Introduction to interfacial waves
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Lecture - 14
Perturbation methods (contd..)

We were looking at the order symbol in our previous example and we had just defined a function f of epsilon to be of the order another function g of epsilon, if this criteria was satisfied.

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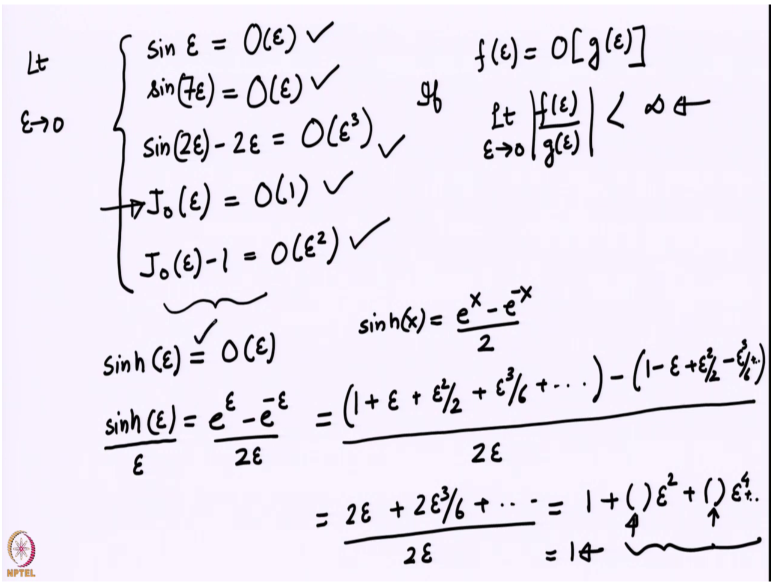

We also looked at a couple of simple examples and we found that sin epsilon is of the order epsilon, sin 7 epsilon is also of the order epsilon, sin 2 epsilon minus 2 epsilon is of the order epsilon cube.

J 0 epsilon is of the order 1 and you can try it for yourself using the same infinite series representation that I had given you earlier for J 0 epsilon, then J 0 epsilon minus 1 is of the order epsilon square. Once again I would like to repeat that the precise value of when we particularly when we say something is of the order 1 the limit actually should be finite.

Whether the limit is actually 1 or not is not important. As long as the limit is finite even, if it the number in the limit comes out to be 1000 the function is still of the order 1. So, now, let us take some more examples. So, for example, sin hyperbolic epsilon is of the order epsilon. This is a claim, let us see whether this is true or not.

So, recall the sin hyperbolic x is defined as e to the power x minus e to the power minus x divided by 2. So, if I use that then, I would like to take this so, sin hyperbolic epsilon would be e to the power epsilon minus e to the power minus epsilon by 2 and if I. So, I would like to divide this by epsilon and then take the limit.

So, there is a two epsilon here. And, then if I expand e to the power epsilon in a Taylor series, then this is 1 plus epsilon plus epsilon square by 2 plus epsilon cube by 6 plus dot dot dot minus 1 minus epsilon plus epsilon square by 2 minus epsilon cube by 6 plus dot dot dot dot. And, this whole thing gets divided by 2 epsilon.

Now, if I add up there will be some things which cancel out. So, the first term would be 2 epsilon. Epsilon square would cancel each other out and then, we would have 2 epsilon cube by 6 plus dot dot dot divided by 2 epsilon. And you can see that the first term is 1, if I divide by the denominator and then, there are terms the first term will be something into epsilon square.

This something is finite plus something into epsilon it would this will be epsilon 5. So, this will be epsilon 4. So, you can see that each of these terms plus dot dot dot, everything here in the limit of epsilon going to 0 is going to 0 and this limit is just going to go to 1. So, sin hyperbolic epsilon by epsilon is indeed of the order epsilon.

Once again, I am emphasizing that even if this limit did not evaluate to 1, but instead evaluate to let us say 100 which is not true in this case, but let us say we took something, where the limit evaluated to 100 as long as we get a finite number it is the statement is correct.

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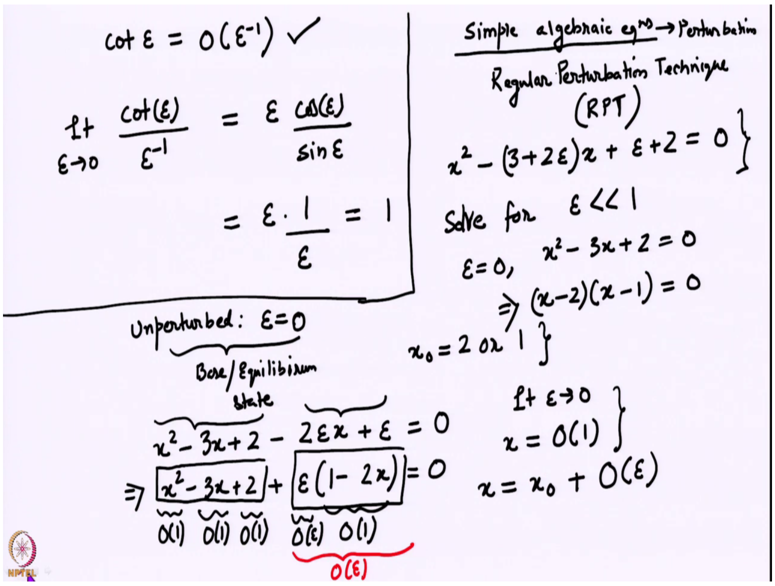

So, this is also correct. Another example, cot epsilon is of the order epsilon to the power minus 1. So, now, how do we check this? So, let us take the ratio cot epsilon by epsilon to the power minus 1. We are interested in this limit. So, you can bring epsilon to the numerator right cot as cos epsilon by sin epsilon.

Now, for sufficiently small epsilon, the first term in the Taylor series approximation of cos epsilon is just 1. The first term in the Taylor series approximation of sin is just epsilon. So, this is 1. So, this is also correct. You can plot this function epsilon cot epsilon epsilon and you will see that it starts at 1.

So, like that we can explore many examples, but let us look at some applications of this order symbol, we are going to use this order symbol and refer to it repeatedly in this course. So, let us look at some applications. I put this in a box ok. So, now, we will solve some simple algebraic equations using perturbation.

These are what are known as regular perturbative techniques or regular perturbation technique. There is also another class of methods, which are called singular perturbation techniques, we will briefly discuss them later on. This spelling is wrong perturbation technique.

So, let us say, we have to we have a cubic equation or rather we have a quadratic equation and we have to find its roots. So, the equation is given by this. We have to solve this equation or solving implies finding its roots for epsilon less than 1. Now, of course, we can solve this exactly, this has been chosen such that there is an exact solution.

So, we all know, what is the formula for finding the root of a quadratic equation. Let us pretend for the time being that we do not remember that formula. So, let us solve this example in a perturbative fashion. So, recall what I said the first step is non dimensionalization that include that step usually brings in an epsilon that step is already done here.

So, you already have epsilon in your equations. So, let us not worry about non-dimensionalization right now. The next thing is to check, whether at epsilon equal to 0 your problem is something which can be solved easily. So, if I set epsilon equal to 0 here, then this becomes x square minus 3 x plus 2 is equal to 0.

This can be factorized easily into this. And so, we know that the unperturbed roots so, unperturbed is when epsilon so, unperturbed is epsilon equal to 0. This is the equivalent of the equivalent of this in mechanical systems that we have been we have been studying so far is the base or the equilibrium state.

So, you can think of epsilon equal to 0 as our base state or equilibrium state that was an easy state to analyze nothing was moving. And in this case, in the equilibrium or the base state, we can solve this equation and find its roots easily. So, x 0 is equal to 2 or 1.

Now, our perturb problem is this with epsilon ok. And, we are interested in finding out an expression for its roots in the limit of epsilon going to 0. Let us say we do not know the formula that we all use for finding the roots of a cubic. So, let see how are we going to do this.

So, let me reorganize this equation and write it like this. So, I am collecting all the epsilon terms and putting them at the end and this is my unperturbed equation. Now, I will write this as. So, now, you can see that if you think of f x. So, you can clearly see that if I change the value of epsilon its roots are going to change for epsilon equal to 0 I have 2 n 1.

For epsilon equal to 0.1, I will have some other root slightly different from 2 n 1. For epsilon equal to 0.2, it will be again slightly different from even more different from 2 n 1 and so on and so forth. I am interested in finding an expression for those roots for the perturbed equation in the limit of epsilon going to 0.

Now, you can see that assuming that all the roots x is a symbol for the roots of this equation. If I say that in the limit of x equal to epsilon equal to 0, let us say that the roots are all order 1, which means that they are essentially finite they do not go away to infinity.

If I make this assumption, then I can evaluate the sizes of various terms in this equation. So, if x is an order 1 term, then x square is also an order 1 term. This is an order 1 term this is of course, an order 1 term this whole thing is an order 1 term, this is an order epsilon term the product of these two is an order epsilon term.

So, you can see that the structure of this perturbed equation is that there was an order one equation which is obtained when epsilon equal to 0. And you have added a perturbation which is probably order epsilon. So, we will say that if you have added a order epsilon perturbation to the equation, the answer to the unperturbed system might have also got perturbed by the same amount ok.

So, we will pose an expansion, we will say that the root x of the perturbed system is the root x 0 of the unperturbed system. x 0 we know is either 2 or 1, plus some correction which are order epsilon. Why is order epsilon? Because, this is order epsilon. So, let me write it in colour this we think is order epsilon. So, how are we going to write these expressions of order epsilon? We will write it like this.

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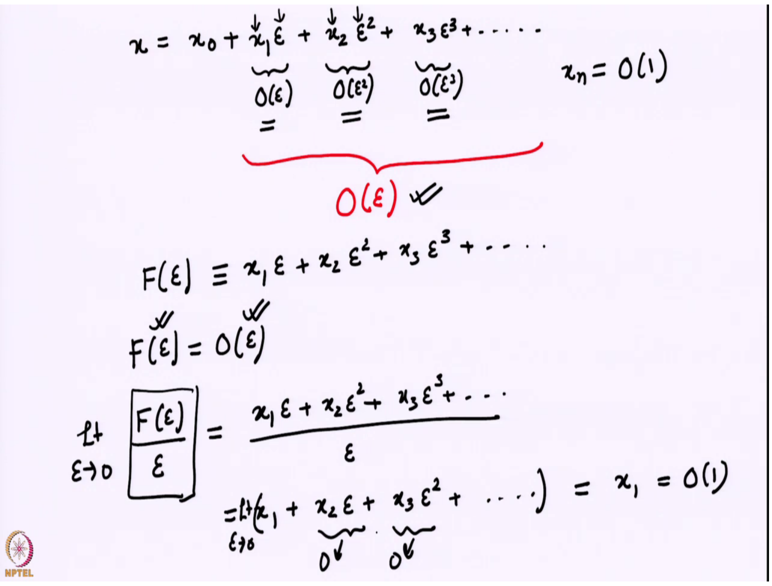

We will say x is equal to x 0 which is my unperturbed root, plus a correction which is an order epsilon term in the limit of epsilon going to 0. So, I will write that correction itself as an infinite series and I will write it as a power series in epsilon. Notice that each of these terms, this is an order epsilon term this is an order epsilon square term, this is an order epsilon cube term and so on.

And that is because we are assuming that all the xn's are order 1. So, here we are having an order 1 term multiplying an order epsilon term. Here, we have an order 1 term multiplying an order epsilon square the product is of the order epsilon square. And now, here it is order epsilon here it is of the order epsilon cube.

Also notice that this entire thing; this entire series this combined thing this infinite thing is of the order epsilon. How do we see that? Let us call this entire thing some function of epsilon. So, F of epsilon by definition is x 1 of epsilon plus x 2 of epsilon square, plus x 3 of epsilon cube, plus dot dot dot. I am claiming that F of epsilon is order epsilon this is what I have written here.

How do I check that I take F of epsilon by epsilon and the limit epsilon goes to 0. In the numerator we have this and if I divide by epsilon, then I get x 1 plus x 2 epsilon plus x 3 epsilon square plus dot dot dot. You can see that if I take the limit epsilon goes to 0, then each of these terms will go to 0. Because the x's are all finite and there is a epsilon to the power something in each of these terms.

So, this limit would just be x 1, which is an order 1 number or a finite number. So, limit F of epsilon by epsilon is a finite number. So, it satisfies our definition and we can say that F of epsilon is order epsilon. I hope this is clear.

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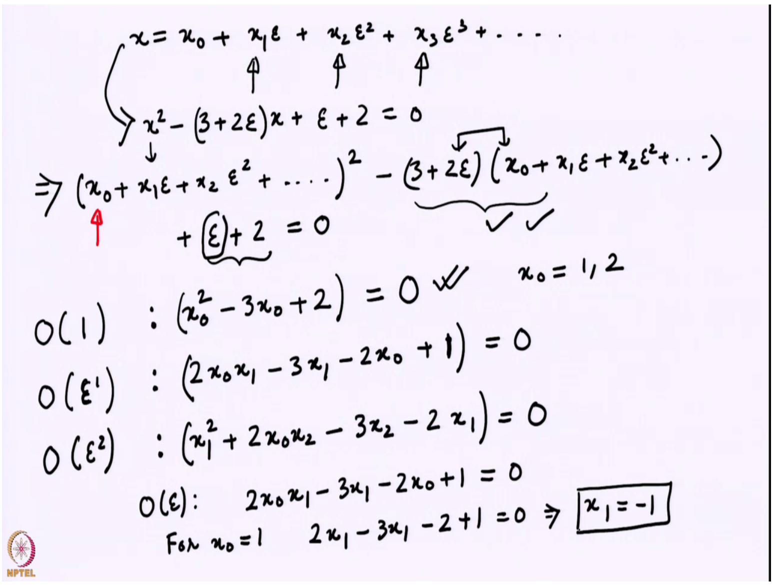

So, what we have said? That x is equal to x 0 plus epsilon plus dot dot dot dot. And, what we want to determine is, we want to determine these corrections x 1 x 2 x 3 and so on and so forth. Let us see how do we do this. The process is simple, we just substitute this into our governing equation and then we collect coefficients for every power of epsilon.

So, our equation let me rewrite it here. So, I am just writing the equation in the old form. So, we have to substitute it here. So, what do we obtain? We get x 0 remember that we know what is the value of x 0. I am just keeping it symbolic here because it will make my algebra a little bit easier.

So, I will get x 0 plus x 1 epsilon plus x 2 epsilon square, I will just go up to x 2 you can go up to higher powers if you want. So, this is squared I am substituting. So, x square is just this term minus 3 plus 2 epsilon into again x 0 plus x 1 plus x 2 epsilon square plus epsilon plus 2 is equal to 0. Now, our task is to collect coefficients of all powers of epsilon, we will start with order epsilon to the power 0 which is 1. So, our order 1 term we will collect.

Even before we do this, we expect the order 1 term to be just proportional to the unperturbed problem that we had written before or in other words, when we took our original equation and put epsilon equal to 0 that would give our unperturbed system. So, our order 1 terms would be I am only writing the coefficients I am not writing in this case there is no its epsilon to the power 0. So, you can see that there will be an x 0 square coming from here.

So, let me use a different colour. So, there is an x 0 square coming from here because this term is squared. So, that is an order 1 term. So, x 0 square there will not be any further order 1 terms from this first term squared. Everything will be of a higher order term, either order epsilon or order epsilon square and so on. Do we get any order 1 term from the product of these 2?

Yes, you can see that if I open up this bracket by multiplying, I will get minus 3 x naught that does not contain any epsilon. So, that is an order 1 term everything else we will contain an epsilon. And so, we will not account for it here. Is there any order 1 term here?

Yes, plus 2 epsilon is an order epsilon term so this. So, this is the coefficient of epsilon to the power 0. What is the coefficients of epsilon or epsilon order epsilon to the power 1? So, let me call it epsilon to the power 1. So, from the first term, what do we find? The product of x 0 and x 1 epsilon will give you a twice x 0 x 1 that will have an epsilon as its coefficient.

We are not going to write the coefficient. Is there any order epsilon term here from the first term? No, what about the second term? If you open up the bracket, then you expect minus 3 x 1 that contains an epsilon, but you also expect the product of these two to be an order epsilon term. So, that is minus 2 x naught. That is all we will get from here and then at the end we will have this term.

I am writing it 1 because I am collecting the coefficients of epsilon. So, this will get multiplied by epsilon, if I put it together order epsilon square. Order epsilon square also we proceed in the same way. So, from the first square term you can see that x 1 squared would have a epsilon square. Then, the product of x 0 and x 2 would there would be a 2 there and then epsilon square there, anything else from here? No.

Then, from the product of this term, you can see that we would get a minus 3 x 2. We will also get a minus 2 x 1 and that is the coefficient of epsilon square. We can continue like this, but the important thing is if you have to solve this equation, then the power of then the coefficient of every power of epsilon has to be separately 0 because this equation as a whole is 0.

So, our equations we get a series of equations for x 0 x 1 x 2. And so, I am just erasing the epsilon and I am just setting it equal to 0. Remember that our goal is to determine x x 1, x 2, x 3 and so on. x 0 is actually known to us. So, this equation is already known to us, we have solved this equation and determined that x 0 is equal to 1 or 2.

So, x 0 is also already known to us. So, now, let us take the order epsilon equation and work out what is the value of x 1. So, at order epsilon, we have our equation twice x 0 x 1 minus thrice x 1 minus twice x naught plus 1 is equal to 0. Now, if I choose for x naught.

So, I have two choices of x naught for x naught equal to 1 this equation becomes twice x 1 minus thrice x 1 minus 2 plus 1 is equal to 0 and this would imply x 1 is equal to minus 1. So, we have determined x 1 is equal to minus 1. Similarly, we can solve for the same x naught is equal to. So, we have to; we have to propagate the value of x 1 here.

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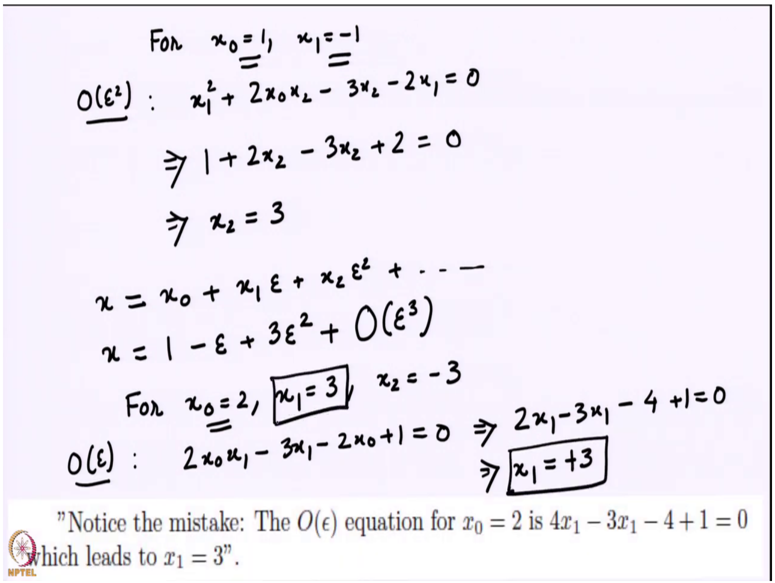

So, for x naught is equal to 1 x 1 is equal to minus 1, which we just found out. Let us write down the expression at order epsilon square; at order epsilon square we found it to be x 1 square plus twice x naught x 2 minus thrice x 2 minus twice x 1 is equal to 0.

If I use this value of x naught and this value of x 1, then this equation becomes 1 plus twice x 2 minus thrice x 2 plus 2 is equal to 0. This implies x 2 is equal to plus 3. So, we have determined our expansion up to order epsilon square for the root x 0 is equal to 1 it is. So, our equation has the form x 0 plus x 1 epsilon plus x 2 epsilon square plus dot dot dot. We are not determining beyond that and we have determined that x for x naught equal to 1 x 1 is minus 1.

So, I get minus epsilon and x 2 is 3. So, I get a plus 3 epsilon square. So, this is a perturbative expansion for one root what I am missing here? Is an order epsilon cube is an infinite series whose order is order epsilon cube. These are further higher order corrections. What about the second root? Our base state or the unperturbed problem had 1 and 2.

So, we will have to redo this exercise for that problem also. You can check that for x 0 is equal to 2 you will get x 1 is equal to 3 and x 2 is equal to minus 3. So, first let us write the order epsilon equation, the order epsilon equation was twice x naught x 1 minus thrice x 1 minus twice x naught plus 1 is equal to 0.

This was our order epsilon equation and if you substitute x naught is equal to 2, then you get twice x 1 minus thrice x 1 minus 4 plus 1 is equal to 0. So, this implies x 1 is equal to plus 3, which is what we have found here. Order epsilon square basically this equation.

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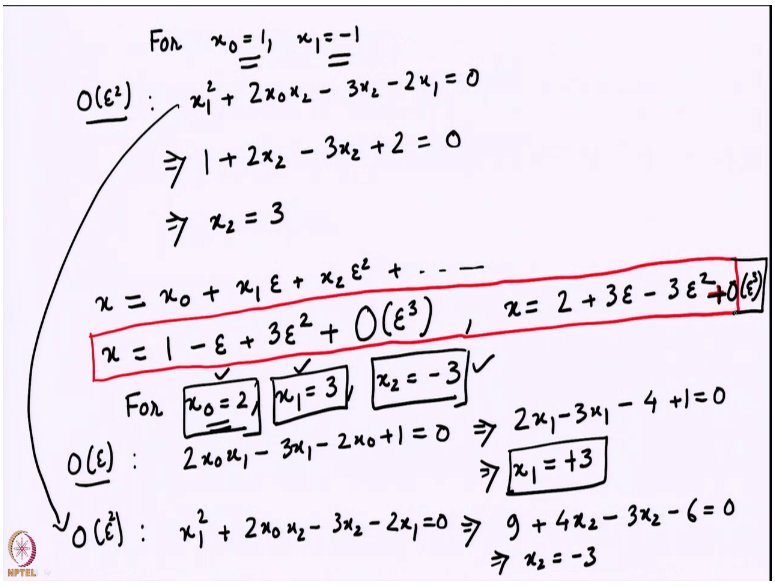

x 1 square plus twice x naught x 2 minus thrice x 2 minus twice x 1 is equal to 0. If you substitute, you will have to substitute x naught is equal to 2 and x 1 is equal to 3. So, x 1 is equal to 3 implies this is 9, x naught is 2. So, plus 4 x 2 minus 3 x 2 minus 6 is equal to 0.

This implies x 2 is equal to minus 3, x 2 is equal to minus 3. So, one can systematically go and find better and better approximations to the problem. Let us try to understand, what exactly is this are these expressions where exactly are these expressions coming from.

So, this is one expression this is corresponding to the root 1 the unperturbed root 1 and then I will have another expression, which is 2 plus 3 epsilon minus 3 epsilon square. I am just using these three values. So, let me put this in red. So, this is one expression and that is another expression.

I have written it up to epsilon square and then I can carry forward to higher powers if I need. So, this 1 has plus order epsilon cube ok. So, let us now try to understand where does this expressions come from. For this we are going to use the exact solution.

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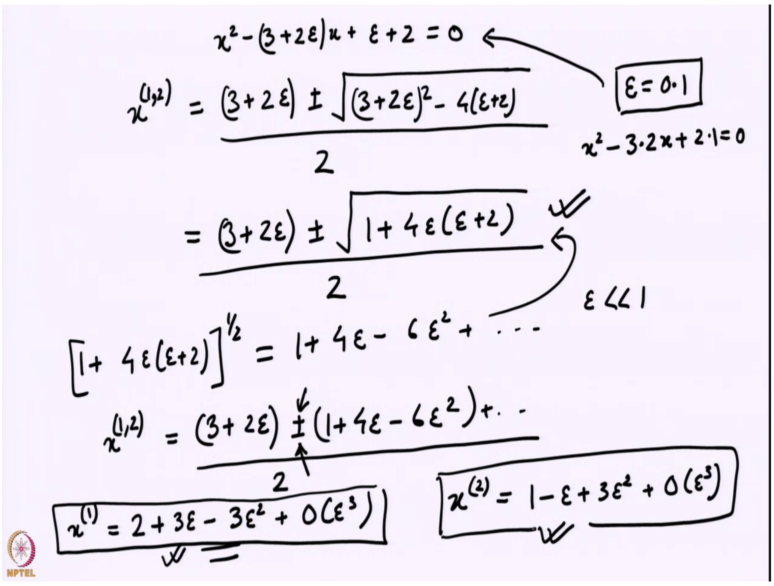

Our equation was x square minus 3 plus twice epsilon into x plus epsilon plus 2 is equal to 0. x 1 comma 2 the 2 roots let us say is minus b plus minus. Now, I am writing down the formula for the roots of a quadratic equation. Now, simplify this, then you can write this as the inside part if you open up the square, then you can write this as 1 plus 4 epsilon into epsilon plus 2.

Now, 1 plus 4 epsilon into epsilon plus 2 to the power half, I am just looking at the discriminant. This you can expand it in using the binomial expansion and this will have 1 plus 4 epsilon minus 6 epsilon square plus dot dot dot. This is all this is for epsilon less than 1.

So, if you now go and plug this expression into this, then you will get x 1 2 is equal to 3 plus 2 epsilon plus minus 1 plus 4 epsilon minus 6 epsilon square plus dot dot dot of course, divided by 2. And so, this will give you for the plus 1, you will get 1 expression for the minus for the plus here you will get 1 for the minus you will get another expression.

So, corresponding to the ok. So, let me not use x 1 2 like this because 1 2 are actually the roots also. So, this is the 1 or 2. So, the first root and this root corresponds to 2 the unperturbed root is 2. So, you will get 2 plus 3 epsilon minus 3 epsilon square plus order epsilon cube.

The second root so, you can check which one comes from which one of them one of this expression will come from taking the plus sign the other will come from taking the minus sign. The other root is 1 minus epsilon plus 3 epsilon square plus order epsilon cube.

So, what perturbation is doing? Is basically it is taking the exact solution and it is expressing it in a series form. And so, we are getting this series. I leave it to you to check the accuracy of this roots. So, for example, take x 1 is equal to a small number compared to 1, let us say 0.1 sorry, epsilon to be a small number 0.1.

Plug it back into this equation, you will get some equation. So, in this case you will get the equation x square minus 3.2 x plus 2.1 is equal to 0. Calculate the exact roots of this equation from this expression using your calculator. Calculate the same roots using this and that and check the accuracy. You will find that these expressions do a reasonably good job at finding the roots.

We will continue our discussion of perturbation methods in the next video.