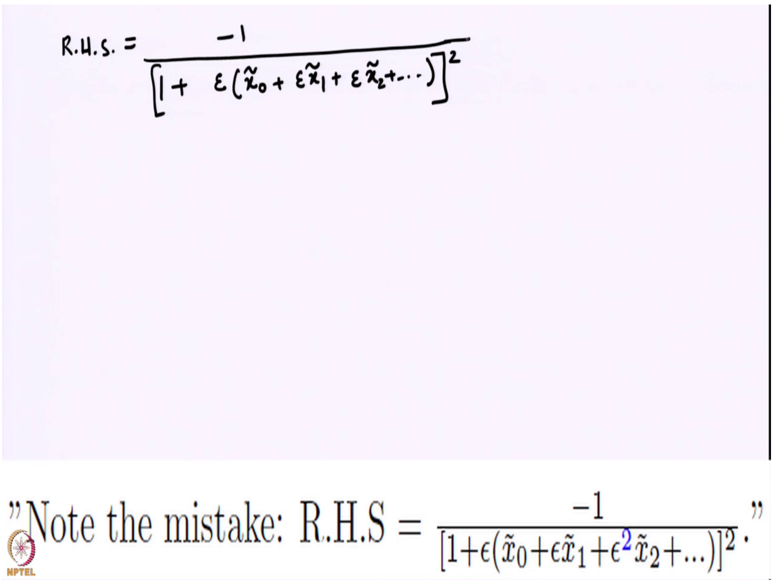

The right hand side has to be worked on. The right hand side is 1 plus x tilde whole square which is or epsilon x tilde whole square. So, this is epsilon into x naught tilde plus epsilon x 1. And now I will have to bring it up in the numerator using binomial expansion.

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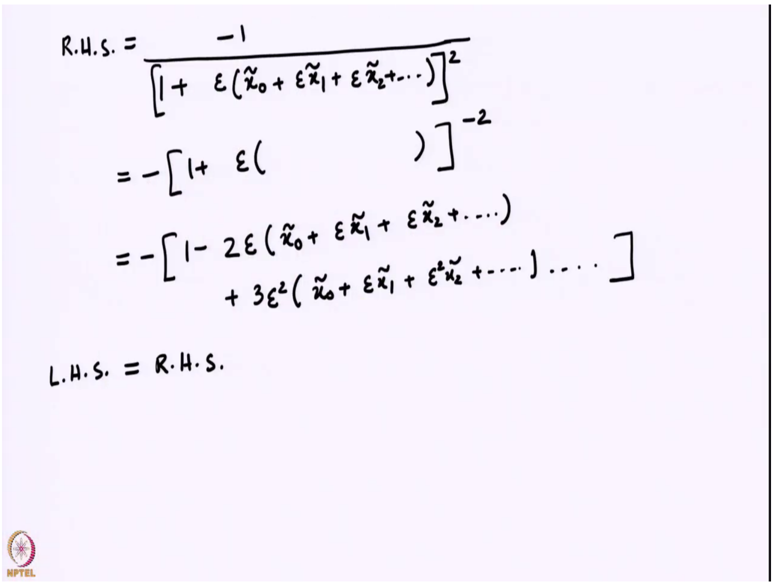

So, this just becomes 1 plus epsilon the same thing to the power minus 2. And if you expand it, you are going to get. So, the power the exponent comes as a coefficient, then we just have this, then there will be more terms in the expansion. So, now, I have to set left hand side is equal to right hand side.

Collect all terms which are order 1 that will give me my equation governing x 0 tilde. Collect all terms order epsilon that will give me an equation governing x 1 tilde. In this equation, you will find that x 0 tilde appears on the right hand side. We cannot solve the order epsilon equation unless we have solve the order 1 equation. This is also true at every order. At every order, we can only solve it at that order provided we have solved at all previous orders.

In the next video, we will do this, we will continue this, we will write down the equations obtained at every order, pair them with the initial conditions that I wrote in the previous slide, and then solve them individually. You will see that at every order, we will get a linear differential equation, but it will become inhomogeneous as we go to higher and higher orders, and that will give us once we solve those equations it we will recover our perturbative solution to the non-linear ordinary differential equation.